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On Graphs Associated with Character Degrees and Conjugacy Class Sizes of Direct Products of Finite Groups

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Abstract. The prime vertex graph, $\Delta(X)$, and the common divisor graph, $\Gamma(X)$, are two graphs that have been defined on a set of positive integers X. Some properties of these graphs have been studied in the cases where either X is the set of character degrees of a group or X is the set of conjugacy class sizes of a group. In this paper, we gather some results on these graphs arising in the context of direct product of two groups.

1 Introduction

Throughout this paper, G will be a finite group. The set of irreducible characters of G is written $\mathrm{Irr}(G)$. The integers $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ are called the character degrees of G, and the integers $\mathrm{cs}(G) = \{|C| \mid C \in \mathrm{class}(G)\}$ are the class sizes of G. These are both finite sets of positive integers that include 1.

Let X be a set of positive integers. The set of prime numbers that divide integers in X is denoted by $\rho(X)$. The *prime vertex graph* $\Delta(X)$ has $\rho(X)$ as vertex set, and two such distinct primes p, q are joined by an edge if and only if pq divides some number $x \in X$.

These graphs arise when G is a finite group and X is either the set of irreducible character degrees of G, denoted by $\mathrm{cd}(G)$, or the set of conjugacy class sizes of G, denoted by $\mathrm{cs}(G)$. In this paper, we focus in on the case where $G = H \times K$ for nonabelian groups H and K. In particular, we prove the following theorem. The terms used will be defined in Section 2.

Theorem 1 *Let H and K be nonabelian groups, and suppose $G = H \times K$. Let X be either $\mathrm{cd}(G)$ or $\mathrm{cs}(G)$. Then $\Delta(X)$ is a connected graph of diameter 2. Also, the independent number, clique number, and chromatic number of $\Delta(X)$ are determined by H and K.*

We obtain our result in a more general context. Suppose X is a set of positive integers such that X has a decomposition $X = YZ$, where $YZ = \{yz \mid y \in Y, z \in Z\}$.

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We will require throughout that Y and Z be nontrivial sets of positive integers. We say a set is a *trivial set* if it has the single element 1.

2 Notation and Definitions

In this section, we outline our notation, particularly our notation for graphs. Any other notations are standard and taken mainly from [1, 4, 5]. Throughout, all graphs are considered to be simple and to have at most one edge between pairs of vertices.

Suppose that $\mathcal{G}$ is a graph. The graph $\mathcal{G}$ is called connected if there is a path between any two vertices. If u and v are vertices lying in $\mathcal{G}$, then the *distance* between the vertices u and v of $\mathcal{G}$ the number of edges on a minimum path connecting them and is denoted by $d(u,v)$. The diameter of a connected graph $\mathcal{G}$ is the maximum distance between vertices.

Let $\mathcal{G}$ and $\mathcal{H}$ be two graphs with vertex sets $V(\mathcal{G})$, $V(\mathcal{H})$ and edge sets $E(\mathcal{G})$, $E(\mathcal{H})$, respectively. The *union* of graphs $\mathcal{G}$ and $\mathcal{H}$ is the graph $\mathcal{G} \cup \mathcal{H}$ with vertex set $V(\mathcal{G}) \cup V(\mathcal{H})$ and edge set $E(\mathcal{G}) \cup E(\mathcal{H})$. If $\mathcal{G}$ and $\mathcal{H}$ are disjoint, we refer to their union as a *disjoint union*. The *join* $\mathcal{G} + \mathcal{H}$ of graphs $\mathcal{G}$ and $\mathcal{H}$ is the graph union $\mathcal{G} \cup \mathcal{H}$ together with all the edges joining $V(\mathcal{G})$ and $V(\mathcal{H})$. The *composition* $\mathcal{G}[[\mathcal{H}]]$ of two graphs $\mathcal{G}$ and $\mathcal{H}$ is the graph with vertex set $V(\mathcal{G}) \times V(\mathcal{H})$, and the vertex $u = (u_1, v_1)$ is adjacent to the vertex $v = (u_2, v_2)$ whenever either $u_1u_2 \in E(\mathcal{G})$ or $u_1 = u_2$ and $v_1v_2 \in E(\mathcal{H})$ (see [4, p. 185]).

