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C^*-Crossed-Products by an Order-Two Automorphism

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Abstract. We describe the representation theory of C^*-crossed-products of a unital C^*-algebra A by the cyclic group of order 2. We prove that there are two main types of irreducible representations for the crossed-product: those whose restriction to A is irreducible and those who are the sum of two unitarily unequivalent representations of A. We characterize each class in term of the restriction of the representations to the fixed point C^*-subalgebra of A. We apply our results to compute the K-theory of several crossed-products of the free group on two generators.

1 Introduction

This paper explores the structure of the representation theory of C^*-crossed-products [5] of unital C^*-algebras by order-two automorphisms. We show that irreducible representations of the C^*-crossed-products $A \rtimes \mathbb{Z}_2$ of a unital C^*-algebra A by $\mathbb{Z}_2$ fall into two categories: either their restriction to A is already irreducible, or it is the direct sum of two irreducible representations of A that are related by the automorphism and not unitarily equivalent to each other.

Given a unital C^*-algebra A and an order-two automorphism σ of A, the C^*-crossed-product $A \rtimes_\sigma \mathbb{Z}_2$ is the C^*-algebra generated by A and a unitary W with $W^2 = 1$ satisfying the following universal property: if $\psi\colon A \to B$ is a unital $*$-morphism for some unital C^*-algebra B such that B contains a unitary u satisfying $u^2 = 1$ and $u\psi(a)u^* = \psi\circ\sigma(a)$ for all $a \in A$, then ψ extends uniquely to $A\rtimes_\sigma\mathbb{Z}_2$ with $\psi(W) = u$. The general construction of $A \rtimes_\sigma \mathbb{Z}_2$ can be found in [5]. In particular, $W = W^*$ since W is unitary, the spectrum of W is $\{-1, 1\}$ and $WaW^* = \sigma(a)$ for all $a \in A$. We call the unitary W the canonical unitary of $A \rtimes_\sigma \mathbb{Z}_2$. Proposition 2.2 will offer an alternative description of $A \rtimes_\sigma \mathbb{Z}_2$.

The questioqn raised in this paper is: what is the connection between the representation theory of $A \rtimes_\sigma \mathbb{Z}_2$ and the representation theory of A? Of central importance is the fixed point C^*-algebra A_1 for σ defined by $A_1 = \{a \in A : \sigma(a) = a\}$ and the natural decomposition $A = A_1 + A_{-1}$ where $A_{-1} = \{a \in A : \sigma(a) = -a\}$, with $A_1 \cap A_{-1} = \{0\}$. We obtain a complete description of the irreducible representations of $A \rtimes_\sigma \mathbb{Z}_2$ from the representation theory of A and A_1.

Note that, if we considered the crossed-product $A \rtimes_\sigma \mathbb{Z}$ instead of $A \rtimes_\sigma \mathbb{Z}_2$, then our work applies as well, thanks to a simple observation made at the end of Section 2.

The rest of the paper focuses on applications to examples. We are interested in several natural order-two automorphisms of the full C^*-algebra of free group $\mathbb{F}_2$,

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namely the universal C^*-algebra generated by two unitaries U and V. We define the automorphism α by $\alpha(U) = U^*$ and $\alpha(V) = V^*$, while β is the automorphism defined by $\beta(U) = -U$ and $\beta(V) = -V$. We compute the K-theory of the C^*-crossed-products for these two automorphisms, relying in part on our structure theory for their representations. A third natural automorphism, γ, is defined uniquely by $\gamma(U) = V$ and $\gamma(V) = U$. It is the subject of the companion paper [1], which emphasizes the interesting structure of the associated fixed point C^*-algebra and uses different techniques from the representation approach of this paper.

2 Representation Theory of Crossed-Products

In this section, we derive several general results on the irreducible representations of the crossed-product C^*-algebra $A \rtimes_\sigma \mathbb{Z}_2$ where σ is an order-2 automorphism of the unital C^*-algebra A. We recall that $A \rtimes_\sigma \mathbb{Z}_2$ is the universal C^*-algebra generated by A and a unitary W such that $W^2 = 1$ and $WaW^* = \sigma(a)$.

