Let $L$ be a finite-dimensional Lie algebra over the field $F$. The Ado-Iwasawa Theorem asserts the existence of a finite-dimensional $L$-module which gives a faithful representation $\rho$ of $L$. Let $S$ be a subnormal subalgebra of $L$, let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras and suppose that $S \in \mathfrak{F}$. I show that there exists a module $V$ with the extra property that it is $\mathfrak{F}$-hypercentral as $S$-module. Further, there exists a module $V$ which has this extra property simultaneously for every such $S$ and $\mathfrak{F}$, along with the Hochschild extra that $\rho(x)$ is nilpotent for every $x \in L$ with $\text{ad}(x)$ nilpotent. In particular, if $L$ is supersoluble, then it has a faithful representation by upper triangular matrices.


Keywords and phrases: Lie algebras, faithful representations, saturated formations.

1. Introduction

Let $L$ be a finite-dimensional Lie algebra over the field $F$, which may be of any characteristic. The Ado-Iwasawa Theorem asserts that there exists a faithful finite-dimensional $L$-module $V$. In this paper, I consider some extra properties which we may require of $V$ and of the representation $\rho$ given by $V$. Harish-Chandra [6] and Jacobson [9, Remark, page 203] have proved the characteristic 0 case with the extra property that $\rho(x)$ is nilpotent for all $x$ in the nil radical $N(L)$. Hochschild [7] proved, for any characteristic, that there is a module $V$ with the stronger extra property that $\rho(x)$ is nilpotent for all $x \in L$ for which $\text{ad}(x)$ is nilpotent.

The theory of saturated formations, set out in Barnes and Gastineau-Hills [5] and of $\mathfrak{F}$-hypercentral modules, set out in Barnes [1], provides a means of generalising this.

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A saturated formation of soluble Lie algebras over $F$ is a class $\mathfrak{F}$ of finite-dimensional soluble Lie algebras over $F$ such that

1. if $L \in \mathfrak{F}$ and $A < L$, then $L/A \in \mathfrak{F}$;
2. if $A, B < L$ and $L/A, L/B \in \mathfrak{F}$, then $L/(A \cap B) \in \mathfrak{F}$; and
3. if $L/\Phi(L) \in \mathfrak{F}$, then $L \in \mathfrak{F}$,

where $\Phi(L)$ is the Frattini subalgebra of $L$. An irreducible finite-dimensional $L$-module $V$ is called $\mathfrak{F}$-central if the split extension of $V$ by $L/\mathcal{C}_L(V)$ is in $\mathfrak{F}$, where $\mathcal{C}_L(V)$ denotes the centraliser of $V$ in $L$. Otherwise, it is called $\mathfrak{F}$-excentric. An $L$-module $V$ is called $\mathfrak{F}$-hypercentral if every composition factor of $V$ is $\mathfrak{F}$-central. It is called $\mathfrak{F}$-hyperexcentric if every composition factor is $\mathfrak{F}$-excentric.

If $S$ is an ideal of $L$, we write $S < L$. A subalgebra $S$ of $L$ is called subnormal in $L$, written $S \triangleleft\triangleleft L$, if there exists a chain of subalgebras $S = S_0 < S_1 < \cdots < S_r = L$, each an ideal in the next. Let $S$ be a subnormal subalgebra of $L$. Any $L$-module $V$ can be regarded as an $S$-module. To simplify terminology, we say that $V$ is $S\mathfrak{F}$-hypercentral if it is $\mathfrak{F}$-hypercentral as $S$-module and $S\mathfrak{F}$-hyperexcentric if it is $\mathfrak{F}$-hyperexcentric as $S$-module.

For any field $F$, the class $\mathfrak{N}$ of nilpotent algebras is a saturated formation. If $N$ is a nilpotent Lie algebra, an $N$-module $V$ is $\mathfrak{N}$-hypercentral if and only if every element of $N$ acts nilpotently on $V$. Thus the Harish-Chandra extension of Ado's Theorem asserts, for a finite-dimensional Lie algebra $L$ over a field of characteristic 0, that there exists a faithful, finite-dimensional $L$-module which is $\mathfrak{N}$-hypercentral as $N(L)$-module, where $N(L)$ denotes the nil radical of $L$. We shall generalise this to arbitrary saturated formations $\mathfrak{F}$, with arbitrary subnormal subalgebras $S \in \mathfrak{F}$ in place of $N(L)$. A special case of some interest is that of the saturated formation $\mathcal{U}$ of supersoluble Lie algebras, that is, of Lie algebras of whose chief factors are 1-dimensional.

An essential tool for this investigation is the following easy generalisation of Barnes [1, Theorem 4.4].

**Lemma 1.1.** Let $F$ be any field and let $L$ be a Lie algebra over $F$. Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over $F$. Suppose $S \triangleleft\triangleleft L$ and that $S \in \mathfrak{F}$. Let $V$ be a finite-dimensional $L$-module. Then $V$ is the $L$-module direct sum $V = V_0 \oplus V_1$, where $V_0$ is $S\mathfrak{F}$-hypercentral and $V_1$ is $S\mathfrak{F}$-hyperexcentric.

**Proof.** Since $S \triangleleft\triangleleft L$, there exists a chain of subalgebras $S = S_0 < S_1 < \cdots < S_r = L$. By Barnes [1, Theorem 4.4], $V$ has an $S$-module direct decomposition $V = V_0 \oplus V_1$ with $V_0$ $\mathfrak{F}$-hypercentral and $V_1$ $\mathfrak{F}$-hyperexcentric. We prove by induction over $i$ that $V_0$ and $V_1$ are $S_i$-submodules of $V$.

Let $W$ be any $S_i$-submodule of $V$. For $s \in S_i$, $x \in S_{i+1}$ and $w \in W$, we have $s(xw) = x(sw) + (sx)w \in xW + W$. Thus $xW + W$ is also an $S_i$-submodule of $V$, and $(xW + W)/W$ is a homomorphic image of $W$. If $W$ is $S\mathfrak{F}$-hypercentral, then
so is \( x W + W \). In particular, for \( W = V_0 \), this implies that \( x V_0 \subseteq V_0 \). Thus \( V_0 \) is invariant under the action of \( S_{i+1} \) and, by induction, under the action of \( L \). Similarly, \( V_1 \) is invariant under the action of \( L \).

Also of use are the following two lemmas proved in Hochschild [7] in the course of proving his main result.

