REGULAR ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

Yupaporn Kemprasit and Thawhat Changphas

The semigroup $OT(X)$ of all order-preserving total transformations of a finite chain $X$ is known to be regular. We extend this result to subchains of $Z$; and we characterise when $OT(X)$ is regular for an interval $X$ in $R$. We also consider the corresponding idea for partial transformations of arbitrary chains and posets.

1. Introduction

If $X$ is a set, we let $P(X)$ denote the semigroup under composition of all partial transformations of $X$ (that is, mappings $\alpha : A \to B$ where $A,B \subseteq X$); and we note that $P(X)$ is regular (that is, for each $\alpha \in P(X)$, there exists $\beta \in P(X)$ such that $\alpha = \alpha \beta \alpha$). Following standard notation, we let $\text{dom} \alpha$ and $\text{ran} \alpha$ denote the domain and range of $\alpha \in P(X)$.

If $(X, \leq)$ is a poset, we say $\alpha \in P(X)$ is order-preserving if for all $x,y \in \text{dom} \alpha$, $x \leq y$ implies $x \alpha \leq y \alpha$; and we let $OP(X)$ denote the semigroup under composition of all order-preserving partial transformations of $X$. Similarly, $T(X)$ denotes the semigroup of all total transformations of $X$ (that is, mappings $\alpha : X \to X$) and likewise it is regular. Also, if $(X, \leq)$ is a poset, we let $OT(X)$ denote the subsemigroup of $T(X)$ consisting of all order-preserving total transformations of $X$.

It is known [9, p.203, Exercise 6.1.7] that $OT(X)$ is regular if $(X, \leq)$ is a finite chain. In Section 2, we extend this to any chain which is order-isomorphic to a subset of $Z$, the set of integers with their natural order. We also prove that if $X$ is an interval in $R$, the set of real numbers, then $OT(X)$ is regular if and only if $X$ is closed and bounded. Then we answer similar questions for $OP(X)$ and some of its subsemigroups for arbitrary chains $X$. And in Section 3, we suppose $X$ is not a chain and characterise when $OP(X)$ is regular. We also list some conditions under which $OT(X)$ is (or is not) regular when $X$ is not a chain.

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2. ORDER-PRESERVING TRANSFORMATIONS OF CHAINS

If \((X, \leq)\) is a poset, we define the opposite partial order \(\leq_{\text{opp}}\) on \(X\) via:

\[ x \leq_{\text{opp}} y \text{ if and only if } y \leq x. \]

Note that if \(\alpha \in P(X)\) then \(\alpha\) preserves \(\leq\) if and only if it preserves \(\leq_{\text{opp}}\). Consequently, \(OP(X, \leq) = OP(X, \leq_{\text{opp}})\) and the regularity of \(OP(X, \leq)\) holds equally for \(OP(X, \leq_{\text{opp}})\). Similar statements are true for \(OT(X, \leq)\) and for \(OI(X, \leq)\), the semigroup of all order-preserving partial transformations of \(X\) which are one-to-one ("injective").

For the chain \(Z\) of integers, we let \(Z^+\) denote the set of positive integers and \(Z^-\) the set of negative integers. If \(X\) is a chain which is order-isomorphic to a subset of \(Z\) with its natural order then \(X\) has one of the following forms:

1. \(\{x_1, x_2, \ldots, x_n\}\) where \(n \in \mathbb{Z}^+\) and \(x_1 < x_2 < \cdots < x_n\),
2. \(\{x_i : i \in \mathbb{Z}^+\}\) where \(x_i < x_j\) if \(i < j\),
3. \(\{x_i : i \in \mathbb{Z}\}\) where \(x_i < x_j\) if \(i < j\), or
4. \(\{x_i : i \in \mathbb{Z}^-\}\) where \(x_i < x_j\) if \(i < j\).

Our first result will be needed often in what follows: its simple proof is omitted.

**Lemma 2.1.** Let \(X\) be a chain. If \(\alpha \in OP(X)\) and \(a, b \in \text{ran } \alpha\) satisfy \(a < b\) then \(x < y\) for all \(x \in \text{ran } \alpha^{-1}\) and \(y \in b\alpha^{-1}\).

For any \(A \subseteq X\), we let \(\min(A)\) and \(\max(A)\) denote the minimum and the maximum elements of \(A\) if they exist.

**Theorem 2.2.** Let \(X\) be a chain. If \(X\) is order-isomorphic to a subset of \(Z\) then the semigroup \(OT(X)\) is regular.

**Proof:** We regard \(X\) as being one of the chains (1)-(4) listed above. Then, if \(A\) is any nonempty subset of \(X\), \(\max(A)\) exists if \(A\) has an upper bound in \(X\), and \(\min(A)\) exists if \(A\) has a lower bound in \(X\). It follows from this and Lemma 2.1 that, if \(\alpha \in OP(X)\) and \(a \in \text{ran } \alpha\), then \(\max(a\alpha^{-1})\) exists if \(a < b\) for some \(b \in \text{ran } \alpha\), and \(\min(a\alpha^{-1})\) exists if \(b < a\) for some \(b \in \text{ran } \alpha\).