2.1 Representations from the Algebra

A central feature of the crossed-products by finite groups is their connection with the associated fixed point C^*-algebra [4]. In our case, the following easy lemma will prove useful:

Lemma 2.1 *Let A be a unital C^*-algebra and σ an order-2 automorphism of A. The set $A_1 = \{a + \sigma(a) : a \in A\}$ is the fixed point C^*-algebra of A for σ and the set $A_{-1} = \{a - \sigma(a) : a \in A\}$ is the space of elements $b \in A$ such that $\sigma(b) = -b$. Then $A = A_1 + A_{-1}$ and $A_1 \cap A_{-1} = \{0\}$.*

Proof If a is any element in A then $a+\sigma(a)$ (resp. $a-\sigma(a)$) is a fixed point for σ (resp. an element $b \in A$ such that $\sigma(b) = -b$). Conversely, let $x \in A$. Then $x = \frac{1}{2}\bigl(x + \sigma(x)\bigr) + \frac{1}{2}\bigl(x - \sigma(x)\bigr)$. If x is σ-invariant then $x - \sigma(x) = 0$ so $x = \frac{1}{2}\bigl(x+\sigma(x)\bigr)$ indeed, and thus the fixed point C^*-algebra is A_1 (similarly $\{b \in A : \sigma(b) = -b\} = A_{-1}$). Of course, if $a \in A_1 \cap A_{-1}$, then $\sigma(a) = a = -a$ so $a = 0$. ∎

We exhibit a simple algebraic description of the crossed-product.

Proposition 2.2 *Let σ be an order 2-automorphism of a unital C^*-algebra A. Then the C^*-crossed-product $A \rtimes_\sigma \mathbb{Z}_2$ is $*$-isomorphic to*

$$\left\{ \begin{bmatrix} a & b \\ \sigma(b) & \sigma(a) \end{bmatrix} : a, b \in A \right\} \subseteq M_2(A)$$

via the following isomorphism: $a \in A \mapsto \left[\begin{smallmatrix} a & 0 \\ 0 & \sigma(a) \end{smallmatrix}\right]$ and $W \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, where W is the canonical unitary of $A \rtimes_\sigma \mathbb{Z}_2$.

Proof Let $\psi\colon a \in A \mapsto \left[\begin{smallmatrix} a & 0 \\ 0 & \sigma(a) \end{smallmatrix}\right] \in M_2(A)$ and set $\psi(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in M_2(A)$. Since $\psi(W)\psi(a)\psi(W) = \psi\bigl(\sigma(a)\bigr)$ we deduce by universality that ψ extends to a (unique) $*$-automorphism of $A \rtimes_\sigma \mathbb{Z}_2$ valued in $M_2(A)$. Now, let $c \in A \rtimes_\sigma \mathbb{Z}_2$. By construction of $A \rtimes_\sigma \mathbb{Z}_2$, there exists a sequence $(a_n + b_n W)_{n\in\mathbb{N}}$ with $a_n, b_n \in A$ such

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that $c = \lim_{n\to\infty} a_n + b_n W$ in $A \rtimes_\sigma \mathbb{Z}_2$. Now, $\psi(a_n + b_n W) = \left[\begin{smallmatrix} a_n & b_n \\ \sigma(b_n) & \sigma(a_n)\end{smallmatrix}\right]$ for all $n \in \mathbb{N}$, and converges to $\psi(c) = \left[\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{smallmatrix}\right]$ when $n \to \infty$. In particular, $(a_n)_{n\in\mathbb{N}}$ converges to $c_{11} \in A$ and $(b_n)_{n\in\mathbb{N}}$ converges to $c_{12} \in A$. Consequently, $c = c_{11} + c_{12}W$. Hence $A + AW$ is a closed dense $*$-subalgebra of $A \rtimes_\sigma \mathbb{Z}_2$ and thus $A \rtimes_\sigma \mathbb{Z}_2 = A + AW$.

