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ALEPH-ZERO CATEGORICAL STONE ALGEBRAS

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Abstract

This paper is a contribution to the problem of characterizing the $\aleph_0$-categorical Stone algebras. If the dense set is either finite or a chain, the problem is solved by reducing it to the $\aleph_0$-categoricity of the skeleton and the dense set, solutions for these being known. If the dense set is a Boolean algebra, we show that this type of reduction works for certain subclasses but not for all such algebras. For generalized Post algebras the characterization problem is solved completely.

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A Stone algebra is a distributive lattice with 0 and 1, with pseudo-complementation x^* where x^* is the largest element disjoint from x, and satisfying $x^*+x^{**} = 1$. This latter property is weaker than the requirement $x+x^* = 1$ which would make the algebra a Boolean algebra. There is an extensive literature of Stone algebras (see (Grätzer (1971), Balbes and Dwinger (1974), and further references therein).

A structure is categorical if it is determined up to isomorphism by its first-order logical properties. It is a classic result that a structure is categorical if and only if its set of elements is finite. If α is an infinite cardinal, a structure is α-categorical if its set of elements has cardinality α and if any other structure having α elements and with the same first-order logical properties is isomorphic to it. A reference for categoricity results is Chang and Keisler (1973), Sections 2.3 and 7.1.

Because any infinite distributive lattice must contain an infinite chain, a result of Shelah (see Chang and Keisler (1973), p. 424) shows that no distributive lattice

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can be α-categorical, where α is uncountable. So we shall be concerned with Stone algebras which are $\aleph_0$-categorical.

There has been considerable interest in characterizing algebraically those structures which are $\aleph_0$-categorical. For example, Rosenstein (1973) showed that a denumerable Abelian group is $\aleph_0$-categorical if and only if there is a finite bound on the orders of its elements. A well-known result, which we shall use frequently, states that a denumerable Boolean algebra is $\aleph_0$-categorical if and only if its set of atoms is finite. Other results on $\aleph_0$-categoricity are referred to in the introduction to Baur, Cherlin and Macintyre (1977).

Suppose L is a Stone algebra. Chen and Grätzer (1969) obtained the following fundamental results. The skeleton of L, denoted S_L or just S, is $\{x^* \mid x \in L\}$. The dense set of L, denoted D_L or just D, is $\{x \mid x^* = 0\}$. Define the function φ_L (or just φ) from S into the lattice of filters in D, ordered by set inclusion, as follows: $\varphi(x) = \{y \in D \mid y \geqslant x^*\}$. For the triple (S, D, φ) thus associated with L, S is a Boolean algebra, D is a distributive lattice with 1, and φ is a Boolean algebra homomorphism. We say that two such triples (S, D, φ) and (S_1, D_1, φ_1) are isomorphic if there are isomorphisms θ from S onto S_1 and μ from D onto D_1 such that if μ' is the isomorphism induced by μ from the lattice of filters of D onto the lattice of filters of D_1 then $\mu'\varphi = \varphi_1 \theta$. Chen and Grätzer showed that two Stone algebras are isomorphic if and only if their associated triples are isomorphic, every triple is isomorphic to a triple associated with a Stone algebra, and in fact if S is any Boolean algebra with at least two elements and D is any distributive lattice with 1, then there is a φ and a Stone algebra L such that (S, D, φ) is isomorphic to the triple associated with L.

It follows easily (Lemma 2(ii)) that if L is a Stone algebra then D_L is a first-order definable subset of L. Hence a characterization of all $\aleph_0$-categorical Stone algebras might yield a characterization of all $\aleph_0$-categorical distributive lattices with 1. This latter problem seems quite difficult.

On p. 164 of Grätzer (1971), Grätzer formulates "the goal of research for Stone algebras" as reducing a given problem to two corresponding problems, one for Boolean algebras and one for distributive lattices with 1. He points out that there are "examples in which this program works and others in which it does not". We shall show that there are classes of Stone algebras for which the $\aleph_0$-categoricity problem can be handled this way, but that for the class of all Stone algebras of order 3 it cannot.