Let \(\alpha \in OT(X)\). If \(\alpha\) is a constant map, it is clearly regular. Therefore, suppose \(\text{ran } \alpha\) contains at least two elements and note that, since it is a subchain of \(X\), it takes one of the forms listed in (1)-(4) above. In cases (1)-(3), we define \(\beta : X \rightarrow X\) as follows:

1. if \(\text{ran } \alpha = \{a_1, a_2, \ldots, a_n\}\), let
2. \(x\beta = \begin{cases} 
\max(a_1\alpha^{-1}) & \text{if } x \leq a_1, \\
\min(a_i\alpha^{-1}) & \text{if } a_i < x \leq a_{i+1} \text{ and } i \neq n, \\
\min(a_n\alpha^{-1}) & \text{if } x > a_n.
\end{cases}\)
(2) if ran $\alpha = \{a_i : i \in \mathbb{Z}^+\}$, let

$$x\beta = \begin{cases} 
\max (a_i \alpha^{-1}) & \text{if } x \leq a_1, \\
\min (a_i+1 \alpha^{-1}) & \text{if } a_i < x \leq a_{i+1} \text{ for some } i \in \mathbb{Z}^+.
\end{cases}$$

(3) if ran $\alpha = \{a_i : i \in \mathbb{Z}\}$, let

$$x\beta = \max (a_i+1 \alpha^{-1}) \quad \text{if } a_i < x \leq a_{i+1} \text{ for some } i \in \mathbb{Z}.$$

Now, if $x \in X$ then $x\alpha = a_k$ for some $k \in \mathbb{Z}$. By definitions (1)–(3), $a_k\beta$ equals $\max (a_k \alpha^{-1})$ or $\min (a_k \alpha^{-1})$: in each case, this means $(a_k\beta)\alpha = a_k$, and hence $x(\alpha\beta\alpha) = x\alpha$ for all $x \in X$; that is, $\alpha = \alpha\beta\alpha$.

To show $\beta$ is order-preserving, suppose $x < y$ in $X$. In cases (1) and (2), if $y \leq a_1$ then $x\beta = y\beta$, and the same conclusion holds in case (1) if $x \geq a_n$. On the other hand, in each of (1)–(3), if $a_k < x < y \leq a_{k+1}$ for some $k$ then $x\beta = y\beta$, by the definition of $\beta$. Suppose instead that $a_k < x \leq a_{k+1} \leq a_\ell < y \leq a_{\ell+1}$

for some $k, \ell$. Then $a_{k+1} < a_{\ell+1}$ and Lemma 2.1 imply that $u < v$ for all $u \in a_{k+1} \alpha^{-1}$ and $v \in a_{\ell+1} \alpha^{-1}$. But, by the definition of $\beta$, $x\beta \in a_{k+1} \alpha^{-1}$ and $y\beta \in a_{\ell+1} \alpha^{-1}$, hence $x\beta < y\beta$ as required. The remaining possibilities: $x \leq a_1 < a_n \leq y$ in case (1), and $x \leq a_1 \leq a_\ell < y \leq a_{\ell+1}$ in cases (1) and (2), lead to the same conclusion. Therefore, $\beta \in OT(X)$ is regular in cases (1)–(3).

For case (4), we recall from the start of this section that $OT(\mathbb{Z}^+, \leq) = OT(\mathbb{Z}^+, \leq_{opp})$, and clearly $(\mathbb{Z}^-, \leq)$ is order-isomorphic to $(\mathbb{Z}^+, \leq_{opp})$. Hence, from our conclusion in case (3), if $X$ is order-isomorphic to $\mathbb{Z}^-$ then $OT(X)$ is regular. \[ \square \]

We now show that $OT(\mathbb{R})$ is not regular when $\mathbb{R}$ is the set of real numbers with their natural order. In fact, the following sequence of Lemmas will eventually characterise when $OT(X)$ is regular for an interval $X$ in $\mathbb{R}$.

**Lemma 2.3.** The semigroup $OT(\mathbb{R})$ is not regular.

**Proof:** Fix $r \in (1, \infty)$ and let $\alpha \in OT(\mathbb{R})$ be the map: $x\alpha = r^x$ for all $x \in \mathbb{R}$. Then ran $\alpha = \mathbb{R}^+$ and $\alpha$ is one-to-one. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(\mathbb{R})$. Then, since $\alpha$ is one-to-one, $x = x\alpha\beta$ for each $x \in \mathbb{R}$. Thus, $\mathbb{R}^+ \beta = \mathbb{R}$. Hence, since $0\beta \in \mathbb{R}$, there exists $d \in \mathbb{R}^+$ such that $0\beta = d\beta$. Choose $c \in (0, d)$. Then

$$0\beta \leq c\beta \leq d\beta = 0\beta,$$

and we deduce that $c\beta = d\beta$. Since $c, d \in \mathbb{R}^+ = \text{ran} \alpha$, we can choose $x, y \in \mathbb{R}$ with $x\alpha = c$ and $y\alpha = d$. Then

$$c = x\alpha = x\alpha\beta\alpha = c\beta\alpha = d\beta\alpha = y\alpha\beta\alpha = y\alpha = d.$$
which contradicts \( c < d \). Hence, \( \alpha \) is not a regular element of \( OT(\mathbb{R}) \), and the result follows.