Moreover, if $\psi(c) = 0$ then, writing $c = a + bW$, by definition of ψ, we get $\psi(c) = \left[\begin{smallmatrix} a & b \\ \sigma(b) & \sigma(a)\end{smallmatrix}\right] = 0$, so $a = b = 0$, hence $c = 0$. Thus ψ is a $*$-isomorphism from $A \rtimes_\sigma \mathbb{Z}_2$ onto the C^*-algebra $\left\{ \left[\begin{smallmatrix} a & b \\ \sigma(b) & \sigma(a)\end{smallmatrix}\right] : a, b \in A \right\} \subseteq M_2(A)$. ■

In other words, the abstract canonical unitary W of $A \rtimes_\sigma \mathbb{Z}_2$ can be replaced by the concrete unitary $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]$ and $A \rtimes_\sigma \mathbb{Z}_2$ can be seen as the C^*-algebra $\psi(A) + \psi(A)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]$ in $M_2(A)$ with $\psi\colon a \in A \mapsto \left[\begin{smallmatrix} a & 0 \\ 0 & \sigma(a)\end{smallmatrix}\right]$. Equivalently, the $*$-subalgebra $A + AW$ in $A \rtimes_\sigma \mathbb{Z}_2$ is in fact equal to $A \rtimes_\sigma \mathbb{Z}_2$.

From the algebraic description of Proposition 2.2 we get a family of representations of the crossed-product described in the following proposition. These representations are in fact induced representations from the sub-C^*-algebra A to the C^*-algebra $A \rtimes_\sigma \mathbb{Z}_2$ in the sense of [3].

Proposition 2.3 *Let A be a unital C^*-algebra and σ be an order two automorphism of A. Let W be the canonical unitary of the crossed-product $A \rtimes_\sigma \mathbb{Z}_2$ such that $WaW = \sigma(a)$ for all $a \in A$. Then for each representation π of A on some Hilbert space $\mathcal{H}$ there exists a representation $\widetilde{\pi}$ of $A \rtimes_\sigma \mathbb{Z}_2$ on $\mathcal{H} \oplus \mathcal{H}$ defined by $\widetilde{\pi}(a) = \pi(a) \oplus \pi \circ \sigma(a)$ for all $a \in A$ and $\widetilde{\pi}(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]$.*

Moreover, the following are equivalent:

- *the representation $\widetilde{\pi}$ is irreducible;*
- *the representation π is irreducible and not unitarily equivalent to $\pi \circ \sigma$;*
- *there does not exist a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $U\pi U^* = \pi \circ \sigma$ and $U^2 = 1$.*

If π is a faithful representation of A then $\widetilde{\pi}$ is faithful for $A \rtimes_\sigma \mathbb{Z}_2$. In particular, if A has a faithful representation that is a direct sum of finite representations, so does $A \rtimes_\sigma \mathbb{Z}_2$.

Proof Let π be a given representation of A. Then by setting $\widetilde{\pi}(a) = \pi(a) \oplus \pi\big(\sigma(a)\big)$ and $\widetilde{\pi}(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]$, we define a $*$-representation of $A \rtimes_\sigma \mathbb{Z}_2$ by universality of $A \rtimes_\sigma \mathbb{Z}_2$. In fact, $\widetilde{\pi} = \left[\begin{smallmatrix} \pi & 0 \\ 0 & \pi\end{smallmatrix}\right] \circ \psi$, where ψ is the isomorphism of Proposition 2.2.