THEOREM 1. *Suppose L is a denumerable Stone algebra whose dense set D_L is a chain. Then L is $\aleph_0$-categorical if and only if S_L and D_L are $\aleph_0$-categorical.*

An algebraic characterization of the $\aleph_0$-categorical chains is given in Rosenstein (1969). Theorem 1 is proved below.

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THEOREM 2. *Suppose L is a denumerable Stone algebra whose dense set D_L is finite. Then L is $\aleph_0$-categorical if and only if S_L is $\aleph_0$-categorical.*

We shall not give a proof of Theorem 2. Such a proof would combine ideas from the proofs of Theorems 1 and 3 below, and would also use the obvious fact that any filter in a finite lattice is principal.

There is another class of Stone algebras for which we can completely solve the $\aleph_0$-categoricity problem. Generalized Post algebras were defined by Chang and Horn (see the first paragraph on p. 204 of Balbes and Dwinger (1974)). A generalized Post algebra L is a free product $C * B$ in the variety of all bounded distributive lattices of a bounded chain C and a Boolean algebra B. If C is finite then this is equivalent to the usual definition of a Post algebra of finite order. That L is a Stone algebra follows from the second paragraph on p. 206 of Balbes and Dwinger (1974). Note that $S_L = B$, but in general D_L is a strictly larger set than C. The following theorem will be proved later by piecing together known results.

THEOREM 3. *If $L = C * B$ is a denumerable generalized Post algebra then L is $\aleph_0$-categorical if and only if C and B are $\aleph_0$-categorical.*

The remaining results in this paper are concerned with Stone algebras whose dense set is a Boolean algebra. These are called Stone algebras of order 3 (see Balbes and Dwinger (1974), pp. 205–210; the only Stone algebra of order 1 is trivial, and the Stone algebras of order 2 are the Boolean algebras). If F is a filter in a countable Boolean algebra B and if F has a complement in the lattice of filters in B then F is principal. (This simple fact is proved in Lemma 1 below.) If L is a Stone algebra of order 3 we can associate with L the simplified triple (S_L, D_L, ν_L) where S_L and D_L are Boolean algebras and ν_L maps each x in S_L to the complement in D_L of the element $y \in D_L$ such that $\varphi_L(x)$ is the principal filter in D_L generated by y. The function ν_L is a Boolean algebra homomorphism from S_L into D_L. A simplified triple (S, D, ν) has S and D Boolean algebras and ν a Boolean algebra homomorphism from S into D. Two simplified triples (S, D, ν) and (S_1, D_1, ν_1) are isomorphic if there are isomorphisms θ from S onto S_1 and μ from D onto D_1 such that $\nu_1 \theta = \mu\nu$. The fundamental results of Chen and Grätzer now apply to simplified triples and Stone algebras of order 3.

THEOREM 4. *Suppose L is a denumerable Stone algebra of order 3 such that $\nu_L(S_L)$ is finite. Then L is $\aleph_0$-categorical if and only if S_L and D_L are $\aleph_0$-categorical.*

Examples 1 and 2 below show that the hypothesis of Theorem 4 that $\nu_L(S_L)$ be finite cannot be weakened to $\nu_L(S_L)$ being $\aleph_0$-categorical. Theorem 4 and Theorem 5 are proved below.

THEOREM 5. *Suppose L is a denumerable Stone algebra of order* 3 *such that* $\nu_L(S_L) = D_L$ *and the kernel of* ν_L *has a supremum in* S_L. *Then L is* $\aleph_0$*-categorical if and only if* S_L *and* D_L *are* $\aleph_0$*-categorical.*

Example 1 below shows that the hypothesis of Theorem 5 that $\nu_L(S_L) = D_L$ cannot be weakened to $\nu_L(S_L)$ being $\aleph_0$-categorical. Example 2 below shows that the hypothesis of Theorem 5 that the kernel of ν_L have a supremum in S_L cannot be deleted.

In Example 1 a Stone algebra L of order 3 is constructed such that S_L, D_L and $\nu_L(S_L)$ are denumerable atomless Boolean algebras, ν_L is one-to-one, and yet L is not $\aleph_0$-categorical (because $\nu_L(S_L)$ is "badly" embedded in D_L). In Example 2 a Stone algebra of order 3 is constructed such that S_L and D_L are denumerable atomless Boolean algebras, $\nu_L(S_L) = D_L$, and yet L is not $\aleph_0$-categorical (because the kernel of ν_L is "badly" embedded in S_L).