**Lemma 2.4.** The semigroup \( OT((a, \infty)) \) is not regular for any \( a \in \mathbb{R} \).

**Proof:** Fix \( r \in [1, \infty) \) and let \( \alpha \in OT((a, \infty)) \) be the map: \( x\alpha = x + r \) for all \( x \in (a, \infty) \). Then \( \text{ran} \alpha = (a + r, \infty) \) and \( \alpha \) is one-to-one. Suppose \( \alpha = \alpha\beta\alpha \) for some \( \beta \in OT((a, \infty)) \). Then, since \( \alpha \) is one-to-one, \( x = x\alpha\beta \) for each \( x \in (a, \infty) \). Thus, \( (a + r, \infty)\beta = (a, \infty) \). Hence, since \( (a + r)\beta \in (a, \infty) \), there exists \( d \in (a + r, \infty) \) such that \( (a + r)\beta = d\beta \). Choose \( c \in (a + r, d) \). Then

\[
(a + r)\beta \leq c\beta \leq d\beta = (a + r)\beta,
\]
and we deduce that \( c\beta = d\beta \). Since \( c, d \in (a + r, \infty) = \text{ran} \alpha \), we can choose \( x, y \in (a, \infty) \) with \( x\alpha = c \) and \( y\alpha = d \). Then

\[
c = x\alpha = x\alpha\beta\alpha = c\beta\alpha = d\beta\alpha = y\alpha\beta\alpha = y\alpha = d
\]
which contradicts \( c < d \). Hence, \( \alpha \) is not a regular element of \( OT((a, \infty)) \), and the result follows.

**Lemma 2.5.** The semigroup \( OT([a, \infty)) \) is not regular for any \( a \in \mathbb{R} \).

**Proof:** Fix \( r \in [1, \infty) \) and let \( \alpha \) be the map:

\[
x\alpha = a + \frac{x - a}{x - a + r} \quad \text{for all} \ x \in [a, \infty).
\]

Then \( \text{ran} \alpha = [a, a + 1) \). Moreover, since the derivative of \( \alpha \) is strictly positive on \( (a, \infty) \), we know \( \alpha \) is increasing and hence it is one-to-one. Therefore, \( \alpha \in OT([a, \infty)) \). Suppose \( \alpha = \alpha\beta\alpha \) for some \( \beta \in OT([a, \infty)) \). Then, since \( \alpha \) is one-to-one, \( x = x\alpha\beta \) for each \( x \in [a, \infty) \), and so \( [a, a + 1)\beta = [a, \infty) \). Hence, since \( (a + 1)\beta \in [a, \infty) \), we know \( (a + 1)\beta = d\beta \) for some \( d \in [a, a + 1) \). Choose \( c \in (d, a + 1) \). Then

\[
d\beta \leq c\beta \leq (a + 1)\beta = d\beta,
\]
and we deduce that \( c\beta = d\beta \). Since \( c, d \in [a, a + 1) = \text{ran} \alpha \), we can choose \( x, y \in [a, \infty) \) with \( x\alpha = c \) and \( y\alpha = d \). The result then follows as in the proofs of the last Lemmas.

**Lemma 2.6.** The semigroup \( OT((a, b)) \) is not regular for any \( a, b \in \mathbb{R} \) with \( a < b \).

**Proof:** Fix \( r \in (0, b - a) \) and let \( \alpha \) be the map:

\[
x\alpha = \left(1 - \frac{r}{b - a}\right)x + \frac{rb}{b - a} \quad \text{for all} \ x \in (a, b).
\]
That is, the graph of $\alpha$ is a line segment with positive slope, and clearly $\text{ran}\,\alpha = (a + r, b)$. Therefore, $\alpha \in \text{OT}((a, b))$. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in \text{OT}((a, b))$. Then, since $\alpha$ is one-to-one, $x = x\alpha\beta$ for each $x \in (a, b)$, and so $(a + r, b)\beta = (a, b)$. Hence, since $(a + r)\beta \in (a, b)$, we know $(a + r)\beta = d\beta$ for some $d \in (a + r, b)$. Choose $c \in (a + r, d)$. Then

$$d\beta = (a + r)\beta \leq c\beta \leq d\beta,$$

and we deduce that $c\beta = d\beta$. Since $c, d \in (a + r, b) = \text{ran}\,\alpha$, we can choose $x, y \in (a, b)$ with $x\alpha = c$ and $y\alpha = d$, and the result follows as before.

**Lemma 2.7.** The semigroup $\text{OT}((a, b))$ is not regular for any $a, b \in \mathbb{R}$ with $a < b$.

**Proof:** Fix $r \in (0, b-a)$ and let $\alpha$ be the map:

$$x\alpha = \frac{rx}{b-a} + a - \frac{ra}{b-a} \quad \text{for all } x \in [a, b).$$

That is, the graph of $\alpha$ is a line segment with positive slope, and clearly $\text{ran}\,\alpha = [a, a + r)$. Therefore, $\alpha \in \text{OT}((a, b))$. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in \text{OT}((a, b))$. Then, since $\alpha$ is one-to-one, $x = x\alpha\beta$ for each $x \in [a, b)$, and so $[a, a + r)\beta = [a, b)$. Hence, since $(a + r)\beta \in [a, b)$, we know $(a + r)\beta = d\beta$ for some $d \in [a + r, b)$. Choose $c \in (d, a + r)$. Then

$$d\beta \leq c\beta \leq (a + r)\beta = d\beta,$$

and we deduce that $c\beta = d\beta$. Since $c, d \in [a + r, b) = \text{ran}\,\alpha$, we can choose $x, y \in [a, b)$ with $x\alpha = c$ and $y\alpha = d$, and the result follows as before.

With the notation at the start of this section, if $\leq$ is the natural order on $\mathbb{R}$ then

1. $((-\infty, a), \leq)$ is order-isomorphic to $((-a, \infty), \leq_{\text{opp}})$ for each $a \in \mathbb{R}$,
2. $((-\infty, a], \leq)$ is order-isomorphic to $([-a, \infty), \leq_{\text{opp}})$ for each $a \in \mathbb{R}$, and
3. $([a, b], \leq)$ is order-isomorphic to $([-b, -a), \leq_{\text{opp}})$ for each $a, b \in \mathbb{R}$.