Let us now assume that π is irreducible and not unitarily equivalent to $\pi \circ \sigma$. Assume V is an operator commuting with $\widetilde{\pi}$. Then since V commutes with $\widetilde{\pi}(W)$, we have $V = \left[\begin{smallmatrix} a & b \\ b & a\end{smallmatrix}\right]$ for some $a, b \in \mathcal{B}(\mathcal{H})$. Now, since V commutes with $\pi \oplus (\pi \circ \sigma)$ we conclude that a commutes with π and, as π is irreducible, this implies that $V = \left[\begin{smallmatrix} \lambda 1 & b \\ b & \lambda 1\end{smallmatrix}\right]$ for some $\lambda \in \mathbb{C}$ and where 1 is the identity on $\mathcal{H}$. Hence, V commutes with $\widetilde{\pi}$ if and only if $\left[\begin{smallmatrix} 0 & b \\ b & 0\end{smallmatrix}\right]$ does. Now, if $\left[\begin{smallmatrix} 0 & b \\ b & 0\end{smallmatrix}\right]$ commutes with $\widetilde{\pi}(A \rtimes_\sigma \mathbb{Z}_2)$ then so does its square $\left[\begin{smallmatrix} b^2 & 0 \\ 0 & b^2\end{smallmatrix}\right]$. Hence again $b^2 = \mu 1$ by irreducibility of π, and up to replacing b by $\frac{1}{2}(b + b^*)$ we can assume that b is self-adjoint and thus $\mu \geq 0$.

Assume that $\mu \neq 0$. Set $u = (\sqrt{\mu})^{-1} b$: then $u = u^*$ and $u^2 = 1$, so u is unitary. Moreover, as $\frac{1}{\sqrt{\mu}} V = \left[\begin{smallmatrix} 0 & u \\ u & 0\end{smallmatrix}\right]$ commutes with $\pi(c) \oplus \big(\pi \circ \sigma(c)\big)$ for all $c \in A$, we check that $u\pi\big(\sigma(c)\big) = \pi(c)u$, so $u^*\pi(c)u = \pi\big(\sigma(c)\big)$ for all $c \in A$. Hence, we have reached a contradiction, as we assumed that π is not unitarily equivalent to $\pi \circ \sigma$. Therefore $\mu = 0$ and thus $V = \lambda(1 \oplus 1)$, so $\widetilde{\pi}$ is irreducible.

Conversely, if there exists a unitary u such that $u^2 = 1$ and $u\pi u^* = \pi \circ \sigma$, then the operator $V = \left[\begin{smallmatrix} 0 & u \\ u & 0 \end{smallmatrix}\right]$ commutes with $\widetilde{\pi}$, so $\widetilde{\pi}$ is not irreducible.

On the other hand, if π is reducible, then let p be a nontrivial projection of $\mathcal{H}$ such that $p\pi = \pi p$. Then $p \oplus p$ is a nontrivial projection commuting with $\widetilde{\pi}$, as can easily be checked (it is obvious on $\widetilde{\pi}(A)$ and easy for $\widetilde{\pi}(W)$). Hence $\widetilde{\pi}$ is reducible as well. This proves the first two equivalences.

Now, we observe that π is unitarily equivalent to $\pi \circ \sigma$ if and only if there exists a unitary u with $u^2 = 1$ such that $u\pi u^* = \pi \circ \sigma$. One implication is trivial; let us check the easy other one. Let v be unitary such that $v\pi v^* = \pi \circ \sigma$. Then $v^2\pi v^{*2} = \pi \circ \sigma^2 = \pi$. Hence, as π is irreducible, $v^2 = \exp(2i\pi\theta)1$ for some $\theta \in [0, 1)$. Hence, $u = \exp(-i\pi\theta)v$ satisfies both $u^2 = 1$ and $\pi \circ \sigma = u\pi u^*$. ■

Proposition 2.3 describes a family of representations and gives us a criterion for their irreducibility. Conversely, given an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_2$, what can be said about its structure relative to the representation theory of A and its fixed point algebra A_1? This is the subject of the next section, which establishes a sort of converse for Proposition 2.3.

2.2 Irreducible Representations

The following theorem is the main result of this paper and shows that any irreducible representation of $A \rtimes_\sigma \mathbb{Z}_2$ is built from either a single unitary representation of A (and is then just an extension of it) or from two non-equivalent irreducible representations of A.