Preliminaries

Our notation for Stone algebras for the most part follows Section 14 of Grätzer (1971). Our model theoretic notation is standard (see Chang and Keisler (1973)). Many of the proofs will be variations on the classic back-and-forth argument which shows that a denumerable atomless Boolean algebra is $\aleph_0$-categorical, and which uses the method of "bits". We refer the reader to Chang and Keisler (1973), especially Section 5.5.

Our language has two two-place functions (xy for infimum and $x+y$ for supremum), two constant symbols 0, 1 and a one-place function symbol (x^* for pseudocomplement). If B is a Boolean algebra and $b \in B$ then $B|b$ is the Boolean algebra $\{x \in B \mid x \leqslant b\}$ with $+$, $\cdot$, 0 as usual, with largest element b, and with the complement of x defined to be the infimum of b and the complement in B of x (see Chang and Keisler (1973), p. 294). $\mathscr{P}(\omega)$ denotes the power set of ω (the set of all non-negative integers), considered as a Boolean algebra in the usual way.

LEMMA 1. *If a filter F in a countable Boolean algebra B has a complement in the lattice of filters in B then F is principal.*

PROOF. We have F, F' such that $F \cap F' = \{1\}$ and the filter generated by $F \cup F'$ is B. Since B is countable we get F generated by $x_0 \geqslant x_1 \geqslant x_2 \geqslant \ldots$, and F' generated by $y_0 \geqslant y_1 \geqslant y_2 \geqslant \ldots$. Since $F \cap F' = \{1\}$, $x_i + y_j = 1$ for all i,j. Since $F \cup F'$ generates all of B, there is an x_i and a y_j such that $x_i y_j = 0$. Without loss of generality we can assume $x_0 y_0 = 0$, and thus $x_i y_j = 0$ for all i,j. It now follows that each x_i is the complement of each y_j, and thus $x_i = x_j$ and $y_i = y_j$ for all i,j. So F is generated by x_0 and F' by $y_0 = \bar{x}_0$, as required.

LEMMA 2. *There are first-order formulas* $\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x,y), \psi_5(x)$ *such that if* $L = (S, D, \varphi)$ *is any Stone algebra then*

(i) $\psi_1(x)$ *defines* S *in* L,

(ii) $\psi_2(x)$ *defines* D *in* L,

(iii) $\psi_3(x)$ *defines the kernel of* φ *in* L;

and if $L = (S, D, \nu)$ *is any countable Stone algebra of order* 3 *then*

(iv) $\psi_3(x)$ *defines the kernel of* ν *in* L,

(v) $\psi_4(x,y)$ *defines* $\{(x,y) \mid \nu(x) = y\}$ *in* L,

(vi) $\psi_5(x)$ *defines* $\nu(S)$ *in* L.

PROOF. Let $\psi_1(x)$ be $(\exists y)(x = y^*)$, and let $\psi_2(x)$ be $x^* = 0$. More informally, $\psi_3(x)$ is the formula

$$x \in S \quad \text{and} \quad (\forall y) \quad (\text{if } y \in D \text{ and } y \geqslant x^* \text{ then } y = 1).$$

Again informally, $\psi_4(x,y)$ is the formula

$$x \in S,\ y \in D \text{ and } y \text{ is the complement in } D \text{ of the least } z \text{ in } D \text{ such that } z \geqslant x^*.$$

Now let $\psi_5(x)$ be $(\exists z)\psi_4(z,x)$.

The proofs of (i), (ii) and (iii) are immediate from the definitions of S, D, φ. As to (iv), it follows from the definition of ν that the kernel of ν is the same as the kernel of φ. Using Lemma 1 and the definition of ν, (v) follows. Then (vi) is immediate.

COROLLARY 3. *If* L *is an* $\aleph_0$*-categorical Stone algebra then* S_L *is an* $\aleph_0$*-categorical Boolean algebra and* D_L *is an* $\aleph_0$*-categorical distributive lattice with* 1. *If* L *is an* $\aleph_0$*-categorical Stone algebra of order* 3 *then* S_L, D_L, $\nu_L(S_L)$ *are* $\aleph_0$*-categorical Boolean algebras and the kernel of* ν_L *is an* $\aleph_0$*-categorical distributive lattice with* 0.