Consequently, Lemmas 2.4, 2.5 and 2.7 show that the semigroups $\text{OT}((-\infty, a))$, $\text{OT}((-\infty, a])$ and $\text{OT}((a, b))$ are not regular for any $a, b \in \mathbb{R}$. This covers all nonempty intervals of $\mathbb{R}$ except one: namely, $[a, b)$ with $a < b$ and we shall prove that the semigroup $\text{OT}([a, b))$ is regular. But, to do this, we need one more Lemma.

**Lemma 2.8.** Let $\alpha \in \text{OT}([a, b])$ where $a, b \in \mathbb{R}$ and $a < b$, and suppose $x \in (a\alpha, b\alpha)$. If $A_x = [a, x]\alpha^{-1}$ and $B_x = (x, b]\alpha^{-1}$ then $\{A_x, B_x\}$ is a partition of $[a, b]$ such that $c < d$ for all $c \in A_x$ and $d \in B_x$.

**Proof:** Since $x \in (a\alpha, b\alpha)$, we know $a \leq a\alpha < x$ and $x < b\alpha \leq b$, so $a \in [a, x]\alpha^{-1} = A_x$ and $b \in (x, b]\alpha^{-1} = B_x$. The result then follows from Lemma 2.1 and the fact that $[a, b] = [a, x] \cup (x, b)$. $\square$
For the next result, we recall: if $I$ is an interval in $\mathbb{R}$ and if $\{A, B\}$ is a partition of $I$ such that $x < y$ for all $x \in A$ and $y \in B$ then either $\max(A)$ or $\min(B)$ exists (but not both).

**Lemma 2.9.** The semigroup $OT([a, b])$ is regular for any $a, b \in \mathbb{R}$ with $a < b$.

**Proof:** Let $\alpha \in OT([a, b])$ and note that $a\alpha \leq b\alpha$ and $\text{ran }\alpha \subseteq [a\alpha, b\alpha]$. Define $d_x$ for each $x \in [a, b]$ as follows:

$$
\begin{align*}
    d_x =
    \begin{cases}
        a & \text{if } x \in [a, a\alpha), \\
        b & \text{if } x \in (b\alpha, b], \\
        xa^{-1} & \text{if } x \in \text{ran }\alpha.
    \end{cases}
\end{align*}
$$

To complete the definition, suppose $x \in (a\alpha, b\alpha) \setminus \text{ran }\alpha$ and put $A_x = [a, x]a^{-1}$ and $B_x = (x, b]a^{-1}$. By Lemma 2.8, $\{A_x, B_x\}$ is a partition of $[a, b]$ with a special property; and, by a remark above, either $\max(A_x)$ or $\min(B_x)$ exists (but not both). Hence, we can define:

$$
    d_x = \begin{cases}
        \max(A_x) & \text{if } x \in (a\alpha, b\alpha) \setminus \text{ran }\alpha \text{ and } \max(A_x) \text{ exists}, \\
        \min(B_x) & \text{if } x \in (a\alpha, b\alpha) \setminus \text{ran }\alpha \text{ and } \min(B_x) \text{ exists}.
    \end{cases}
$$

Finally, we let $\beta : [a, b] \to [a, b], x \to d_x$. If $x \in [a, b]$ then $x\alpha \in \text{ran }\alpha$, so the definition of $d_x$ implies that $d_{x\alpha}\alpha = x\alpha$. Hence,

$$
    x\alpha\beta\alpha = ((x\alpha)\beta)\alpha = d_{x\alpha}\alpha = x\alpha
$$

which shows that $\alpha = \alpha\beta\alpha$ on $[a, b]$.

To show that $\beta$ is order-preserving, let $x, y \in [a, b]$ and $x < y$. Then $x \in [a, y]$ and $y \in (x, b]$, and we consider six cases.

**Case 1.** $x < a\alpha$. Then $x\beta = d_x = a$, so $x\beta \leq y\beta$.

**Case 2.** $y > b\alpha$. Then $y\beta = d_y = b$, so $x\beta \leq y\beta$.

**Case 3.** $x, y \in \text{ran }\alpha$. By Lemma 2.1, $u < v$ for all $u \in x\alpha^{-1}$ and $v \in y\alpha^{-1}$. In particular, by definition, $x\beta = d_x < d_y = y\beta$.

**Case 4.** $x \in \text{ran }\alpha$ and $y \in (a\alpha, b\alpha) \setminus \text{ran }\alpha$. Then $d_x \in x\alpha^{-1} \subseteq [a, y]\alpha^{-1} = A_y$. Hence, if $\max(A_y)$ exists then

$$
    x\beta = d_x \leq \max(A_y) = d_y = y\beta.
$$

On the other hand, if $\min(B_y)$ exists then, by Lemma 2.8, we have:

$$
    x\beta = d_x < d_y = \min(B_y) = y\beta.
$$

**Case 5.** $x \in (a\alpha, b\alpha) \setminus \text{ran }\alpha$ and $y \in \text{ran }\alpha$. An argument similar to that in case (4) shows $x\beta \leq y\beta$ in this case also.
CASE 6. \(x, y \in (a\alpha, b\alpha) \setminus \text{ran } \alpha\). If \([x, y] \cap \text{ran } \alpha = \emptyset\) then

\[
A_x = [a, x]\alpha^{-1} = [a, y]\alpha^{-1} = A_y \quad \text{and} \quad B_x = (x, b]\alpha^{-1} = (y, b]\alpha^{-1} = B_y
\]

and hence, by definition, \(x\beta = d_x = d_y = y\beta\). However, if \([x, y] \cap \text{ran } \alpha \neq \emptyset\) then there exists \(c \in \text{ran } \alpha\) such that \(x < c < y\). In this event, we can choose \(p \in [a, b]\) with \(p\alpha = c\). Then

\[
p \in [a, y]\alpha^{-1} \cap (x, b]\alpha^{-1} = A_y \cap B_x
\]

and so, using a standard property of \(\mathbb{R}\), we have:

\[
\sup(A_x) \leq \inf(B_x) \leq p \leq \sup(A_y) \leq \inf(B_y).
\]