Theorem 2.4 *Let σ be an order-two-automorphism of a unital C^*-algebra A. We denote by W the canonical unitary of the C^*-crossed-product $A \rtimes_\sigma \mathbb{Z}_2$ such that $WaW = \sigma(a)$ for all $a \in A$.*

Let π be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_2$ on a Hilbert space $\mathcal{H}$. Let π' be the restriction of π to A and π'' be the restriction of π to the fixed point C^*-algebra A_1. Then one and only one of the following two alternatives hold:

(i) *The operator $\pi(W)$ is either the identity* Id *or* $-$Id *and $\pi(A \rtimes_\sigma \mathbb{Z}_2) = \pi'(A) = \pi''(A_1)$.*

(ii) *The spectrum of $\pi(W)$ is $\{-1, 1\}$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_{-1}$ where $\mathcal{H}_\varepsilon$ is the spectral Hilbert space of $\pi(W)$ for the eigenvalue ε. With this decomposition of $\mathcal{H}$, we have $\pi(W) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Let us write $\pi'(a) = \left[\begin{smallmatrix} \alpha(a) & \beta(a) \\ \gamma(a) & \delta(a) \end{smallmatrix}\right]$ for $a \in A$. Then α, δ restrict to irreducible representations of A_1, and $\alpha(A_{-1}) = \delta(A_{-1}) = \{0\}$. Moreover, $\beta(A_1) = \gamma(A_1) = \{0\}$.*

Furthermore, the representation π' is irreducible if and only if α and δ are not unitarily equivalent.

Proof Let π be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_2$ on $\mathcal{H}$. Let $w = \pi(W)$. Since w is unitary and $w^2 = 1$, the spectrum of w is either $\{-1, 1\}$ or $w = 1$ or $w = -1$. In the latter two cases, w commutes with $\pi(A \rtimes_\sigma \mathbb{Z}_2)$. Since $A \rtimes_\sigma \mathbb{Z}_2 = A + AW$ from

Proposition 2.2, we have $\pi(A \rtimes_\sigma \mathbb{Z}_2) = \pi'(A) + \pi'(A)w = \pi'(A)$ (as $w = \pm 1$). Thus as π is irreducible, so is π'. Moreover, since $w\pi'(a)w = \pi'(a) = \pi' \circ \sigma(a)$, we see that π' is null on A_{-1} and thus $\pi' = \pi''$. Conversely if $\pi(A_{-1}) = 0$ then w must commute with $\pi(A) = \pi(A_1)$ and thus with $\pi(A \rtimes_\sigma \mathbb{Z}_2) = \pi(A) + \pi(A)w$. Therefore, as π is irreducible, w is scalar, and as w unitary and $w^2 = 1$ we conclude w is 1 or -1.

Assume now that the unitary w has spectrum $\{-1, 1\}$. Write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_{-1}$ accordingly. In this decomposition, we have

$$w = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \pi(a) = \begin{bmatrix} \alpha(a) & \beta(a) \\ \gamma(a) & \delta(a) \end{bmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are linear maps on A. Thus,

$$w\pi(a)w^* = \begin{bmatrix} \alpha(a) & -\beta(a) \\ -\gamma(a) & \delta(a) \end{bmatrix}.$$

In particular, if $a \in A_{-1}$, then $\pi \circ \sigma(a) = -\pi(a)$ so $\alpha(a) = -\alpha(a) = 0$. Since $A = A_1 \oplus A_{-1}$ as a vector space, we conclude that $\alpha(a) \in \alpha(A_1)$ for all $a \in A$. Similarly $\delta(a) \in \delta(A_1)$, $\beta(a) \in \beta(A_{-1})$ and $\gamma(a) \in \gamma(A_{-1})$ for all $a \in A$ while $\gamma(A_1) = \beta(A_1) = \{0\}$.

Consequently, $\pi'' = \alpha \oplus \beta$ and α, β are representations of A_1 (but not of A).