Example 1 shows that the converse of each statement in Corollary 3 is false.

LEMMA 4. *Suppose* A_0, A_0', A_1, A_1' *are denumerable atomless Boolean algebras and for each* $i \in \{0, 1\}$, τ_i *is a homomorphism from* A_i *onto* A_i' *such that the supremum in* A_i *of the kernel of* τ_i *is* 1. *Then there exist isomorphisms* θ *from* A_0 *onto* A_1 *and* μ *from* A_0' *onto* A_1' *such that* $\tau_1\theta = \mu\tau_0$.

PROOF. Let I_i denote the kernel of τ_i. It suffices to find an isomorphism θ from A_0 onto A_1 such that $\theta(I_0) = I_1$, since μ can then be defined as follows: for x in A_0', choose any y in A_0 such that $\tau_0(y) = x$, and let $\mu(x)$ be $\tau_1\theta(y)$.

We use a back-and-forth argument to get θ. Suppose we have defined θ on a finite subalgebra B_0 of A_0 so that θ maps B_0 isomorphically onto $B_1 \subset A_1$, and

$\theta(B_0 \cap I_0) = B_1 \cap I_1$. Let $b_1, \ldots, b_n$ be the atoms in B_0, and $c_1 = \theta(b_1), \ldots, c_n = \theta(b_n)$ the atoms in B_1. Now let x be any member of A_0. So $x = xb_1 + \ldots + xb_n$. We will define $y = y_1 + \ldots + y_n$ in A_1 as follows. If $xb_i = 0$, let $y_i = 0$. If $xb_i = b_i$, let $y_i = c_i$. If $xb_i > 0$, $\bar{x}b_i > 0$, and both are in I_0 then $b_i \in I_0$ and so $c_i \in I_1$. Since A_1 is atomless, get $c_i = u + v$ with $uv = 0$, $u > 0$, $v > 0$. Let $y_i = u$. If $xb_i \notin I_0$ and $\bar{x}b_i \notin I_0$ then $b_i \notin I_0$ and so $c_i \notin I_1$. Since $A_1/I_1 \cong A_1'$ is atomless we can get $c_i = u + v$ with $uv = 0$, $u \notin I_1$, $v \notin I_1$. Let y_i be u.

Now suppose $xb_i \in I_0$, $xb_i > 0$ and $\bar{x}b_i \notin I_0$. Hence $b_i \notin I_0$ and thus $c_i \notin I_1$. Using formula (9) on p. 60 of Sikorski (1964), we have

$$c_i = c_i \cdot 1 = c_i\left(\sum_{t \in I_1} t\right) = \sum_{t \in I_1} (c_i t).$$

Choose one such $c_i t$ which is not 0 and let it be y_i. Note $\bar{y}_i c_i \notin I_1$. The case $xb_i \notin I_0$, $\bar{x}b_i \in I_0$, $\bar{x}b_i > 0$ is handled similarly.

Defining $\theta(x) = y$, the construction ensures that θ maps the subalgebra C_0 generated by $B_0 \cup \{x\}$ isomorphically onto the subalgebra C_1 generated by $B_1 \cup \{y\}$ with $\theta(C_0 \cap I_0) = C_1 \cap I_1$. The remainder of the proof proceeds as in the classic back-and-forth argument.

Proof of Theorem 1. One direction follows from Corollary 3 above. Now assume S and D are $\aleph_0$-categorical. Suppose L_1 is a countable structure and $L_1 \equiv L$. We require $L_1 \cong L$. We get at once that $L_1 = (S_1, D_1, \varphi_1)$ is a Stone algebra. By Lemma 2, $S \equiv S_1$ and $D \equiv D_1$. Hence $S \cong S_1$ and $D \cong D_1$. If $D = \{1\}$, $L \cong S \cong S_1 \cong L_1$. So now assume D has at least two elements. Let μ be any isomorphism from D onto D_1. It is easy to see that a chain with largest element 1 has only two complemented filters, $\{1\}$ and the chain itself. Hence the induced map μ' on the lattice of filters in D maps D to D_1 and $\{1_D\}$ to $\{1_{D_1}\}$. We also know that the Boolean algebra homomorphism φ maps S onto the two-element Boolean algebra, and similarly for φ_1 and S_1. We will be done if we can find an isomorphism θ from S onto S_1 such that θ(kernel of φ) = kernel of φ_1.