Note that \(\sup(A_x)\) equals \(\max(A_x)\) if this maximum exists, and it equals \(\min(B_x)\) if this minimum exists; and a similar comment can be made for \(\inf(B_y)\). Hence, we conclude from the above that \(d_x \leq d_y\) and so \(x\beta \leq y\beta\). \(\square\)

The combination of Lemmas 2.3-2.9, and the remarks between them, give us the following result.

**Theorem 2.10.** For any interval \(X\) of \(\mathbb{R}\), the semigroup \(\mathcal{O}T(X)\) is regular if and only if \(X\) is closed and bounded.

In passing, we note that in [11] Howie showed that \(\mathcal{O}T(X)\) is also idempotent-generated if \(X\) is a finite chain, and in [8] the authors extended this to \(\mathcal{O}P(X)\), while in [7] Garba investigated the same idea for various subsemigroups of \(\mathcal{O}P(X)\); see [12] for a brief survey of this and related work; and see [10] for an alternative approach to the same idea for \(\mathcal{O}T(X)\) and its subsemigroup consisting of all decreasing transformations (that is, \(x\alpha \leq x\) for all \(x \in X\)). For an arbitrary chain \(X\), the elements of \(\mathcal{O}T(X)\) which are products of idempotents were described in [14]; and the corresponding notion for products of "nilpotents" in \(\mathcal{O}P(X)\) and \(\mathcal{O}I(X)\) has been examined in [6] and [5] for finite chains.

In [3] the authors considered the semigroup \(\mathcal{O}P'(X)\) consisting of all order-preserving transformations \(\alpha\) whose domains are final segments in a chain \(X\) (that is, \(x \in \text{dom } \alpha\) and \(x \leq y\) imply \(y \in \text{dom } \alpha\)); and they observed that this semigroup need not be regular. By contrast with this fact and the above results for \(\mathcal{O}T(X)\), we prove the following Theorem.

**Theorem 2.11.** If \(X\) is a chain then the semigroup \(\mathcal{O}P(X)\) is regular.

**Proof:** Let \(\alpha \in \mathcal{O}P(X)\) and, for each \(a \in \text{ran } \alpha\), choose \(d_a \in a\alpha^{-1}\). Define a partial transformation \(\beta\) via: \(\text{dom } \beta = \text{ran } \alpha\) and \(a\beta = d_a\) for each \(a \in \text{ran } \alpha\). Then \(x(a\beta\alpha) = xa\) for all \(x \in \text{dom } \alpha\) (since \(d_a\alpha = a\) for all \(a \in \text{ran } \alpha\)) and in fact
dom (αβα) = dom α. Hence, α = αβα. Also, if a < b in dom β = ran α then da < db by Lemma 2.1, so β is order-preserving, and the result follows.

As usual, if X is a set and α ∈ P(X), we define the shift of α to be

\[ s(\alpha) = |S(\alpha)| \]

where

\[ S(\alpha) = \{ x \in \text{dom } \alpha : x \alpha \neq x \} \]

and we write

\[ P(X, \aleph_0) = \{ \alpha \in P(X) : s(\alpha) < \aleph_0 \} \]

It is well-known that P(X, \aleph_0) is a semigroup: it is sometimes called the semigroup of almost identical partial transformations of X [15]. If X is a poset, we let OP(X, \aleph_0) denote the semigroup of all order-preserving partial transformations of X with finite shift, and OI(X, \aleph_0) will denote the semigroup of all one-to-one transformations in OP(X, \aleph_0).

If X is a chain and α ∈ OP(X, \aleph_0), we can define a map β ∈ OP(X) as in the proof of Theorem 2.11 so that α = αβα. In fact, since αα⁻¹ = {a} for all a ∈ ran α \ S(α)α, we have aβα = a for all a ∈ ran α \ S(α)α and so S(β) ⊆ S(α)α (since dom β = ran α). Hence, β ∈ OP(X, \aleph_0) and we have proved that OP(X, \aleph_0) is regular.

In passing, we note that Lavers [13] considered certain subsemigroups of OP(\mathbb{Z}^+, \aleph_0) in a different context, with the aim of giving presentations for them and describing their principal left (right) ideals.

In [4, Proposition 1.4], Fernandes noted that OI(X) is regular if X is a finite chain. In fact, following the proof of Theorem 2.11, it is clear that if X is any chain and α ∈ OI(X) then there exists β ∈ OI(X) with α = αβα. So, OI(X) is regular for any chain X. Indeed, since the idempotents of OI(X) are simply those transformations which fix a subchain of X pointwise, and hence they commute, we deduce that OI(X) is an inverse semigroup. The significance of OI(X) is illustrated by a result in [2]: namely, any set X with |X| ≠ 3 can be ordered so that OI(X) forms a transversal of the set of H–classes in I(X).

Finally, we consider the semigroup:

\[ OT(X, \aleph_0) = \{ \alpha \in OT(X) : s(\alpha) < \aleph_0 \} \]

and aim to show it is regular if X is a chain. However, for this we need another three Lemmas.