We observe that $A \rtimes_\sigma \mathbb{Z}_2 = A + AW$ by Proposition 2.2, so

$$\pi(A \rtimes_\sigma \mathbb{Z}_2) = \left\{ \begin{bmatrix} \alpha(a_1) + \alpha(a_2) & \beta(a_1) - \beta(a_2) \\ \gamma(a_1) + \gamma(a_2) & \delta(a_1) - \delta(a_2) \end{bmatrix} : a_1, a_2 \in A \right\}.$$

(Note that w is given in this form by $a_1 = 1$ and $a_2 = 0$, since $1 \in A_1$ so $\beta(a_1) = \gamma(a_1) = 0$.) Now, $\alpha(a) \in \alpha(A_1)$ for all $a \in A$, so $\{\alpha(a_1) + \alpha(a_2) : a_1, a_2 \in A\}$ is the set $\alpha(A_1)$. Furthermore, since π is irreducible, we have $\pi(A \rtimes_\sigma \mathbb{Z}_2)'' = \mathcal{B}(\mathcal{H})$, *i.e.*, the range of π is SOT-dense, and in particular $\alpha(A_1)$ is SOT-dense in $\mathcal{B}(\mathcal{H}_1)$, so α is an irreducible representation of A_1 on $\mathcal{H}_1$. The same applies to δ.

We now distinguish according to the two following cases: either α and δ are unitarily equivalent as representations of A_1 or they are not.

Assume that α and δ are not unitarily equivalent. Let us assume P is a projection which commutes with π'. Then in particular, P commutes with π''. Writing $P = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{smallmatrix}\right]$, this gives the relations

$$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \alpha p_{11} & \alpha p_{12} \\ \delta p_{21} & \delta p_{22} \end{bmatrix},$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} = \begin{bmatrix} p_{11}\alpha & p_{12}\delta \\ p_{21}\alpha & p_{22}\delta \end{bmatrix}.$$

Hence, since both α and δ are irreducible, we deduce that p_{11}, p_{22} are scalar. Now, as P is a projection, $p_{11} = p_{11}^*$ and $\|p_{11}\| \leq 1$ so $p_{11} \in [-1, 1]$. Again since $P = P^* = P^2$, we have $p_{12} = p_{21}^*$ and $p_{12}p_{12}^* + p_{11}^2 = p_{11} \in \mathbb{R}$. Assume $\lambda = p_{11}(1 - p_{11}) \neq 0$. Since $p_{11} \in [-1, 1]$, we have $\lambda \in [0, 1]$. Then $\nu = \frac{1}{\sqrt{\lambda}} p_{12}$ is a unitary operator and

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since $\alpha p_{12} = p_{12}\delta$, we obtain $\nu\alpha\nu^* = \delta$. This contradicts our assumption that α and δ are not unitarily equivalent. Hence $\lambda = 0$ and so $p_{11} = 1$ or 0 and $p_{12} = 0$ (since $p_{12}p_{12}^* = 0$). Now, again since P is a projection, $p_{22}^2 + p_{12}p_{12}^* = p_{22}$ yet $p_{12} = 0$ and p_{22} is a scalar so $p_{22} = 0$ or 1 as well. Thus, in the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_{-1}$ the projection P is either $0 \oplus 0$, $1 \oplus 0$, $0 \oplus 1$ or $1 \oplus 1$.

Now, the first part of this proof established that $\pi(W)$ must be scalar if π is irreducible and $\pi(A) = \pi(A_1)$. Since we assume that $\pi(W)$ is not scalar, we conclude that $\pi(A) \neq \pi(A_1)$. Consequently, there exists $a_0 \in A \backslash A_1$ such that $\pi(a_0)$ is not diagonal in the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_{-1}$. Thus $\pi(a_0)$ does not commute with $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. So P is scalar, and thus π' is irreducible. Note that $\pi(W)\pi'(a)\pi(W) = \pi' \circ \sigma(a)$ for all $a \in A$, so π' is unitarily equivalent to $\pi' \circ \sigma$.