A subgraph obtained by vertex deletions only is called an *induced subgraph*. If X is a subset of $V(\mathcal{G})$, the subgraph of $\mathcal{G}$ that is induced by X is denoted by $\mathcal{G}[X]$ and it is the graph whose vertex set is X and whose edge set consists of all edges of $\mathcal{G}$ which have both vertices in X.

An *independent set* in a graph is a set of vertices no two of which are adjacent. An independent set in a graph is called *maximum* if the graph contains no larger independent set and *maximal* if the set cannot be extended to a larger independent set. A maximum independent set is necessarily maximal, but not conversely. The cardinality of a maximum independent set in a graph $\mathcal{G}$ is called the *independent number* of $\mathcal{G}$ and is denoted by $\alpha(\mathcal{G})$. A *clique* in a graph is a set of mutually adjacent vertices. The maximum size of a clique in a graph $\mathcal{G}$ is called the *clique number* of $\mathcal{G}$ and is denoted by $\omega(\mathcal{G})$. The *chromatic number* of a graph $\mathcal{G}$ is the smallest number of colors, $\chi(\mathcal{G})$, needed to color the vertices of $\mathcal{G}$ so that no two adjacent vertices share the same color.

3 Results

We let K_n denote the complete graph with n vertices.

Theorem 3.1 *If X and Y are two nontrivial sets of positive integers that satisfy $\rho(X) \cap \rho(Y) = F$ where $|F| = n$, then*

$$\Delta(XY) = K_n + \Delta(X)[\rho(X) - F] + \Delta(Y)[\rho(Y) - F].$$

Proof Let $P = \rho(X) - F$ and $Q = \rho(Y) - F$ so that $\rho(X) = P \cup F$, $\rho(Y) = Q \cup F$, and $P \cap Q = \varnothing$. It follows that $\rho(XY)$ is the disjoint union $F \cup P \cup Q$.

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Let pq be an edge in $\Delta(X)$ where $p, q \in P$, so there is some integer x in X such that $pq \mid x$. This implies that $pq \mid xy$ for each integer $y \in Y$. Therefore, pq is an edge in the graph $\Delta(XY)$. Conversely, suppose pq is an edge in $\Delta(XY)$ where $p, q \in P$, then there are integers $x \in X$ and $y \in Y$ so that $pq \mid xy$. Since $P \cap Q = \varnothing$, we conclude that $pq \mid x$. In other words, pq is an edge in $\Delta(X)$. So the induced graph in $\Delta(XY)$ on the set P is isomorphic to $\Delta(X)[\rho(X) - F]$. Similarly, the induced graph in $\Delta(XY)$ on the set of Q is isomorphic to $\Delta(Y)[\rho(Y) - F]$.

Let $p \in \rho(X)$ and $q \in \rho(Y)$ be vertices in $\Delta(XY)$. So there exist integers $x \in X$ and $y \in Y$ so that $p \mid x$ and $q \mid y$. It follows that $pq \mid xy$, and so, pq is an edge in $\Delta(XY)$; *i.e.*, each vertex in $\rho(X)$ is adjacent to the all vertices in $\rho(Y)$. Thus every vertex in $P \cup F$ is adjacent to every vertex in $Q \cup F$ (except itself if it is in F), this implies that the subgraph of $\Delta(XY)$ induced by $P \cup Q$ is isomorphic to $\Delta(X)[P] + \Delta(Y)[Q]$. Also, the subgraph induced by F will be a complete graph and every vertex in F is adjacent to every vertex in $P \cup Q$. This yields the desired graph. ∎

Recently, Hafezieh *et al.* in [3] have shown for two nonempty sets of positive integers X and Y that the diameter of $\Delta(XY)$ is less than or equal to 3. In the following corollary, we improve this bound. Note that if H and K are nonabelian groups and either $X = \mathrm{cd}(H)$ and $Y = \mathrm{cd}(K)$ or $X = \mathrm{cs}(H)$ and $Y = \mathrm{cs}(K)$, then this yields Theorem 1.