**Lemma 1.2.** Let \( F \) be any field and let \( L \) be a Lie algebra over \( F \) whose derived algebra \( L' \) is nilpotent. Suppose \( x \in L \) and that \( \text{ad}(x) \) is nilpotent. Then \( x \) is in the nilpotent radical \( N(L) \).

**Lemma 1.3.** Suppose \( \text{char}(F) = 0 \). Let \( V \) be a finite-dimensional \( L \)-module giving representation \( \rho \). Suppose \( N(L) \) acts nilpotently on \( V \). Let \( x \in L \) with \( \text{ad}(x) \) nilpotent. Then \( \rho(x) \) is nilpotent.

If \( L \) is a soluble Lie algebra over a field \( F \) of characteristic 0, then \( L' \) is nilpotent. Every subalgebra of a nilpotent Lie algebra is subnormal, so \( x \in N(L) \) implies that the subspace \( \langle x \rangle \) spanned by \( x \) is a subnormal subalgebra of \( L \). Even in non-zero characteristic, the following weak form of Lemma 1.2 holds.

**Lemma 1.4.** Let \( L \) be a soluble Lie algebra over any field \( F \). Suppose \( x \in L \) and that \( \text{ad}(x) \) is nilpotent. Then \( \langle x \rangle \ll L \).

**Proof.** Suppose the result holds for algebras of smaller dimension than \( L \). Let \( A \) be a minimal ideal of \( L \). Then \( A_1 = \langle x \rangle + A \ll L \). But \( A \) is abelian and \( x \) acts nilpotently on \( A \). Thus \( A_1 \) is nilpotent and \( \langle x \rangle \ll A_1 \ll L \). \( \square \)

It follows that, for a module \( V \) giving representation \( \rho \) of a soluble Lie algebra \( L \), the condition that \( \rho(x) \) be nilpotent for all \( x \in L \) with \( \text{ad}(x) \) nilpotent is equivalent to the condition that \( V \) be \( S \)-hypercentral for every nilpotent subnormal subalgebra \( S \) of \( L \).

Suppose \( S \ll L \) and that \( S \in \mathfrak{F} \). A straightforward approach to proving the existence of a faithful finite-dimensional \( L \)-module which is \( S \mathfrak{F} \)-hypercentral easily reduces to the case where \( L \) has a unique minimal ideal. We take a faithful finite-dimensional \( L \)-module \( V \). By Lemma 1.1, this is the direct sum of an \( S \mathfrak{F} \)-hypercentral \( L \)-module \( V_0 \) and an \( S \mathfrak{F} \)-hyperexcentric \( L \)-module \( V_1 \). One (at least) of these must be faithful. Unfortunately, it need not be \( V_0 \). That this difficulty is a serious obstruction to the straightforward approach is shown by the results of Section 2.
2. Faithful $\mathfrak{F}$-hyperexcentric modules

To construct faithful $\mathfrak{F}$-hyperexcentric modules, we will use tensor products. The following lemma will help to determine the kernel of a tensor product.

**Lemma 2.1.** Let $L$ be a Lie algebra over any field $F$. Suppose $V, W$ are finite-dimensional $L$-modules and that $x$ is in the kernel of $V \otimes W$. Then there exists $\lambda \in F$ such that $xv = \lambda v$ and $xw = -\lambda w$ for all $v \in V$ and $w \in W$.

**Proof.** Let $v, w$ be any non-zero elements of $V$ and $W$. Take bases $v = v_0, \ldots, v_m$ and $w = w_0, \ldots, w_n$ of $V$ and $W$. Then $xv = \sum \lambda_i v_i$ and $xw = \sum \mu_j w_j$. Now $0 = x(v \otimes w) = \sum \lambda_i v_i \otimes w_0 + \sum \mu_j v_0 \otimes w_j$. Therefore $\lambda_i = 0$ for $i \neq 0$, $\mu_j = 0$ for $j \neq 0$ and $\lambda_0 + \mu_0 = 0$. Since every non-zero element of $V$ is an eigenvector, $\lambda_0$ is independent of the choice of $v$. $\square$

**Corollary 2.2.** Suppose $x$ is in the kernel of $(W \otimes V) \oplus (W \otimes V \otimes V)$. Then $x$ is in the kernel of $V$.

**Proof.** For $v \in V$ and $w \in W$, we have $xv = \lambda v$ and $xw = -\lambda w$. Then $x(w \otimes v \otimes v) = \lambda (w \otimes v \otimes v)$. Therefore $\lambda = 0$. $\square$

If $\text{char}(F) = 0$, then, by Barnes [2, Theorem 2], for some normal $F$-subspace $\Lambda$ of the algebraic closure $\bar{F}$ of $F$, $\mathfrak{F}$ is the class of all soluble finite-dimensional Lie algebras $S$ over $F$ with the property that for all $x \in S$, the eigenvalues of $\text{ad}(x)$ all lie in $\Lambda$. It follows that, if the degree of $\bar{F}$ over $F$ is finite, there exist Lie algebras $L$ for which the smallest saturated formation $\mathfrak{F}$ containing $L$ is the formation of all soluble Lie algebras.

**Theorem 2.3.** Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over the field $F$ of characteristic 0. Suppose $\mathfrak{F}$ is not the formation of all soluble Lie algebras. Let $S \in \mathfrak{F}$ be a non-nilpotent soluble subnormal subalgebra of $L$. Then $L$ has a faithful, finite-dimensional $S\mathfrak{F}$-hyperexcentric module giving representation $\rho$ with $\rho(x)$ nilpotent for all $x \in L$ for which $\text{ad}(x)$ is nilpotent.

**Proof.** Let $N = N(L)$ be the nil radical of $L$. By Lemma 1.3, the condition on an $L$-module $V$ giving representation $\rho$ that $\rho(x)$ be nilpotent for all $x \in L$ with $\text{ad}(x)$ nilpotent is equivalent to $V$ being $N\mathfrak{M}$-hypercentral. By Hochschild [7], $L$ has a faithful finite-dimensional $N\mathfrak{M}$-hypercentral module $V$.

Let $R$ be the soluble radical of $L$. Since $R/N$ is abelian and $S \not\leq N$, there exists a maximal ideal $M \geq N$ of $R$ not containing $S$. Since $LR \leq N$, $M < L$. Let $K$ be
the sum of $M$ and a Levy factor of $L$. Then $K$ is an ideal of $L$ of codimension 1 and $K + S = L$.

