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ASYMPTOTIC BEHAVIOUR FOR AN ALMOST-ORBIT OF NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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In this paper, by using the technique of product nets, we are able to prove a weak convergence theorem for an almost-orbit of right reversible semigroups of nonexpansine mappings in a general Banach space X with Opial's condition. This includes many well known results as special cases. Let C be a weakly compact subset of a Banach space X with Opial's condition. Let G be a right reversible semitopological semigroup, $\mathcal{S} = \{T(t) : t \in G\}$ a nonexpansive semigroup on C, and $u(\cdot)$ an almost-orbit of $\mathcal{S}$. Then $\{u(t) : t \in G\}$ is weakly convergent (to a common fixed point of $\mathcal{S}$) if and only if it is weakly asymptotically regular (that is, $\{u(ht) - u(t)\}$ converges to 0 weakly for every $h \in G$).

1. INTRODUCTION

Let C be a nonempty subset of a Banach space X. A mapping $T : C \mapsto C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for every $x, y \in C$. In [8], Opial proved the first weak convergence theorem in a Hilbert space: Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \mapsto C$ be a nonexpansive mapping. Then for each $x \in C, \{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular, that is, $T^{n+1}x - T^n x$ converges weakly to 0.

Let G be a semitopological semigroup , that is, G is a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \mapsto t \cdot s$ and $s \mapsto s \cdot t$ from G to G are continuous. G is called right reversible if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \leqslant)$ is a directed system when the binary relation "$\leqslant$" on G is defined by $a \leqslant b$ if and only if $\{a\} \cup \overline{Ga} \supseteq \{b\} \cup \overline{Gb}$, for $a, b \in G$. Right reversible semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups. Now let $\mathcal{S} = \{T(t) : t \in G\}$ be a family of self-mappings of C. Recall that $\mathcal{S}$ is said to be a nonexpansive semigroup on C if the following conditions are satisfied:

(1) $T(ts)x = T(t)T(s)x$ for all $t, s \in G$ and $x \in C$,
(2) $\|T(t)x - T(t)y\| \leqslant \|x - y\|$ for all $t \in G$ and $x, y \in C$.

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We denote by $\mathcal{F}(\mathcal{S})$ the set of all common fixed points of $T(t)$, $t \in G$.

We say that a function $u(\cdot) : G \mapsto C$ is an almost-orbit of $\mathcal{S}$ if

$$\lim_{t\in G}\Big[\sup_{h\in G}\big\|u(ht) - T(h)u(t)\big\|\Big] = 0. \tag{1}$$

It is clear that for each $x \in C$, the orbit $\{T(t)x\}$ is an almost-orbit of S.

A Banach space X is said to satisfy Opial's condition if $\{x_\alpha\}$ converges weakly to x implies

$$\limsup_{\alpha} \|x_\alpha - x\| < \limsup_{\alpha} \|x_\alpha - y\|$$

for all $y \neq x$.

In recent years, much effort has been devoted to studying asymptotic behaviour for (asymptotically) nonexpansive mappings and semigroups, (See [**1, 2, 3, 4, 5, 6, 7, 9, 10**].) Most of the work was carried out in a uniformly convex Banach space whose norm is either Frechet differentiable or satisfying Opial's condition.

In the present article, by using the technique of product nets, we are able to prove a weak convergence theorem for an almost-orbit of right reversible semigroups of nonexpansine mappings in a general Banach space X with Opial's condition. This includes many known results as special cases. Our theorem seems to be new, even if X is a uniformly convex Banach space, since C need not to be convex and G is a non-commutative semigroup.

2. Main Results

Theorem 1. *Let C be a weakly compact subset of a Banach space X with Opial's condition. Let G be a right reversible semitopological semigroup, let $\mathcal{S} = \{T(t) : t \in G\}$ be a nonexpansive semigroup on C, and let $u(\cdot)$ be an almost-orbit of $\mathcal{S}$. Then $\{u(t) : t \in G\}$ is weakly convergent (to a fixed point) if and only if it is weakly asymptotically regular (that is, $u(ht) - u(t)$ converges weakly to 0 for every $h \in G$).*

To prove Theorem 1, we need the following simple lemmas.