Corollary 3.2 *Suppose X and Y are two nontrivial sets of positive integers, and $|F| = n$ where $F = \rho(X) \cap \rho(Y)$.*

(a) *$\Delta(XY)$ is connected and* $\mathrm{diam}(\Delta(XY)) \leq 2$.
(b) $\chi(\Delta(XY)) = n + \chi(\Delta(X)[\rho(X) - F]) + \chi(\Delta(Y)[\rho(Y) - F])$. *In particular,* $n \leq \chi(\Delta(XY)) \leq n + \chi(\Delta(X)) + \chi(\Delta(Y))$.
(c) $\omega(\Delta(XY)) = n + \omega(\Delta(X)[\rho(X) - F]) + \omega(\Delta(Y)[\rho(Y) - F])$. *In particular,* $n \leq \omega(\Delta(XY)) \leq n + \omega(\Delta(X)) + \omega(\Delta(Y))$.
(d) $\alpha(\Delta(XY)) = \max(\alpha(\Delta(X)[\rho(X) - F]), \alpha(\Delta(Y)[\rho(Y) - F]))$.

Proof Let $P = \rho(X) - F$ and $Q = \rho(Y) - F$, and assume as before that $\rho(XY)$ is a disjoint union $F \cup P \cup Q$. If two vertices lie in different sets, then they are adjacent by Theorem 3.1. If they both lie in F, then they also are adjacent. If they both lie in P, then both are adjacent to a prime in $\rho(Y)$ and have distance at most 2. A similar proof replacing Y by X works if they both lie in Q. This proves (a).

Because of the adjacencies between vertices in the sets P, Q, and F, we must use different colors for each of these sets. Hence, coloring $\Delta(XY)$ is the same as independently coloring each of the induced subgraphs giving the first conclusion in (b). Since $0 \leq \chi(\Delta(Z)[\rho(Z) - F]) \leq \chi(\Delta(Z))$ where Z is either X or Y, the inequality in (b) holds. The proof of (c) is similar to the proof of (b). See Theorem 10 of [2].

Because $\Delta(X)[P] + \Delta(Y)[Q]$ is a subgraph of $\Delta(XY)$, we obtain the inequality $\max(\alpha(\Delta(X)[P]), \alpha(\Delta(Y)[Q])) \leq \alpha(\Delta(XY))$. On the other hand, it is not difficult to see that any independent set in $\Delta(XY)$ must lie in either $\Delta(X)[P]$ or $\Delta(X)[Q]$, and the other inequality follows. ∎

For the remainder of this paper, we consider the common divisor graph of X. The *common divisor graph* $\Gamma(X)$ has vertex set $X^* := X \setminus \{1\}$, and $x, y \in X^*$ form an edge if and only if $\gcd(x, y) > 1$. When X and Y are nontrivial sets of integers, we obtain a relationship between the graph $\Gamma(XY)$ and the graphs $\Gamma(X)$ and $\Gamma(Y)$.

It was shown in Corollary 3.2 in [7] that if X is a set of positive integers so that X is not empty, then $\left|\operatorname{diam}\big(\Gamma(X)\big) - \operatorname{diam}\big(\Delta(X)\big)\right| \leq 1$. Using this result and Corollary 3.2(a), we obtain the following.

Corollary 3.3 *If X and Y are two nontrivial sets of positive integers, then $\Gamma(XY)$ is connected and* $\operatorname{diam}\big(\Gamma(XY)\big) \leq 3$.

A different proof of Corollary 3.3 was given in [3]. Also, taking $X = \{1, 2, 3\}$ and $Y = \{1, 5\}$, we see that the $\operatorname{diam}\big(\Gamma(XY)\big) = 3$ does occur, so this bound is best possible.