Let $\mathcal{F}$ be the saturated formation given by the normal subspace $\Lambda$ of $\bar{F}$. Then $\Lambda \neq \bar{F}$, so there exists $\alpha \in \bar{F} - \Lambda$. For the 1-dimensional Lie algebra $L/K = \langle \bar{x} \rangle$, we can construct an irreducible module $W$ on which $\bar{x}$ has $\alpha$ as an eigenvalue. Then the $L$-module $W$ is $N\mathcal{F}$-hypercentral and $S\mathcal{F}$-excentric.

Let $V_0$ and $V_1$ be the $S\mathcal{F}$-hypercentral and $S\mathcal{F}$-hyperexcentric components of $V$. Put $V^* = (W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0) \oplus V_1$. Then $V^*$ is $N\mathcal{F}$-hypercentral and $S\mathcal{F}$-hyperexcentric by Barnes [1, Theorem 2.1] and [4, Theorem 2.3]. If $x$ is in the kernel of $V^*$, then $x$ is in the kernel of $V_1$ and of $(W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0)$. By Corollary 2.2, $x$ is also in the kernel of $V_0$, so $x = 0$. Thus $V^*$ is faithful.

The situation in non-zero characteristic is different. The Lie algebras of nilpotent length at most $n$ form a saturated formation $\mathcal{F}^n$. Thus it is not possible for the smallest saturated formation containing $L$ to be the formation of all soluble Lie algebras. If $L \in \mathcal{F}^n$, then every irreducible $L$-module is $\mathcal{F}^{n+1}$-central. Thus $L$ has no $\mathcal{F}^{n+1}$-hyperexcentric modules. Even when $\mathcal{F}$ is the smallest saturated formation containing the non-nilpotent algebra $L$, there may not be $\mathcal{F}$-hyperexcentric $L$-modules with the Hochschild property. For example, if $L = \langle x, y \rangle$ with $xy = y$ and $F$ is algebraically closed, any irreducible module on which $y$ acts nilpotently is 1-dimensional and so $\mathcal{U}$-central.

**Theorem 2.4.** Suppose $\text{char}(F) \neq 0$. Let $S$ be a soluble subnormal subalgebra of the Lie algebra $L$ over $F$. Let $\mathcal{F}$ be the smallest saturated formation containing $S$. Then $L$ has a faithful finite-dimensional $S\mathcal{F}$-hyperexcentric module.

**Proof.** Let $V$ be a faithful finite-dimensional $L$-module with $V_0$ and $V_1$ its $S\mathcal{F}$-hypercentral and $S\mathcal{F}$-hyperexcentric components. Let $K$ be a minimal ideal of $L$. Let $\mathcal{F}_0$ be the smallest saturated formation containing $(S + K)/K$. If $\mathcal{F}_0 = \mathcal{F}$, then by induction, there exists an irreducible $L/K$-module $W$ which is $(S + K/K)\mathcal{F}$-hypercentral. If not, then $S$ is not nilpotent, and since, by Schenkman [10, Theorem 3], $S^\infty \triangleleft L$, we can take $K \subseteq S^\infty$. Since $S \triangleleft L$, the $S$-composition factors of $K$ are isomorphic. As $S \not\subseteq \mathcal{F}_0$, $K$ is $S\mathcal{F}_0$-hyperexcentric. Let $\mathcal{F}_1$ be the saturated formation locally defined by $\mathcal{F}_0$, that is, the class of all soluble Lie algebras $M$ with $M/N(M) \in \mathcal{F}_0$. (See [5, Theorem 4.6].) Then $S \in \mathcal{F}_1$. Since by Jacobson [9, Theorem VI.12, page 205], $L$ has a faithful completely reducible module, there exists an irreducible $L$-module $W$ on which $K$ acts faithfully. The $S$-composition factors of $W$ are all isomorphic. Thus $K$ acts non-trivially on each $S$-composition factor $W_i$, $S/\mathcal{F}_1(W_i) \not\subseteq \mathcal{F}_0$ and $W$ is $S\mathcal{F}_1$-hyperexcentric. Hence, in either case, we have an irreducible $S\mathcal{F}$-hypercentric $L$-module $W$. Put $V^* = (W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0) \oplus V_1$. Then $V^*$ is $S\mathcal{F}$-hyperexcentric. By Corollary 2.2, $V^*$ is faithful.

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3. Splitting algebras

To get around the difficulty pointed out above, we follow Iwasawa’s use of a splitting module in the construction of the desired faithful module.

**DEFINITION 3.1.** Let $A$ be an abelian ideal of the Lie algebra $L$. A *splitting algebra* for $L$ relative to $A$ is a Lie algebra $M$ together with an abelian ideal $B$ of $M$ such that $L \leq M$, $L + B = M$, $L \cap B = A$ and such that $M$ splits over $B$.

In the above, we can regard both $A$ and $B$ as $L/A$-modules. Choosing coset representatives in $L$ for the elements of $\tilde{L} = L/A$ by a linear map $u : \tilde{L} \to L$, we can identify $L$ with $\tilde{L} \times A$, identifying $(\tilde{x}, a)$ with the element $u(\tilde{x}) + a \in L$ for $\tilde{x} \in \tilde{L}$ and $a \in A$. We then have the multiplication given by

$$(\tilde{x}_1, a_1)(\tilde{x}_2, a_2) = (\tilde{x}_1 \tilde{x}_2, \tilde{x}_1 a_2 - \tilde{x}_2 a_1 + f(\tilde{x}_1, \tilde{x}_2),$$

where $f(\tilde{x}_1, \tilde{x}_2) = u(\tilde{x}_1)u(\tilde{x}_2) - u(\tilde{x}_1 \tilde{x}_2)$. Then $f : \tilde{L} \wedge \tilde{L} \to A$ is a 2-cocycle. Let $h$ be the cohomology class of $f$. Let $j^* : H^2(\tilde{L}, A) \to H^2(\tilde{L}, B)$ be the map induced by the module inclusion $j : A \to B$. Then $M$ is the extension of $B$ by $\tilde{L}$ constructed using the cocycle $jf$, that is, $M = \tilde{L} \times B$ with multiplication given by

$$(\tilde{x}_1, b_1)(\tilde{x}_2, b_2) = (\tilde{x}_1 \tilde{x}_2, \tilde{x}_1 b_2 - \tilde{x}_2 b_1 + f(\tilde{x}_1, \tilde{x}_2),$$

for $\tilde{x}_1, \tilde{x}_2 \in \tilde{L}$ and $b_1, b_2 \in B$. The requirement that $M$ splits over $B$ is equivalent to $j^*(h) = 0$.