Lemma 2. (See [**2**].) *Let $\{v_1(t) : t \in G\}$ and $\{v_2(t) : t \in G\}$ be almost-orbits of $\mathcal{S}$. Then, $\lim_{t\in G}\big\|v_1(t) - v_2(t)\big\|$ exists. In particular, for each $f \in \mathcal{F}(\mathcal{S})$, $\lim_{t\in G}\big\|v_1(t) - f\big\|$ exists.*

Lemma 3. *Let C be a weakly compact subset of a Banach space X with Opial's condition, $\mathcal{S} = \{T(t) : t \in G\}$ a nonexpansive semigroup on C, and $u(\cdot)$ an almost-orbit of $\mathcal{S}$. Suppose that every weak limit point of $\{u(t) : t \in G\}$ is a common fixed point of $\mathcal{S}$. Then $\{u(t) : t \in G\}$ converges weakly.*

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PROOF: Let $\omega_w(u)$ be the set of all weak limit points of a subnet of the net $\{u(t) : t \in G\}$. Clearly, $\omega_w(u)$ is nonvoid since C is a weakly compact subset. Let $v_i \in \omega_w(u)$, $i = 1, 2$ and $v_1 = w - \lim_{\alpha \in A} u(t_\alpha)$, $v_2 = w - \lim_{\beta \in B} u(t_\beta)$, where $\{t_\alpha : \alpha \in A\}$ and $\{t_\beta : \beta \in B\}$ are two subnets of G, for directed sets A and B. Suppose that $v_1 \neq v_2$. Then, by Lemma 2 and Opial's condition

$$\begin{aligned}\lim_{t \in G} \|u(t) - v_1\| &= \lim_{\alpha \in A} \|u(t_\alpha) - v_1\| \\ &< \lim_{\alpha \in A} \|u(t_\alpha) - v_2\| \\ &= \lim_{t \in G} \|u(t) - v_2\|.\end{aligned}$$

In the same way, we have $\lim_{t \in G} \|u(t) - v_2\| < \lim_{t \in G} \|u(t) - v_1\|$. This is a contradiction. Consequently, $v_1 = v_2$ and hence we have the desired result. □

PROOF OF THEOREM 1: We only need to prove the "if" part. Suppose that $u(ht) - u(t)$ converges weakly to 0 for every $h \in G$. In view of Lemma 3, it is enough to show that $\omega_w(u) \subset \mathcal{F}(\mathcal{S})$. Let $y \in \omega_w(u)$ and $\{t_\alpha : \alpha \in A\}$ be a subnet of G such that

$$w - \lim_{\alpha \in A} u(t_\alpha) = y.$$

Then for any $h \in G$

$$w - \lim_{\alpha \in A} u(ht_\alpha) = y.$$

Let I_1 be the family of all finite nonempty subset of X^* (the dual of X), N the set of positive integers, and $I = I_1 \times N = \{(B, n) : B \in I_1,\ n \in N\}$. Then, for any $\beta = (B, n) \in I$, we write $P_1\beta = B$ and $P_2\beta = n$. In this case, $(I, \leqslant)$ is a directed system when the binary relation "$\leqslant$"on I is defined by $\beta_1 \leqslant \beta_2$ if and only if

$$P_1\beta_1 \subseteq P_1\beta_2 \text{ and } P_2\beta_1 \leqslant P_2\beta_2.$$

Let τ be the weak topology on C. Let

$$O_\beta = \left\{x \in C : |f(x) - f(y)| < \frac{1}{P_2\beta},\ \forall f \in P_1\beta\right\} \text{ for } \beta \in I.$$

It is easily seen that $\{O_\beta : \beta \in I\}$ is a τ-open base at y and $O_{\beta_1} \supseteq O_{\beta_2}$ if $\beta_1 \leqslant \beta_2$.

Put

$$\varphi(t) = \sup_{h \in G} \|u(ht) - T(h)u(t)\|,$$

$$b(t) = \limsup_{\alpha \in A} \|u(tt_\alpha) - y\|,$$

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and

$$b = \inf\{b(s) : s \in G\}.$$

For $\beta \in I$, one can choose $s_\beta \in G$ such that

(2)
$$b(s_\beta) \leqslant b + \frac{1}{P_2\beta}.$$

Since

$$\begin{aligned} b(st) &= \limsup_{\alpha \in A} \left\| u(stt_\alpha) - y \right\| \\ &\leqslant \limsup_{\alpha \in A} \left\| u(stt_\alpha) - T(s)y \right\| \quad \text{(Opial's condition)} \\ &\leqslant \limsup_{\alpha \in A} \left(\left\| u(stt_\alpha) - T(s)u(tt_\alpha) \right\| + \left\| T(s)u(tt_\alpha) - T(s)y \right\| \right) \\ &\leqslant \limsup_{\alpha \in A} \left\| u(tt_\alpha) - y \right\| \\ &= b(t) \end{aligned}$$

for all $t, s \in G$, we have

(3)
$$b \leqslant b(ts_\beta) \leqslant b(s_\beta) \leqslant b + \frac{1}{P_2\beta}$$

for all $t \in G$ and $\beta \in I$. Since for $h \in G$, $u(hs_\beta t_\alpha)$ converges weakly to y as $\alpha \in A$, for each $\beta \in I$ there exists $\alpha_\beta^1 \in A$ such that