**Lemma 2.12.** Let X be a poset, α ∈ OP(X) and a ∈ dom α. Then

\[ \{ x \in \text{dom } \alpha : aa < x < a \} \subseteq S(\alpha) \quad \text{and} \quad \{ x \in \text{dom } \alpha : a < x < aa \} \subseteq S(\alpha). \]

**Proof:** If x ∈ dom α and aa < x < a then xα ≤ aaα. Thus, if xα = x, we have x ≤ aaα, a contradiction; so, x ∈ S(α) as required. The other containment follows similarly. \[ \square \]
LEMMA 2.13. Let X be a poset, \( \alpha \in OP(X) \) and \( A \subseteq \text{ran} \alpha \). If \( \max(A) \) and \( \max(A\alpha^{-1}) \) exist then \( \max(A) = [\max(A\alpha^{-1})]\alpha \).

PROOF: Since \( \max(A) \in A \subseteq \text{ran} \alpha \), there exists \( x \in \text{dom} \alpha \) such that \( \max(A) = x\alpha \). Then \( x \in A\alpha^{-1} \), so \( x \leq \max(A\alpha^{-1}) \) and, since \( \alpha \) is order-preserving, we deduce that \( \max(A) \leq [\max(A\alpha^{-1})]\alpha \). Since \( \max(A\alpha^{-1}) \in A\alpha^{-1} \) and \( A \subseteq \text{ran} \alpha \), we know \([\max(A\alpha^{-1})]\alpha \in A\) and this implies \( [\max(A\alpha^{-1})]\alpha \leq \max(A) \). Hence, equality holds as required.

LEMMA 2.14. Let X be a poset, \( \alpha \in OP(X) \) and \( A,B \subseteq \text{ran} \alpha \), and suppose \( \max A, \max B, \max(A\alpha^{-1}) \) and \( \max(B\alpha^{-1}) \) exist.

1. If \( \max A = \max B \) then \( \max(A\alpha^{-1}) = \max(B\alpha^{-1}) \).
2. If X is a chain and \( \max A < \max B \) then \( \max(A\alpha^{-1}) < \max(B\alpha^{-1}) \).

PROOF: By Lemma 2.13, \([\max(A\alpha^{-1})]\alpha = \max A\). Therefore, if \( \max A = \max B \), we have \([\max(A\alpha^{-1})]\alpha \in B\) and hence \( \max(A\alpha^{-1}) \in B\alpha^{-1} \). Consequently, \( \max(A\alpha^{-1}) \leq \max(B\alpha^{-1}) \), and a dual argument establishes equality in (1). On the other hand, if \( \max A < \max B \) then Lemma 2.13 implies \([\max(A\alpha^{-1})]\alpha < [\max(B\alpha^{-1})]\alpha \). Therefore, if X is a chain, we must have \( \max(A\alpha^{-1}) < \max(B\alpha^{-1}) \)(since \( \alpha \) is order-preserving).

We can now prove the following result.

THEOREM 2.15. If X is a chain then the semigroup \( OT(X, \mathbb{N}_0) \) is regular.

PROOF: Let \( \alpha \in OT(X, \mathbb{N}_0) \) and note that \( \text{dom} \alpha = X \). For each \( x \in X \), we define \( d_x \in X \) as follows.

CASE I. \( x \in \text{ran} \alpha \). Since \( x\alpha^{-1} \) is nonempty and finite for each \( x \in \text{ran} \alpha \), and X is a chain, we know \( \max(x\alpha^{-1}) \) always exists in this case. So, we put
\[
d_x = \max(x\alpha^{-1}) \quad \text{if} \ x \in \text{ran} \alpha.
\]

CASE II. \( x \notin \text{ran} \alpha \). In this case, \( x \in S(\alpha) \) and this implies \( x\alpha < x \) or \( x < x\alpha \) since X is a chain. Then, from Lemma 2.12, we deduce that \( \{z \in X : x\alpha < z < x\} \) and \( \{z \in X : x < z < x\alpha\} \) are finite sets. In turn this implies \( \{y \in \text{ran} \alpha : x\alpha \leq y < x\} = C \) say, and \( \{y \in \text{ran} \alpha : x < y \leq x\alpha\} = D \) say, are finite sets. Therefore, since \( y\alpha^{-1} \) is finite for each \( y \in \text{ran} \alpha \), the inverse images of C and D are also finite. Hence, if \( x\alpha < x \) then \( C \neq \emptyset \) and \( \max(C\alpha^{-1}) \) exists; and if \( x < x\alpha \) then \( D \neq \emptyset \) and \( \min(D\alpha^{-1}) \) exists. So, in this case, we put
\[
d_x = \begin{cases} 
\max(\{y \in \text{ran} \alpha : x\alpha \leq y < x\}\alpha^{-1}) & \text{if } x\alpha < x, \\
\min(\{y \in \text{ran} \alpha : x < y \leq x\alpha\}\alpha^{-1}) & \text{if } x < x\alpha.
\end{cases}
\]
Now let $\beta : X \to X, x \to d_x$ and note that $\alpha = \alpha \beta \alpha$ as in the proof of Theorem 2.11. Also note that

$$\left\{ x \in \text{ran } \alpha : \max (x \alpha^{-1}) \neq x \right\} \subseteq \left\{ x \in \text{ran } \alpha : x \alpha^{-1} \neq \{x\} \right\}$$

and the second set in this display is finite since $S(\alpha)$ is finite. By definition of $\beta$, this means $\{ x \in \text{ran } \alpha : x \beta \neq x \}$ is finite. But we have:

$$S(\beta) \subseteq (X \setminus \text{ran } \alpha) \cup \{ x \in \text{ran } \alpha : x \beta \neq x \}$$

where both sets in this union are finite. Hence, $\beta \in T(X, \mathbb{N}_0)$.