Since the development of homological algebra, the existence of a splitting algebra has become a triviality. Any $\tilde{L}$-module $A$ has an embedding $j : A \to B$ in an injective module $B$ and we then have $H^2(\tilde{L}, B) = 0$. Except in the trivial case where $\tilde{L} = 0$, the splitting algebra so obtained is infinite-dimensional. The original existence proof constructed the module $B$ from $A$ and the universal enveloping algebra of $\tilde{L}$, also giving an infinite-dimensional splitting algebra. In [8], Iwasawa modified this construction to obtain the following result which was the key to his proof of the Ado-Iwasawa Theorem.

**THEOREM 3.2.** Let $A$ be an abelian ideal of the finite-dimensional Lie algebra $L$ over any field $F$. Then there exists a finite-dimensional splitting algebra for $L$ relative to $A$.

This result can be strengthened in the special case where we have a soluble subnormal subalgebra $S$ of $L$ with $S \in \mathcal{F}$ for some saturated formation $\mathcal{F}$ of soluble Lie algebras.
Lemma 3.3. Let \( L \) be a Lie algebra over any field \( F \). Suppose \( S \lhd L \) and that \( S \in \mathfrak{F} \) where \( \mathfrak{F} \) is a saturated formation of soluble Lie algebras. Let \( A \) be an abelian ideal of \( L \) which is \( S\mathfrak{F} \)-hypercentral. Let \( h \) be the cohomology class of \( L \) as an extension of \( A \). Then

(1) there exists a finite-dimensional splitting algebra \((M, B)\) for \( L \) relative to \( A \) with \( B \) \( S\mathfrak{F} \)-hypercentral;

(2) there exists an embedding \( j : A \rightarrow B \) of \( A \) in a finite-dimensional \( L/A \)-module \( B \) which is \( S\mathfrak{F} \)-hypercentral and such that \( j^*(h) = 0 \).

Proof. The two assertions are equivalent. By Iwasawa’s Theorem 3.2, there exists a finite-dimensional splitting algebra \( M \) with ideal \( B \). For the \( L/A \)-module inclusion \( j : A \rightarrow B \), we have \( j^*(h) = 0 \). By Lemma 1.1, \( B = B_1 \oplus B'_1 \) where \( B_1 \) is \( S\mathfrak{F} \)-hypercentral and \( B'_1 \) is \( S\mathfrak{F} \)-hyperexcentric. As \( A \) is \( S\mathfrak{F} \)-hypercentral, \( j(A) \subseteq B_1 \) and \( j \) is the composite of the inclusion \( j_1 : A \rightarrow B_1 \) and the inclusion \( i_1 : B_1 \rightarrow B \). As the induced map \( i_1^* \) of cohomology is injective, it follows that \( j_1^*(h) = 0 \). Replacing \( B \) by \( B_1 \) gives the result.

The condition that \( A \) be \( S\mathfrak{F} \)-hypercentral is automatically satisfied if \( S \supseteq A \) or if \( A \) is central. As the results about splitting algebras will only be needed in the case where \( A \) is central, I simplify the statements by assuming this from here on.

We can iterate this reduction of the splitting module. If \((S_2, \mathfrak{F}_2)\) is another pair satisfying the conditions of Lemma 3.3, we can decompose the above module \( B_1 = B_2 \oplus B'_2 \) where \( B_2 \) is \( S_2\mathfrak{F}_2 \)-hypercentral and \( B'_2 \) is \( S_2\mathfrak{F}_2 \)-hyperexcentric. This reduction process must terminate since \( B \) is finite-dimensional. We thus have

Theorem 3.4. Let \( A \) be a central ideal of the finite-dimensional Lie algebra \( L \) over any field \( F \). Then there exists a finite-dimensional splitting algebra \((M, B)\) for \( L \) relative to \( A \) such that, for every saturated formation \( \mathfrak{F} \) and subnormal subalgebra \( S \in \mathfrak{F} \), \( B \) is \( S\mathfrak{F} \)-hypercentral.

4. The Hochschild extra

In this section, I show that, if \( A \) is central, then there exists a splitting algebra \((M, B)\) as in Theorem 3.4 with the Hochschild extra property that, for all \( x \in L \), if \( \text{ad}(x) \) is nilpotent, then so is the action \( \psi(x) \) of \( x \) on \( B \). For \( N = N(L) \) and the saturated formation \( \mathfrak{N} \) of nilpotent algebras, by Theorem 3.4, we may suppose that \( B \) is \( N\mathfrak{N} \)-hypercentra. Thus \( \psi(x) \) is nilpotent for all \( x \in N \). By Lemma 1.3, we now have

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LEMMA 4.1. Let $A$ be a central ideal of the finite-dimensional Lie algebra $L$ over a field of characteristic 0. Then there exists a finite-dimensional splitting algebra $(M, B)$ for $L$ with respect to $A$ which satisfies the extra conditions

1. $B$ is $S^S$-hypercentral for every saturated formation $S$ and every $S \triangleleft L$ with $S \in S^S$;

2. the action $\psi(x)$ of $x$ on $B$ is nilpotent for every $x \in L$ with $\text{ad}(x)$ nilpotent.

Now suppose $\text{char}(F) = p \neq 0$. Then $L$ has a finite-dimensional $p$-envelope $\tilde{L}$ by Strade and Farnsteiner [11, Proposition 5.3, page 93]. The $[p]$ operation may be chosen such that $z^{|p|} = 0$ for all $z$ in the centre of $L$. Let $A$ be a central ideal of $L$. Then $A$ is a central $p$-ideal of $\tilde{L}$. If $S \triangleleft L$, then $S \triangleleft \tilde{L}$. If $B$ is a finite-dimensional $p$-module of $\tilde{L}$ which is a splitting module for $\tilde{L}$, and so for $L$, with respect to $A$, then it follows as in the proof of Strade and Farnsteiner [11, Theorem 5.4, page 94], that the action $\psi(x)$ of $x$ on $B$ is nilpotent for every $x \in L$ with $\text{ad}(x)$ nilpotent. The following lemma enables us to prove the existence of such a splitting module.