(4)
$$u(hs_\beta t_\alpha) \in O_\beta$$

for all $\alpha \geqslant \alpha_\beta^1$. One can also choose $\alpha_\beta^2 \in A$ such that

(5)
$$\sup_{t \in G} \varphi(tt_\alpha) < \frac{1}{P_2\beta}$$

for all $\alpha \geqslant \alpha_\beta^2$. It then follows from (2) that there exists $\alpha_\beta^3 \in A$ such that

(6)
$$\left\| u(s_\beta t_\alpha) - y \right\| \leqslant b + \frac{2}{P_2\beta},$$

for all $\alpha \geqslant \alpha_\beta^3$. Since (3) implies that

$$b(hs_\beta) = \limsup_{\alpha \in A} \left\| u(hs_\beta t_\alpha) - y \right\| \geqslant b,$$

there exists $\alpha_\beta \in A$ such that

(7)
$$\alpha_\beta \geqslant \alpha_\beta^i \ (i = 1, 2, 3)$$

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and

$$\|u(hs_\beta t_{\alpha_\beta}) - y\| \geqslant b - \frac{1}{P_2\beta}. \tag{8}$$

Now (4), (6) and (7) imply that

$$u(hs_\beta t_{\alpha_\beta}) \in O_\beta \tag{9}$$

and

$$\|u(s_\beta t_{\alpha_\beta}) - y\| \leqslant b + \frac{2}{P_2\beta}. \tag{10}$$

(9) implies that $u(hs_\beta t_{\alpha_\beta})$ is convergent weakly to y. Combining (5) with (8) and (10), we have

$$\begin{aligned}\|u(hs_\beta t_{\alpha_\beta}) - T(h)y\| &\leqslant \|u(hs_\beta t_{\alpha_\beta}) - T(h)u(s_\beta t_\alpha)\| + \|T(h)u(s_\beta t_\alpha) - T(h)y\| \\ &\leqslant \frac{1}{P_2\beta} + \|u(s_\beta t_\alpha) - y\| \\ &\leqslant b + \frac{3}{P_2\beta} \\ &\leqslant \|u(hs_\beta t_{\alpha_\beta}) - y\| + \frac{4}{P_2\beta}.\end{aligned}$$

This implies that $\limsup_{\beta\in I}\|u(hs_\beta t_{\alpha_\beta}) - T(h)y\| \leqslant \limsup_{\beta\in I}\|u(hs_\beta t_{\alpha_\beta}) - y\|$. By Opial's condition, we have $T(h)y = y$. This completes the proof. ∎

Now, using Theorem 1, we provide weak convergence theorems for almost-orbits of nonexpansive mappings and semigroups. Let T be a nonexpansive mapping from C into itself and let $\{x_n\}$ be an almost-orbit of T, that is,

$$\lim_{n\to\infty}\left[\sup_{m\geqslant 0}\|x_{n+m} - T^m x_n\|\right] = 0.$$

Let $\mathcal{S} = \{T(t) : t \geqslant 0\}$ be a nonexpansive semigroup on C and $u(\cdot) : R^+ \to C$ an almost-orbit of $\mathcal{S}$, that is,

$$\lim_{s\to}\left[\sup_{t\geqslant 0}\|u(t+s) - T(t)u(s)\|\right] = 0.$$

Put $G = \{0, 1, 2, \ldots\}$, $\mathcal{S} = \{T^i : i \in G\}$ in Theorem 1. Then we get the following weak convergence theorem for nonexpansive mappings.

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Theorem 4. *Let X be a Banach space with Opial's condition, C a weakly compact subset of X, $T : C \mapsto C$ a nonexpansive mapping, and $\{x_n\}$ an almost-orbit of T. Then $\{x_n\}$ is weakly convergent (to a fixed point of T) if and only if it is weakly asymptotically regular (that is, $\{x_{n+1} - x_n\}$ converges to 0 weakly).*

Put $G = R^+$, $\mathcal{S} = \{T(t) : t \in G\}$ in Theorem 1. We get a weak convergence theorem for nonexpansive semigroups.

Theorem 5. *Let X be a Banach space with Opial's condition, C a weakly compact subset of X, $\mathcal{S} = \{T(t) : T \geqslant 0\}$ a nonexpansive semigroup, and $u(\cdot)$ an almost-orbit of $\mathcal{S}$. Then $\{u(t)\}$ converges weakly to some point of $\mathcal{F}(\mathcal{S})$ if and only if it is weakly asymptotically regular (that is, $\{u(t+h) - u(t)\}$ converges weakly to 0 as $t \to 0$ for all $h \geqslant 0$).*

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