To show $\beta$ is order-preserving, suppose $a < b$ in $X$ and consider four cases.

CASE 1. $a, b \in \text{ran } \alpha$. Then, by definition, $d_a \in \alpha a^{-1}$ and $d_b \in \alpha b^{-1}$, so $d_a < d_b$ by Lemma 2.1.

CASE 2. $a \in \text{ran } \alpha$ and $b \notin \text{ran } \alpha$. Then $b \alpha \neq b$, so $b \alpha < b$ or $b < b \alpha$ since $X$ is a chain, and we consider two possibilities.

(i) Suppose $b \alpha < b$. If $a \in \{ y \in \text{ran } \alpha : b \alpha < y < b \} = B$ say, then

$$\max (a \alpha^{-1}) \leq \max (B \alpha^{-1})$$

which implies $d_a \leq d_b$. On the other hand, if $a \notin B$ then, since $a \in \text{ran } \alpha$ and $a < b$ by supposition, we must have $a < b \alpha$ and so $a < y$ for all $y \in B$. Hence, Lemma 2.1 implies $u < v$ for all $u \in a \alpha^{-1}$ and $v \in B \alpha^{-1}$, and so $\max (a \alpha^{-1}) < \max (B \alpha^{-1})$: that is, $d_a < d_b$.

(ii) Suppose $b < b \alpha$. Then $a < b < b \alpha$, so $a < y$ for all $y \in \text{ran } \alpha$ such that $b < y \leq b \alpha$. Hence, Lemma 2.1 implies

$$\max (a \alpha^{-1}) < \min (\{ y \in \text{ran } \alpha : b < y \leq b \alpha \})$$

and we have shown $d_a < d_b$.

CASE 3. $a \notin \text{ran } \alpha$ and $b \in \text{ran } \alpha$. In this case, $a \alpha < a$ or $a < a \alpha$. Since $a < b$, the first possibility leads to

$$\max (\{ y \in \text{ran } \alpha : a \alpha \leq y < a \}) < \max (b \alpha^{-1})$$

and so $d_a < d_b$. If the second possibility occurs then $a < b \leq a \alpha$ or $a \alpha < b$, and an argument similar to that in the first paragraph of Case 2 leads to

$$\min (\{ y \in \text{ran } \alpha : a < y \leq a \alpha \}) \leq \max (b \alpha^{-1})$$

and it follows that $d_a \leq d_b$. 

Case 4. $a \notin \text{ran } \alpha$ and $b \notin \text{ran } \alpha$. Then $a \alpha < a$ or $a < a \alpha$, and similarly for $b$. So, we put

$$A = \{ y \in \text{ran } \alpha : a \alpha \leq y < a \} \quad \text{and} \quad B = \{ y \in \text{ran } \alpha : b \alpha \leq y < b \}$$

and consider four possibilities.

(i) Suppose $a \alpha < a$ and $b \alpha < b$. Since $X$ is a chain, we have

$$a \alpha \leq y < b \quad \text{if and only if} \quad b \alpha \leq y < b \quad \text{or} \quad a \alpha \leq y < b \alpha.$$ 

Moreover, $\max B$ exists (as in the definition of $d_x$ in Case II) and this is greater than all $y \in \text{ran } \alpha$ such that $a \alpha \leq y < b \alpha$. Hence, $\max \{ y \in \text{ran } \alpha : a \alpha \leq y < b \}$ exists and it equals $\max B$. Then $a < b$ implies

$$\max A \leq \max \{ y \in \text{ran } \alpha : a \alpha \leq y < b \} = \max B$$

and so Lemma 2.14 implies $\max (A \alpha^{-1}) \leq \max (B \alpha^{-1})$, so $d_a \leq d_b$ as required.

(ii) Suppose $a \alpha < a$ and $b < b \alpha$. Then $a \alpha < a < b < b \alpha$ and so $u < u'$ for all $u \in A$ and $u' \in \{ y \in \text{ran } \alpha : b < y \leq b \alpha \}$. Hence, $v < v'$ for all $v \in A \alpha^{-1}$ and $v' \in \{ y \in \text{ran } \alpha : b < y \leq b \alpha \} \alpha^{-1}$, and it follows that

$$\max (A \alpha^{-1}) < \min \{ \{ y \in \text{ran } \alpha : b < y \leq b \alpha \} \alpha^{-1} \}.$$ 

Thus, $d_a < d_b$ for this possibility.

(iii) Suppose $a < a \alpha$ and $b \alpha < b$. Then $a < a \alpha \leq b \alpha < b$ and so $u \leq u'$ for all $u \in \{ y \in \text{ran } \alpha : a < y \leq a \alpha \}$ and $u' \in B$. Hence,

$$\max \{ y \in \text{ran } \alpha : a < y \leq a \alpha \} \leq \max B$$

and, using Lemma 2.14, we obtain

$$\min \{ \{ y \in \text{ran } \alpha : a < y \leq a \alpha \} \alpha^{-1} \} \leq \max (B \alpha^{-1}).$$

Thus, $d_a \leq d_b$ for this possibility.

(iv) Suppose $a < a \alpha$ and $b < b \alpha$. An argument dual to that in (i), which uses the dual of Lemma 2.14, shows that $d_a \leq d_b$ for this possibility.