We know that S is a countable Boolean algebra with a finite number of atoms, and $S \cong S_1$. Letting I denote the kernel of φ and I_1 the kernel of φ_1, we know that I (and similarly I_1) is proper and is also maximal; that is, for each $x \in S$ exactly one of x, x^* is in I. There are two possibilities for I. If there is an atom z of S not in I then z is the only such atom and, using Lemma 2, S_1 will also have a unique atom z_1 not in I_1. It then follows from the maximality of I and I_1 that any isomorphism θ from S onto S_1 such that $\theta(z) = z_1$ will satisfy $\theta(I) = I_1$. Such a θ is easily found.

Now suppose I contains all the atoms of S. It follows that I is generated in S by the set of atoms of S together with a strictly increasing sequence of atomless elements whose supremum is the largest atomless element in S. S_1 and I_1 have similar properties. The classic argument that any two denumerable atomless Boolean algebras are isomorphic can now be adapted to provide the isomorphism we need. We shall not give the details, but a detailed proof, using the method of "bits" (see Chang and Keisler (1973), p. 300) would use the following facts about S and I, and the corresponding facts about S_1 and I_1:

(i) if $x>0$, $x\in I$, x atomless and $x=y+z$ with $yz=0$, $y>0$, $z>0$, then $y\in I$ and $z\in I$,

(ii) if $x\in S-I$, x atomless and $x=y+z$ with $yz=0$, $y>0$, $z>0$, then exactly one of y, z is in I.

There are some similarities between the detailed proof needed here and the argument given above for Lemma 4. This completes the proof of Theorem 1.

We see from the proof of Theorem 1 that if S is an $\aleph_0$-categorical Boolean algebra and D is an $\aleph_0$-categorical chain with 1 and with at least two elements then

(a) if either S is finite or S contains no atoms then, up to isomorphism, there is a unique $\aleph_0$-categorical Stone algebra L such that $S_L\cong S$ and $D_L\cong D$, and

(b) if S is infinite and contains at least one atom then, up to isomorphism, there are exactly two $\aleph_0$-categorical Stone algebras L such that $S_L\cong S$ and $D_L\cong D$.

Proof of Theorem 3. By Quackenbush (1972), $C*B\cong C[B]^*$, the bounded Boolean power. From Burris (1975), Theorem 4.3 (ix), we get that if C and B are $\aleph_0$-categorical then so is L.

Suppose C is not $\aleph_0$-categorical. So we get C' of the same cardinality as C, $C'\equiv C$, $C'\ncong C$. Using Burris (1975), Theorem 4.3(i), we have

$$C*B\cong C[B]^*\equiv C'[B]^*\cong C'*B.$$

By Balbes and Dwinger (1974), Theorem 7 on p. 141, we get $C*B\ncong C'*B$. So L is not $\aleph_0$-categorical. A similar proof shows that if B is not $\aleph_0$-categorical then neither is L.

Proof of Theorem 4. One direction follows from Corollary 3. We sketch the proof in the other direction. Suppose S, D are $\aleph_0$-categorical Boolean algebras, and $\nu(S)$ is a finite subalgebra of D. Consider the sentences of the following forms which are true in L.

(i) The sentence which states that L is a Stone algebra (S, D, ν) of order 3.

(ii) The sentence which states that S is a Boolean algebra with exactly m atoms (where $m\geqslant 0$ is the appropriate finite number) and which states whether or not S contains an atomless element greater than 0.

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(iii) The sentence which states that D is a Boolean algebra with exactly t atoms (where $t \geqslant 0$ is the appropriate finite number) and which states whether or not D contains an atomless element greater than 0.

(iv) The sentence which states that $\nu(S)$ has exactly 2^n elements (where $n \geqslant 0$ is the appropriate finite number).