We close with a second result. Before we state the next theorem, we give more details about the composition of two graphs. The composition of two graphs is also known as graph substitution, a name that bears witness to the fact that $\mathcal{G}[[\mathcal{H}]]$ can be obtained from $\mathcal{G}$ by substituting a copy of $\mathcal{H}$, labeled $\mathcal{H}_g$, for every vertex g in $V(\mathcal{G})$ and then joining all vertices of $\mathcal{H}_g$ with all vertices of $\mathcal{H}_{g'}$ if and only if $gg' \in E(\mathcal{G})$, and there are no edges between vertices in $\mathcal{H}_g$ and $\mathcal{H}_{g'}$ otherwise.

We show that $\Gamma(XY)$ is the graph union of two subgraphs. We note that there is a very large overlap between the vertices of these subgraphs. In fact, if 1 is not in either X or Y, these subgraphs will each contain all of the vertices in $\Gamma(XY)$.

Theorem 3.4 *Let X and Y be sets of positive integers such that $|X| = r$ and $|Y| = s$. If $\rho(X) \cap \rho(Y) = \varnothing$, then $\Gamma(XY) \cong \Gamma(X)[[K_s]] \cup \Gamma(Y)[[K_r]]$.*

Proof We consider two subgraphs of $\Gamma(XY)$ which we denote by Γ_1 and Γ_2. Set $V(\Gamma_1) = \{xy \mid x \in X^*, y \in Y\}$ and $V(\Gamma_2) = \{xy \mid x \in X, y \in Y^*\}$. Observe that $V(\Gamma_1) \cup V(\Gamma_2) \subseteq (XY)^* = V\big(\Gamma(XY)\big)$. Suppose $x_1, x_2 \in X^*$ and $y_1, y_2 \in Y$, so that $x_1y_1, x_2y_2 \in V(\Gamma_1)$. There is an edge in Γ_1 between x_1y_1 and x_2y_2 if either $x_1x_2 \in E\big(\Gamma(X)\big)$ or $x_1 = x_2$. Since the x_is are in X^*, it follows that the edges in Γ_1 are edges in $\Gamma(XY)$, so Γ_1 is a subgraph of $\Gamma(XY)$. It is not difficult to see that Γ_1 is isomorphic to the graph $\Gamma(X)[[K_s]]$. Similarly, Γ_2 is a subgraph of $\Gamma(XY)$ and Γ_2 is isomorphic to the graph $\Gamma(Y)[[K_r]]$. It follows that $\Gamma_1 \cup \Gamma_2$ is a subgraph of $\Gamma(XY)$.

Since $\rho(X) \cap \rho(Y) = \varnothing$, it is not difficult to see that $V\big(\Gamma(XY)\big) = (XY)^* = V(\Gamma_1) \cup V(\Gamma_2)$ since if $xy \in (XY)^*$ then either $x \in X^*$ or $y \in Y^*$. Thus, to prove the theorem, it suffices to show that $E\big(\Gamma(XY)\big) = E(\Gamma_1) \cup E(\Gamma_2)$. We now assume that there is an edge between x_1y_1 and x_2y_2. Thus, there is a prime p that divides both x_1y_1 and x_2y_2. Since $\rho(X) \cap \rho(Y) = \varnothing$, we see that either p divides x_1 and x_2 or p divides y_1 and y_2. Suppose first that p divides x_1 and x_2. Then $x_iy_i \in V(\Gamma_1)$ for $i = 1$ and 2 and either $x_1x_2 \in E\big(\Gamma(X)\big)$ or $x_1 = x_2$. It follows that there is an edge in Γ_1 between x_1y_1 and x_2y_2 in either case. Similarly, if p divides y_1 and y_2, then there is an edge in Γ_2 between x_1y_1 and x_2y_2. We conclude that $E\big(\Gamma(XY)\big) \subseteq E(\Gamma_1) \cup E(\Gamma_2)$, and this proves the result. ∎

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