LEMMA 4.2. Let $L$ be a restricted Lie algebra over the field $F$ of characteristic $p$. Let $V$ be an $L$-module of dimension $n$ giving the representation $\rho$. Put $\alpha(x) = \rho(x)^p - \rho(x^{[p]})$. Then $V = V_{[p]} \oplus V_{[p]}'$, where $V_{[p]} = \bigcap_{x \in L} \ker \alpha(x)^n$ is a submodule, all of whose composition factors are $p$-representations, and $V_{[p]}' = \sum_{x \in L} \alpha(x)^n V$ is a submodule, none of whose composition factors are $p$-representations.

PROOF. Let $x_1, \ldots, x_r$ be a basis of $L$. Put $\tilde{V} = \tilde{F} \otimes_F V$. We take the character decomposition $\tilde{V} = \bigoplus S_i \tilde{V}$ corresponding to the characters $S_i$ with $S_0 = 0$. The only eigenvalue of $\alpha(x)$ on $\tilde{V}_i$ is $S_i(x)^p$. If this is non-zero, then $\alpha(x)$ acts invertibly on $\tilde{V}_i$. For all $x \in \tilde{L}$, $\alpha(x)^n \tilde{V}_0 = 0$. For each $i > 0$, $S_i \neq 0$ so $S_i(x_{j_i}) \neq 0$ for some $x_{j_i}$. We thus have

$$\sum_{i > 0} \tilde{V}_i = \sum_{i > 0} \alpha(x_{j_i})^n \tilde{V} = \sum_j \alpha(x_j)^n \tilde{V}.$$ 

It follows that

$$\tilde{V}_0 = \bigcap_{x \in \tilde{L}} \ker \alpha(x)^n = \bigcap_i \ker \alpha(x_i)^n.$$ 

The result follows by linearity. \[\square\]

THEOREM 4.3. Let $A$ be a central ideal of the finite-dimensional Lie algebra $L$ over any field $F$. Then there exists a finite-dimensional splitting algebra $(M, B)$ for $L$ with respect to $A$ which satisfies the extra conditions

1. $B$ is $S^S$-hypercentral for every saturated formation $S$ and every $S \triangleleft L$ with $S \in S^S$;
(2) the action \( \psi(x) \) of \( x \) on \( B \) is nilpotent for every \( x \in L \) with \( \text{ad}(x) \) nilpotent.

**Proof.** We already have the result if \( \text{char}(F) = 0 \), so suppose \( \text{char}(F) = p \neq 0 \). We embed \( L \) in a finite-dimensional \( p \)-envelope \( \tilde{L} \) with \( z_p = 0 \) for all \( z \in Z(\tilde{L}) \). By Iwasawa’s Theorem 3.2, there exists a finite dimensional splitting module \( B \) for \( \tilde{L} \) relative to \( A \). Since \( A \) is a \( p \)-module, \( A \subseteq \mathcal{B}_p \), and it follows that \( B_p \) is a splitting module with the property (2). Proceeding as in the proof of Theorem 3.4, we obtain a direct summand of \( B_p \) which also has the property (1). □

5. The main result

**Theorem 5.1.** Let \( L \) be a finite-dimensional Lie algebra over any field \( F \). Then \( L \) has a faithful finite-dimensional module \( V \) which has the extra properties

1. \( V \) is \( S \mathfrak{F} \)-hypercentral for every saturated formation \( \mathfrak{F} \) and every \( S \propto L \) with \( S \in \mathfrak{F} \);

2. the action \( \rho(x) \) of \( x \) on \( V \) is nilpotent for every \( x \in L \) with \( \text{ad}(x) \) nilpotent.

**Proof.** The representation of the 1-dimensional algebra by matrices \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) with \( \lambda \in F \) satisfies all the requirements. By induction, we may suppose that the result holds for algebras of smaller dimension than \( \text{dim}(L) \). If \( A_1 \) and \( A_2 \) are distinct minimal ideals of \( L \), then there exist \( L/A_i \)-modules \( V_i \) which satisfy the requirements with respect to \( L/A_i \). The \( L \)-module \( V_1 \oplus V_2 \) then has all the required properties. Thus we may suppose that \( L \) has a unique minimal ideal \( A \).

Since \( L \) is an \( S \mathfrak{F} \)-hypercentral module for every pair \( S \in \mathfrak{F} \), \( L/Z \) has a faithful simultaneously \( S \mathfrak{F} \)-hypercentral module, where \( Z \) is the centre of \( L \). Thus the result holds if \( Z = 0 \). Hence we may suppose that \( Z \neq 0 \) and is the unique minimal ideal of \( L \). By Theorem 4.3, there exists a finite-dimensional splitting algebra \( (M, B) \) in which \( B \) and the representation \( \psi \) given by \( B \) have the properties (1) and (2). Let \( L_1 \) be a complement to \( B \) in \( M \). Following Iwasawa, we put \( V = \langle e \rangle \oplus B \) as vector space with action of \( M \) on \( V \) given by \( (x + b)e = b \) and \( (x + b)b' = xb' \), (the product of \( x \) and \( b' \) in \( M \)) for \( x \in L_1 \) and \( b, b' \in B \). Then

\[
(x_1 + b_1)((x_2 + b_2)(\lambda e, b')) = (x_1 + b_1)(0, \lambda b_2 + x_2 b') = (0, \lambda x_1 b_2 + x_1 (x_2 b')).
\]

Denoting the commutator of the actions of \( (x_1 + b_1) \) and \( (x_2 + b_2) \) on \( V \) by \( [x_1 + b_1, x_2 + b_2] \), we have

\[
[x_1 + b_1, x_2 + b_2](\lambda e, b') = (0, \lambda x_1 b_2 - \lambda x_2 b_1 + (x_1 x_2)b')
= (x_1 x_2 + x_1 b_2 - x_2 b_1)(\lambda e, b')
= ((x_1 + b_1)(x_2 + b_2))(\lambda e, b').
\]
Thus this action makes $V$ an $M$-module which is clearly finite-dimensional. As $L$ is a subalgebra of $M$, $V$ is an $L$-module. As the unique minimal ideal of $L$ is contained in $B$ which is clearly represented faithfully, $V$ is a faithful $L$-module. $B$ is a submodule of $V$ and is $S\mathfrak{g}$-hypercentral while $V/B$ is the trivial module. Thus $V$ is $S\mathfrak{g}$-hypercentral for every pair $(S, \mathfrak{g})$. As $\rho(x)V \subseteq B$ for all $x \in L$, if $\psi(x)$ is nilpotent on $B$, then $\rho(x)$ is nilpotent on $V$.