(v) The sentence which states that in S there exist $x_0, x_1, \ldots, x_n$ such that $x_0 + x_1 + \ldots + x_n = 1$; for $i \neq j$, $x_i x_j = 0$; for $1 \leqslant i \leqslant n$, $\nu(x_i)$ is an atom in $\nu(S)$; for $1 \leqslant i \leqslant n$, x_i is either an atom in S or is atomless in S; and which states

(a) for $1 \leqslant i \leqslant n$, whether x_i is an atom in S or whether x_i is atomless in S,

(b) whether $x_0 = 0$, how many atoms of S are contained in x_0, and whether x_0 contains a non-zero atomless element of S,

(c) for $1 \leqslant i \leqslant n$, how many atoms of D are contained in $\nu(x_i)$ and whether $\nu(x_i)$ contains a non-zero atomless element of D.

That (i)–(v) are first-order statements follows from Lemma 2.

To complete the proof we have to show that if L_1 is a countable structure and $L_1 \equiv L$ then $L_1 \cong L$. Since $L_1 \equiv L$, we get that L_1 satisfies exactly the same sentences of the form (i)–(v) as L does. We obtain easily from (i)–(iv) that $L_1 = (S_1, D_1, \nu_1)$ is a Stone algebra of order 3, $S_1 \equiv S$, $D_1 \equiv D$ and $\nu_1(S_1) \equiv \nu(S)$. Hence by our hypotheses $S_1 \cong S$, $D_1 \cong D$, $\nu_1(S_1) \cong \nu(S)$. But we must choose these isomorphisms carefully, and (v) is designed to enable us to do so. Let $y_0, y_1, \ldots, y_n$ be the elements in S_1 given by the truth of (v) in L_1. For $1 \leqslant i \leqslant n$, define $\mu(\nu(x_i)) = \nu_1(y_i)$ (both $\nu(x_0)$ and $\nu_1(y_0)$ being 0). Then (v)(c) ensures that we can extend μ to an isomorphism from D onto D_1. For $0 \leqslant i \leqslant n$, define $\theta(x_i) = y_i$. It follows now that θ can be extended to an isomorphism from S onto S_1, with θ(kernel of ν) = kernel of ν_1. If $1 \leqslant i \leqslant n$ and x_i is atomless in S, we would use the method of Theorem 1 to define θ from $S|x_i$ onto $S_1|y_i$. If $1 \leqslant i \leqslant n$ and x_i is an atom in S, θ has already been defined on $S|x_i = \{0, x_i\}$. Since $\nu(x_0) = 0$ and $\nu_1(y_0) = 0$, (v)(b) ensures that θ can be extended from $S|x_0$ onto $S_1|y_0$. Having done these things, θ can now be extended to all of S in only one way. The isomorphism of L and L_1 follows, completing the proof of Theorem 4.

By way of contrast with Theorem 5 below, it can be seen that if $L = (S, D, \nu)$ is a Stone algebra of order 3 with $\nu(S)$ finite, then the kernel of ν has a supremum in S. (In the notation of the proof of Theorem 4 this supremum is x_0 + the sum of those x_i, $1 \leqslant i \leqslant n$, such that x_i is atomless in S.)

Proof of Theorem 5. Again, one direction follows from Corollary 3. Now assume S and D are $\aleph_0$-categorical. Remark that because ν maps S onto D, ν maps an atom of S to either 0 in D or to an atom of D. Consider the sentences of the following forms which are true in L.

(i) The sentence which states that L is a Stone algebra (S, D, ν) of order 3, and that $\nu(S) = D$.

(ii) The sentence which states that S has exactly m atoms, and that exactly r of these atoms are mapped by ν to 0 in D (where $m \geqslant r \geqslant 0$, and m and r are the appropriate finite numbers).

(iii) The sentence which states that D has exactly t atoms (where $t \geqslant 0$ is the appropriate finite number, and where it must be the case that $t \geqslant m-r$).