Hence, $a < b$ implies $a \beta \leq b \beta$ in all possible cases, and the Theorem is completely proved.
3. ORDER-PRESERVING TRANSFORMATIONS OF NON-CHAINS

Very little (if any) research appears to have been done on semigroups of order-preserving transformations of arbitrary posets. In this section, we suppose $X$ is not a chain and determine when $OP(X)$ is regular, and then we state conditions under which $OT(X)$ is regular. We need two preliminary results, the first of which we are unable to find in the literature. If $X$ is a poset, we say $a \in X$ is isolated if for every $x \in X$, $x \leq a$ or $x \geq a$ implies $x = a$, and $X$ is isolated if all its elements are isolated.

**Lemma 3.1.** Suppose $X$ is a poset which is not a chain. If $X$ is not isolated then it contains a subposet with one of the following forms:

- $\Pi_1 = \{a, b, c : \{a, c\} \text{ and } \{b, c\} \text{ are isolated, and } a < b\}$
- $\Pi_2 = \{a, b, c : \{b, c\} \text{ is isolated, and } a < b, a < c\}$
- $\Pi_3 = \{a, b, c : \{b, c\} \text{ is isolated, and } b < a, c < a\}$

**Proof:** Since $X$ is not isolated, it contains a non-isolated element $a$, say. Then there exists $b \in X$ with $a < b$ or $b < a$ and, without loss of generality, we assume $a < b$. By Zorn's Lemma, there is a maximal chain $M$ in $X$ containing $a$ and $b$; and, since $X$ is not a chain, we can choose $c \in X \setminus M$. If $c$ is not comparable with any element of $M$, we obtain $\Pi_1$. On the other hand, suppose $d < c$ for some $d \in M$ and note that, in this case, if $y \in M$ and $y < d$ then $y < c$. Therefore, if for every $x \in M$, $d < x$ implies $c < x$ or $x < c$, we deduce that $M \cup \{c\}$ is a chain, contradicting the maximality of $M$. Hence, there exists $e \in M$ such that $d < e$ and $c \not< e$ and $e \not< c$, and we obtain $\Pi_2$. Finally, if $c < d$ for some $d \in M$, the dual of the above argument gives us $\Pi_3$. □

**Lemma 3.2.** Suppose $X$ is a poset which contains a subposet of the form $\Pi_1, \Pi_2$ or $\Pi_3$. If $S$ denotes one of the semigroups $OP(X),OI(X),OP(X,\mathbb{N}_0)$ or $OI(X,\mathbb{N}_0)$ then $S$ is not regular.

**Proof:** We consider three cases. □

**Case 1.** $X$ contains $\Pi_1$. Let $\alpha \in P(X)$ satisfy: $\text{dom} \alpha = \{a, c\}$ and $\alpha a = b, \alpha c = a$. Then, $\alpha \in S$ since $\{a, c\}$ is isolated. Suppose $\alpha = \alpha \beta \alpha$ for some $\beta \in S$. Then $b = \alpha \alpha = (b\beta)\alpha$ and $a = \alpha c = (a\beta)\alpha$ which implies $b\beta = a$ and $a\beta = c$. But, since $a < b$ and $c \not< a$, this means $\beta$ is not order-preserving, a contradiction. Hence, $\alpha$ is not a regular element of $S$.

**Case 2.** $X$ contains $\Pi_2$. Let $\alpha \in P(X)$ satisfy: $\text{dom} \alpha = \{b, c\}$ and $\beta a = b, \beta c = a$. Then, $\alpha \in S$ since $\{b, c\}$ is isolated. Suppose $\alpha = \alpha \beta \alpha$ for some $\beta \in S$. Then $b = \beta \alpha = (b\beta)\alpha$ and $a = \alpha c = (a\beta)\alpha$ which implies $b\beta = b$ and $a\beta = c$. But, since $a < b$ and $c \not< b$, this means $\beta$ is not order-preserving, a contradiction. Hence, $\alpha$ is not a regular element of $S$.  

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CASE 3. $X$ contains $\Pi_3$. The result follows in this case by recalling the notation at the start of Section 2 and applying Case 2 to $S(X, \leq_{\text{opp}})$.

The next result now follows easily from the above Lemmas.

**Theorem 3.3.** Let $X$ be a poset which is not a chain and suppose $S$ is one of the semigroups $OP(X), OI(X), OP(X, \mathfrak{R}_0)$ or $OI(X, \mathfrak{R}_0)$. Then $S$ is regular if and only if $X$ is isolated.

In [1, pp.27-33], Changphas provides various conditions under which $OT(X)$ is or is not regular when $X$ is not a chain. We summarise some of that work in the following three results without proof.

**Theorem 3.4.** Suppose $X$ is a poset. Then $OT(X)$ is not regular if $X$ contains a subposet of the form

\[ \{a, b, c, d : \{a, b\} \text{ is isolated, and } d < c < a \text{ and } d < c < b\} \]

or

\[ \{a, b, c, d : \{a, b\} \text{ and } \{b, c\} \text{ are isolated, and } d < c < a \text{ and } d < b\}. \]

**Theorem 3.5.** Suppose $X$ is a poset and let $m(X) [M(X)]$ denote the set of all minimal [maximal] elements of $X$. Then $OT(X)$ is regular if $X = m(X) \cup M(X)$ and $x < y$ for all $x \in m(X)$ and $y \in M(X)$.

**Theorem 3.6.** Suppose $X$ is a poset with a minimum element 0 and a maximum element 1. Then $OT(X)$ is regular if $\{x, y\}$ is isolated for all distinct $x, y \in X \setminus \{0, 1\}$.

**References**


Department of Mathematics  Department of Mathematics
Chulalongkorn University  Khon Kaen University
Bangkok 10330  Khon Kaen 40002
Thailand  Thailand