6. $\mathfrak{g}$-hypercentrality of $p$-modules

Comparison of Lemma 1.1 and Lemma 4.2 suggests a possible link between $p$-modules and $\mathfrak{g}$-hypercentral modules which would make the non-zero characteristic case of Theorem 5.1 an immediate consequence of Strade and Farnsteiner [11, Theorem 5.4, page 94].

In the following, $F$ is a field of characteristic $p \neq 0$, $\mathbb{F}_p$ denotes the field of $p$ elements and $\overline{F}$ the algebraic closure of $F$. A polynomial $f(x)$ over $\overline{F}$ is called $\mathbb{F}_p$-linear if the function $f: \overline{F} \rightarrow \overline{F}$ given by $f(x)$ is $\mathbb{F}_p$-linear. Note that to prove a polynomial $f(x)$ to be $\mathbb{F}_p$-linear, it is sufficient to prove $f(a + b) = f(a) + f(b)$ for all $a, b \in \overline{F}$, as then $f(\lambda a) = \lambda f(a)$ for $\lambda \in \mathbb{F}_p$ follows. Note also that a polynomial of the form $f(x) = a_0 x + a_1 x^p + a_2 x^{p^2} + \cdots + a_n x^{p^n}$ is $\mathbb{F}_p$-linear.

**Lemma 6.1.** If $f(x)$ is $\mathbb{F}_p$-linear, then all roots of $f(x)$ have the same multiplicity.

**Proof.** Let $\alpha_1, \ldots, \alpha_n$ be the (not necessarily distinct) roots of $f(x)$. Then $f(x) = a \prod_{i=1}^{n} (x - \alpha_i)$. For any root $\beta$,

$$f(x) = f(x) + f(\beta) = f(x + \beta) = a \prod_{i=1}^{n} (x + \beta - \alpha_i).$$

Thus $(x - \alpha_i)$ and $(x + \beta - \alpha_i)$ occur as factors of $f(x)$ with the same multiplicity. But every root $\alpha_j$ is $\alpha_i - \beta$ for some root $\beta$.

**Lemma 6.2.** Suppose $f(x)$ is $\mathbb{F}_p$-linear and that the coefficient of $x$ in $f(x)$ is not zero. Then all roots of $f(x)$ are simple.

**Proof.** Since $f(0) = 0$, there is no constant term. If the roots have multiplicity $r$, then $f(x) = g(x)^r$ and the lowest term of $f(x)$ has degree at least $r$. Hence $r = 1$.

**Lemma 6.3.** Let $f(x)$ be an $\mathbb{F}_p$-linear polynomial. Then $f(x)$ has the form

$$f(x) = a_0 x + a_1 x^p + a_2 x^{p^2} + \cdots + a_n x^{p^n}.$$

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The text is from a journal article by Donald W. Barnes. The content covers topics in algebra, specifically related to modules and hypercentral modules. The author discusses the properties of modules under certain conditions and introduces lemmas and propositions to support these discussions. The document is a continuation of previous work in the field, indicating a progression in understanding and proving theorems in algebra. The notation and mathematical concepts are typical of advanced algebraic research, involving elements of field theory and module theory. The text is structured to present theorems with clear and concise explanations, supported by examples and proofs. The use of field $\mathbb{F}_p$ and algebraic closure $\overline{F}$ highlights the relevance of characteristic $p$ theory in algebraic studies. The lemmas and propositions are carefully crafted to ensure that the reader can follow the logical progression of ideas and conclusions.
PROOF. We use induction over the degree of \( f(x) \). The result holds if the degree is 1. By replacing \( f(x) \) with \( f(x) + x \) if necessary, we may suppose that all roots of \( f(x) \) are simple. The roots of \( f(x) \) form a vector space \( V \) of some finite dimension \( n \) over \( \mathbb{F}_p \). The number of roots is \( p^n \) and as all roots are simple, the degree of \( f(x) \) is \( p^n \). If the leading coefficient is \( a \), then \( g(x) = f(x) - ax^{p^n} \) is \( \mathbb{F}_p \)-linear of lower degree. Therefore \( g(x) \) has the asserted form and the result follows.

THEOREM 6.4. Let \((L, [p])\) be a restricted Lie algebra over the field \( F \) of characteristic \( p \neq 0 \) and suppose that \( z^{[p]} = 0 \) for all \( z \) in the centre of \( L \). Let \( \mathfrak{Z} \) be a saturated formation and suppose \( S \triangleleft L, S \neq 0 \) and \( S \not\in \mathfrak{Z} \). Let \( V \) be an irreducible \( p \)-module of \( L \). Then \( V \) is \( S\mathfrak{Z} \)-hypercentral.

PROOF. Let \( L, S, V \) be a counterexample with \( L \) of least possible dimension. We now choose \( V \) such that the kernel \( K \) of the representation \( \rho \) of \( L \) on \( V \) has the least possible codimension. Let \( Z = Z(L) \) be the centre of \( L \). Suppose \( Z \neq 0 \). Then \( Z \) acts nilpotently on \( V \) and as \( V \) is irreducible, \( Z \cdot V = 0 \). But \( Z \) is a \( p \)-ideal of \( L \), so \( V \) is an irreducible \( p \)-module for the restricted Lie algebra \( L/Z \). As \( V \) is \((S + Z/Z)\mathfrak{Z} \)-hyperexcentric, \( L/Z \) must have a central element \( z \) with \( z^{[p]} \neq 0 \), that is, we have \( z \in L \) with \( \text{ad}(z)^2 = 0 \) and \( z^{[p]} \notin Z \). Therefore \( Z = 0 \).

If \( A < B < L \), then the \( p \)-closure \( A_p < B_p \) by Strade and Farnsteiner [11, Proposition 1.3, page 66]. Therefore \( S_p \triangleleft L \). If \( S_p \neq L \), then there exists a \( p \)-ideal \( M \) such that \( S_p \leq M < L \). If \( z \in Z(M) \), then \( \text{ad}(z)^2 = 0 \), so \( z^{[p]} \in Z(L) = 0 \). Thus \( M, S \) and any \( M \)-composition factor of \( V \) form a counterexample. Therefore \( S_p = L \), \( L \) is soluble and \( S \triangleleft L \).