(iv) The sentence which asserts, with $n = t-(m-r)$, the existence in S of atomless elements $x_0, x_1, \ldots, x_n, x_{n+1}$ such that if $i \neq j$ then $x_i x_j = 0$, and that $x_0 + x_1 + \ldots + x_n + x_{n+1}$ is the largest atomless element in S, and which states

(a) that x_0 is the largest atomless element in S such that if $0 < z \leqslant x_0$ then $\nu(z) \neq 0$; and whether $x_0 = 0$;

(b) that for $1 \leqslant i \leqslant n$, $\nu(x_i)$ is an atom in D;

(c) that x_{n+1} is the supremum in S of those $z \leqslant x_{n+1}$ such that $\nu(z) = 0$; and whether $x_{n+1} = 0$; and whether $\nu(x_{n+1}) = 0$.

From Lemma 2, (i)–(iv) are first-order statements.

We must show that if L_1 is a countable structure and $L_1 \equiv L$ then $L_1 \cong L$. From (i) we get that $L_1 = (S_1, D_1, \nu_1)$ is a Stone algebra of order 3. We define θ and μ in stages. Let θ map the set of atoms x of S one-to-one and onto the set of atoms y of S_1, such that $\nu(x) = 0$ if and only if $\nu_1(\theta(x)) = 0$. This is possible by (ii). If x is an atom of S and $\nu(x)$ is an atom of D, define $\mu(\nu(x)) = \nu_1(\theta(x))$.

From (iv) we get $x_0, x_1, \ldots, x_n, x_{n+1}$ in S and the corresponding $y_0, y_1, \ldots, y_n, y_{n+1}$ in S_1. Note $x_0 = 0$ if and only if $y_0 = 0$. If $x_0 > 0$ then ν is an isomorphism from $S|x_0$ onto $D|\nu(x_0)$; in addition $y_0 > 0$ and ν_1 is an isomorphism from $S_1|y_0$ onto $D_1|\nu_1(y_0)$. From (iv)(a), $S|x_0$ and $S_1|y_0$ are denumerable atomless Boolean algebras. Let θ be any isomorphism from $S|x_0$ onto $S_1|y_0$, and let μ be $\nu_1 \theta \nu^{-1}$ on $D|\nu(x_0)$. For $1 \leqslant i \leqslant n$, let $\mu(\nu(x_i)) = \nu_1(y_i)$, and using ideas from the proof of Theorem 1, define θ to be an isomorphism from $S|x_i$ onto $S_1|y_i$ so that $\nu_1 \theta = \mu\nu$.

Finally consider x_{n+1} and y_{n+1}. Note $x_{n+1} = 0$ if and only if $y_{n+1} = 0$. If $x_{n+1} > 0$ and $\nu(x_{n+1}) = 0$ then let θ be any isomorphism from $S|x_{n+1}$ onto $S_1|y_{n+1}$, since each is a denumerable atomless Boolean algebra. Now suppose $\nu(x_{n+1}) \neq 0$, and thus $\nu_1(y_{n+1}) \neq 0$. We notice that $S|x_{n+1}$, $S_1|y_{n+1}$, ν restricted to $S|x_{n+1}$, ν_1 restricted to $S_1|y_{n+1}$, $D|\nu(x_{n+1})$, $D_1|\nu_1(y_{n+1})$ satisfy the hypotheses of Lemma 4. Applying Lemma 4 we get θ defined on $S|x_{n+1}$, and μ defined on $D|\nu(x_{n+1})$ so that $\nu_1 \theta = \mu\nu$.

The maps θ and μ, as defined so far, now extend uniquely to give the required isomorphisms, and complete the proof of Theorem 5.

REMARK. If L is a countable structure and $\{\delta_n\}_{n \in \omega}$ is a denumerable sequence of members of L with the property that for $i \neq j$ there is a first-order formula $\psi_{ij}(x)$ which is true of δ_i in L but false of δ_j in L, then L is not $\aleph_0$-categorical. This fact is an immediate consequence of the fundamental characterization of $\aleph_0$-categorical structures (see Ryll–Nardzewski (1959)). We shall use it in both of the following examples.

EXAMPLE 1. We will construct a Stone algebra of order 3, $L = (S, D, \nu)$, with S, D and $\nu(S)$ denumerable atomless Boolean algebras, ν a one-to-one function and yet L not $\aleph_0$-categorical.