Let \( A \) be a minimal ideal of \( S \). Since \( L = S_p, A \triangleleft L \). If \( a \in A \), then \( \text{ad}(a)^2 = 0 \), so \( a^{[p]} \in Z \). But \( Z = 0 \). Thus \( A \) is a \( p \)-ideal and \( A \cdot V = 0 \) since \( V \) is an irreducible \( p \)-module. There exists an element \( z \) such that \( zL \leq A \), but \( z^{[p]} \notin A \). As \( Z = 0 \), we cannot have \( zA = 0 \), so \( z \) acts invertibly on \( A \). By Barnes [3, Theorem 2.2], \( H^n(L/A; A) = 0 \) for all \( n \) and there exists a subalgebra \( M < L \) which complements \( A \). If \( x \in Z(M) \) and \( xA = 0 \), then \( x \in Z(L) = 0 \). Thus \( Z(M) \simeq Z(L/A) \) acts faithfully on \( A \).

There exists a \( p \)-mapping \( [p]' \) on \( L/A \) which is zero on \( \tilde{Z} = Z(L/A) \). For any \( \tilde{x} \in \tilde{L} = L/A, \tilde{x}^{[p]'} \in \tilde{Z} \). Thus any representation of \( \tilde{L} \) whose kernel contains \( \tilde{Z} \) which is a \( p \)-representation with respect to \([p]\) is also a \( p \)-representation with respect to \([p]'\). If \( \tilde{Z} \subseteq \tilde{K} = K/A \), then \((\tilde{L}, [p]'), \tilde{S}, V \) is a counterexample of smaller dimension. Therefore \( \tilde{Z} \nsubseteq \tilde{K} \).

Take \( \tilde{z} \in \tilde{Z}, \tilde{z} \notin \tilde{K} \). Since \( \tilde{z} \) is not nilpotent on \( V \), for all \( r, \tilde{z}^{[p]'r} \notin \tilde{K} \). By replacing \( \tilde{z} \) with \( \tilde{z}^{[p]'r} \) for some \( r \), we obtain \( \tilde{z} \in \{\tilde{z}^{[p]}, \tilde{z}^{[p]'}, \tilde{z}^{[p]'r}, \ldots\} \). Put \( \tilde{T} = \{\tilde{z}, \tilde{z}^{[p]}, \tilde{z}^{[p]'}, \ldots\} \). Let \( \psi : A \rightarrow A \) be the linear transformation of \( A \) given by \( \tilde{z} \).

Let \( r = \text{dim}(\tilde{T}) \). Then there exists a polynomial \( f(x) = x^{p^r} + a_1x^{p^{r-1}} + \cdots + a_rx \)
over $F$ such that $f(\psi) = 0$. Note that the roots of $f(x)$ in the algebraic closure $\tilde{F}$ are distinct and form a vector space $\Lambda$ of dimension $r$ over the prime field $\mathbb{F}_p$ of $p$ elements. Let $\Lambda_0$ be the $\mathbb{F}_p$-subspace of $\tilde{F}$ spanned by the eigenvalues of $\psi$. Let $m(x)$ be the minimum polynomial of $\psi$ and $\alpha_1, \ldots, \alpha_n$ its roots. Then $\Lambda_0 = (\alpha_1, \ldots, \alpha_n)_{\mathbb{F}_p} \subseteq \Lambda$. Let $s = \text{dim} \Lambda_0$.

Put $g(x) = \prod_{\lambda \in \Lambda_0} (x - \lambda)$. Then $g(x)$ has degree $p^s$. Take any $a \in \tilde{F}$ and set $h_a(x) = g(x + a) - g(x) - g(a)$. Since $g(x) = x^{p^s} + \text{terms of lower degree}$, $g(x + a) = (x + a)^{p^s} + \text{lower degree terms} = x^{p^s} + \text{terms of lower degree in } x$ and so $h_a(x)$ is a polynomial of degree less than $p^s$. If $a$ is a root of $g(x)$, then so is $\lambda + a$ for all $\lambda \in \Lambda_0$ and $h_a(\lambda) = 0$. Thus $h_a(x)$ has at least $p^s$ roots and so must be the zero polynomial. Hence $g(x + a) = g(x) + g(a)$ if $g(a) = 0$. Now consider general $a$. For $\lambda \in \Lambda_0$, $g(a + \lambda) = g(a) + g(\lambda)$, so $h_a(\lambda) = g(a + \lambda) - g(\lambda) - g(a) = 0$, so again $h_a(x)$ has at least $p^s$ roots and must be the zero polynomial. Thus $g(x)$ is $\mathbb{F}_p$-linear. Note also that every automorphism of $\tilde{F}$ which fixes $F$ pointwise fixes $g(x)$ which is therefore a polynomial over $F$ since $F(\Lambda)$ is a separable extension of $F$.

Now $f(x)$ is the $\mathbb{F}_p$-linear polynomial over $F$ of least degree for which $f(\psi) = 0$. But $g(\psi) = 0$, so $s \geq r$. But $\Lambda_0$ is an $s$-dimensional subspace of the $r$-dimensional space $\Lambda$. Therefore $\Lambda_0 = \Lambda$.

We now consider the linear transformation $\rho(z) : V \rightarrow V$. Since $\rho$ is a $p$-representation, $f(\rho(z)) = 0$. Thus if $\mu$ is an eigenvalue of $\rho(z)$, then $\mu \in \Lambda = \Lambda_0$. Thus $\mu = \alpha_1 + \cdots + \alpha_k$ for some eigenvalues $\alpha_i$ (not necessarily distinct) of $\psi$. Let $W$ be the $L$-module $\text{Hom}(A^{\otimes k}, V)$ and let $\theta$ be the representation given by $W$. Then $0$ is an eigenvalue of $\theta(z)$.

Since $A$ is $S^3$-hypercentral and $V$ is $S^3$-hyperexcentric, we have by Barnes [1, Theorem 2.1] and [4, Theorem 2.3], that $W$ is $S^3$-hyperexcentric. But for some composition factor $W_0$ of $W$, the action of $z$ on $W_0$ has $0$ as an eigenvalue. Thus $z$ is in the kernel of the representation of $L$ on $W_0$, contrary to the choice of $V$ as giving a representation with kernel of least possible codimension.

Any Lie algebra $L$ over a field of characteristic $p$ can be embedded as an ideal in a restricted Lie algebra $(\tilde{L}, [p])$ with $z^{[p]} = 0$ for all $z$ in the centre of $\tilde{L}$. By Strade and Farnsteiner [11, Theorem 5.4, page 94], $\tilde{L}$ has a faithful finite-dimensional $p$-module. As $S \prec L$ implies $S \prec \tilde{L}$, the characteristic $p$ case of Theorem 5.1 follows by Theorem 6.4.

7. Special cases

We now consider the significance of Theorem 5.1 for supersoluble algebras. A Lie algebra $S$ is supersoluble if it has a sequence $0 = A_0 < A_1 < \cdots < A_n = S$ of ideals of $S$ with $A_i/A_{i-1}$ of dimension $1$ for all $i$. Let $\Omega$ be the saturated formation
of supersoluble algebras. An $S$-module $V$ is $\mathcal{U}$-hypercentral if it has a composition series with all quotients 1-dimensional.

**Theorem 7.1.** Let $L$ be a finite-dimensional Lie algebra over any field $F$ and let $S \ll L$ be supersoluble. Then $L$ has a faithful finite-dimensional representation in which $S$ is represented by upper triangular matrices.

**Proof.** By Theorem 5.1, $L$ has a faithful $S\mathcal{U}$-hypercentral module $V$. It follows that $S$ fixes a flag in $V$ and for suitable choice of basis, is represented by upper triangular matrices. 

If $S_i \ll L$ are supersoluble, then by Theorem 5.1, there exists a faithful $L$-module $V$ which is simultaneously $S_i\mathcal{U}$-hypercentral. It does not follow in general that all $S_i$ simultaneously can be represented by upper triangular matrices. Each $S_i$ fixes some flag but there need not be any flag fixed by them all. However this does hold in characteristic 0.

**Lemma 7.2.** Let $L$ be a Lie algebra over a field $F$ of characteristic 0 and let $\mathfrak{F}$ be a saturated formation. Let $\{S_i | i \in I\}$ be the set of all subnormal subalgebras $S_i \ll L$ which are in $\mathfrak{F}$. Put $S = \sum_{i \in I} S_i$. Then $S \triangleleft L$ and $S \in \mathfrak{F}$.

**Proof.** Let $R$ be the radical of $L$. Then $LR$ is a nilpotent ideal of $R$. Since $\mathfrak{U} \subseteq \mathfrak{F}$, $LR \in \mathfrak{F}$. Since $S_i$ is soluble and $S_i \ll L$, $S_i \leq R$.

Let $S_1$ be any ideal of $L$ which is in $\mathfrak{F}$ and contains $LR$. Let $S_2$ be any subnormal subalgebra of $L$ which is in $\mathfrak{F}$. Then $S_1 + S_2 \ll L$. We have to prove $S_1 + S_2 \in \mathfrak{F}$. The result then follows.

By Barnes [2, Theorem 2], for some normal $F$-subspace $\Lambda$ of the algebraic closure $\bar{F}$ of $F$, $\mathfrak{F}$ is the class of all soluble finite-dimensional Lie algebras $S$ over $F$ with the property that for all $x \in S$, the eigenvalues of $\text{ad}(x)$ all lie in $\Lambda$.

We may suppose $L = S_1 + S_2$. Then $L$ is soluble. Consider any chief factor $V$ of $L$. Then $L'$ is in the kernel of the representation $\rho$ of $L$ on $V$. We have a set $\rho(S_1) \cup \rho(S_2)$ of commuting linear transformations of $V$, all of whose eigenvalues lie in $\Lambda$. They therefore fix a flag in $\bar{F} \otimes V$. For $s_1 \in S_1$ and $s_2 \in S_2$, it follows that the eigenvalues of $\rho(s_1 + s_2)$ are sums of an eigenvalue of $\rho(s_1)$ and an eigenvalue of $s_2$, thus all in $\Lambda$. 

**Corollary 7.3.** Let $L$ be a finite-dimensional Lie algebra over a field $F$ of characteristic 0. Then $L$ has a faithful finite-dimensional representation in which every supersoluble subnormal subalgebra of $L$ is represented by upper triangular matrices.
PROOF. By Lemma 7.2, there exists a supersoluble ideal $S$ of $L$ which contains every supersoluble subnormal subalgebra. Let $V$ be a faithful $SU$-hypercentral $L$-module. A flag in $V$ fixed by $S$ is fixed by every supersoluble subnormal subalgebra.

EXAMPLE 7.4. Lemma 7.2 and Corollary 7.3 do not hold in characteristic $p$. Let $V = (v_0, \ldots, v_{p-1})$ where the subscripts are integers mod $p$ and let $L = (x, y, z, V)$ with multiplication given by $xy = z, xz = yz = v_i v_j = 0, x v_i = i v_{i-1}, y v_i = v_{i+1}$ and $z v_i = v_i$. Then $S_1 = (x, z, V)$ and $S_2 = (y, z, V)$ are supersoluble ideals of $L$ but $S_1 + S_2$ is not supersoluble. A representation with both $S_1$ and $S_2$ upper triangular would have $S_1 + S_2$ upper triangular, which would imply $S_1 + S_2$ supersoluble.

Over the field $\mathbb{R}$ of real numbers, there is another saturated formation, $\mathcal{I}$ consisting of those soluble Lie algebras $S$ such that, for all $s \in S$, all eigenvalues of $\text{ad}(s)$ are pure imaginary.

THEOREM 7.5. Suppose $S \in \mathcal{I}$ is an ideal of the finite-dimensional Lie algebra $L$ over $\mathbb{R}$. Then $L$ has a faithful finite-dimensional representation in which $S$ is represented by matrices which are block upper triangular, and with the diagonal blocks either 0 or of the form $\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$ for some $r \in \mathbb{R}$.

PROOF. For any soluble Lie algebra $S$ over a field of characteristic 0, the derived subalgebra $S'$ is in the kernel of any irreducible representation. Let $V$ be an $\mathcal{I}$-central irreducible module for $S$ and suppose $s_1 \in S$ acts non-trivially. Let $s_2 \in S$. The actions of $s_1$ and $s_2$ commute, so in the complexification of $V$, they have a common eigenvector. Since the eigenvalues are pure imaginary, for some $r \in \mathbb{R}$, $s_2 - rs_1$ has an eigenvalue 0, thus an eigenvector in $V$. These eigenvectors form a submodule, so by the irreducibility of $V$, $s_2 - rs_1$ acts trivially. It follows that the kernel of the representation has codimension 1 and that $V$ is 2-dimensional with the action of $s_1$ given by $\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$ for some $r \in \mathbb{R}$. The result follows.

References


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