Let $\{\alpha_n\}_{n \in \omega}$ be a partition of ω into denumerably many denumerable and pairwise disjoint sets. For each n, let $\{\beta_i^n\}_{i \in \omega}$ and $\{\gamma_i^n\}_{i \in \omega}$ satisfy

(a) $\emptyset = \beta_0^n \subset \beta_1^n \subset \ldots \subset \gamma_2^n \subset \gamma_1^n \subset \gamma_0^n = \alpha_n$, and

(b) for $i \geqslant 0$, $\beta_{i+1}^n - \beta_i^n$ and $\gamma_i^n - \gamma_{i+1}^n$ are infinite.

For each $n \geqslant 0$, $i \geqslant 0$, let H_i^n be a denumerable collection of subsets of $\beta_{i+1}^n - \beta_i^n$ with the property that if $x \in H_i^n$ then there are y, z in H_i^n such that y, z are denumerable sets, $y \cap z = \emptyset$, $y \cup z = x$. Let K_i^n be defined similarly for $\gamma_i^n - \gamma_{i+1}^n$.

Let T be the subalgebra of $\mathscr{P}(\omega)$ generated by

$$\{\alpha_n\}_{n \in \omega} \cup \{\beta_i^n\}_{i \in \omega, n \in \omega} \bigcup \cup_{i \in \omega, n \in \omega} H_i^n.$$

Let D be the subalgebra of $\mathscr{P}(\omega)$ generated by $T \cup \{\gamma_i^n\}_{i \in \omega, n \in \omega} \bigcup \cup_{i \in \omega, n \in \omega} K_i^n$. Note that D and T are denumerable atomless Boolean algebras. Let S be any denumerable atomless Boolean algebra and let ν be any isomorphism from S onto T. This defines $L = (S, D, \nu)$.

From Lemma 2, D and $\nu(S)$ are definable subsets of L. Consider the sequence $\{\delta_n\}_{n \in \omega}$, where $\delta_n = \alpha_0 + \alpha_1 + \ldots + \alpha_n$. Note that δ_n has been given as the sum of $n+1$ pairwise disjoint elements x, each with the property that $x > 0$, $x \in \nu(S)$ and

$$\{y \in D \mid y < x \text{ and } (\forall z)\,(z \leqslant y \rightarrow z \notin \nu(S))\}$$

has no supremum in D. This first-order property of δ_n is not a property of δ_m for $m < n$. The reason is that any element x as described above must contain at least one element of the form $\alpha_j . \overline{\beta_i^j}$, and any pairwise disjoint collection of such elements contained in δ_m has cardinality at most $m+1$. The result now follows from the remark preceding this example.

EXAMPLE 2. We will construct a Stone algebra of order 3, $L = (S, D, \nu)$, with S, D denumerable atomless Boolean algebras, $\nu(S) = D$, and yet L not $\aleph_0$-categorical.

Let $\alpha_n, \beta_i^n, \gamma_i^n, H_i^n, K_i^n$ be defined as in Example 1. Let S be the subalgebra of $\mathscr{P}(\omega)$ generated by the collection of all $\alpha_n, \beta_i^n, \gamma_i^n$, and all members of each H_i^n and each K_i^n. Note S is denumerable and atomless. Let I be the ideal in S generated by all β_i^n. Note $H_i^n \subset I$. Notice also that any x in $S - I$ must contain some member y of some K_i^n. By the definition of K_i^n we can get $y = u + v$, $uv = 0$, $u \in S - I$, $v \in S - I$.

This shows that the Boolean algebra S/I is denumerable and atomless. Let D be S/I and let ν be the canonical map from S onto D.

Consider the sequence $\{\delta_n\}_{n\in\omega}$, where $\delta_n = \alpha_0+\alpha_1+\ldots+\alpha_n$. By Lemma 2, S and I are definable subsets of $L = (S, D, \nu)$. Note that δ_n has been given as the sum of $n+1$ pairwise disjoint elements x, each with the property that $x \in S-I$ and $\{y\in I \mid y<x\}$ has no supremum in S. This first-order property of δ_n is not a property of δ_m for $m<n$. The reason is that any element x as described above must contain at least one element of the form $\gamma_i^n.\overline{\beta_j^n}$, and any pairwise disjoint collection of such elements contained in δ_m has cardinality at most $m+1$. The result now follows from the remark preceding Example 1.

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