Conversely, assume α and δ are unitarily equivalent. Assume moreover that π' is irreducible. Then $\pi'(A)$ is WOT-dense in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_{-1})$. In particular, $\pi(W) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is the limit of a WOT-converging net $\left(\pi'(a_\lambda)\right)_{\lambda \in \Lambda}$ in $\pi'(A)$. Since we assume that there exists a unitary $u \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ such that for all $a \in A$ we have $\alpha(a) = u\delta(a)u^*$, and since the map $c \in \mathcal{B}(\mathcal{H}_1) \mapsto ucu^*$ is WOT-continuous, we get the contradiction $1 = \lim_{\lambda \in \Lambda} \alpha(a_\lambda) = u\big(\lim_{\lambda \in \Lambda} \delta(a_\lambda)\big) u^* = u(-1)u^* = -1$. Hence, if α and δ are unitarily equivalent then π' is reducible. ■

We can extend Theorem 2.4 with the following description of some irreducible representations of $A \rtimes_\sigma \mathbb{Z}_2$ which completes the statement of Proposition 2.2.

Proposition 2.5 *Let σ be an order-two automorphism of A and let $A_1 = \{a \in A : \sigma(a) = a\}$ be the associated fixed point C^*-algebra. Let π be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_2$ on a Hilbert space and π'' its restriction A_1. Then the following statements are equivalent:*

- *π'' is the direct sum of exactly two unitarily equivalent representations, where each is an irreducible representation on A_1.*
- *π is unitarily equivalent to a representation ρ such that $\rho(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\rho(a) = \left[\begin{smallmatrix} \rho'(a) & 0 \\ 0 & \rho' \circ \sigma(a) \end{smallmatrix}\right]$, where ρ' is an irreducible representation of A and W is the canonical unitary in $A \rtimes_\sigma \mathbb{Z}_2$ and ρ' is not unitarily equivalent to $\rho' \circ \sigma$.*

The proof of this proposition relies upon the following lemma.

Lemma 2.6 *Let $\mathcal{H}$ be a Hilbert space. Let A, B be two bounded linear operators on $\mathcal{H}$ such that $BTA = ATB$ for all bounded linear operators T on $\mathcal{H}$. Then A and B are linearly dependent.*

Proof The result is obvious if $A = 0$ or $B = 0$, so we assume henceforth that $A \neq 0$ and $B \neq 0$. Let $\gamma \in \mathcal{H}$ such that $A\gamma \neq 0$. Assume that there exists $x_0 \in \mathcal{H}$ such that $\{Ax_0, Bx_0\}$ is linearly independent. Then let T be any bounded linear operator such that $T(Ax_0) = 0$ and $T(Bx_0) = \gamma$. Such a T is well defined by the Hahn–Banach theorem. But then $0 = BTAx_0 = ATBx_0 = A\gamma$ which is a contradiction. Hence for all $x \in \mathcal{H}$ there exists $\lambda_x \in \mathbb{C}$ such that $Bx = \lambda_x Ax$.

Now, let $y \in \mathcal{H}$. Let T be any bounded operator on $\mathcal{H}$ such that $TA\gamma = y$. Then we compute $By = BTA\gamma = ATB\gamma = AT(\lambda_\gamma A\gamma) = \lambda_\gamma Ay$. Hence $B = \lambda_\gamma A$. This concludes our theorem. ■

Note that we can prove the following similarly.

Lemma 2.7 *Let A, B be two bounded operators on a Hilbert space $\mathcal{H}$ and assume that for all bounded operators T of $\mathcal{H}$ we have $ATA^* = BTB^*$. Then there exists $\theta \in [0,1)$ such that $B = \exp(2i\pi\theta)A$.*

We can then prove Proposition 2.5.

Proof of Proposition 2.5 We use the same notations as in Theorem 2.4 and its proof. We can now work out in greater detail the decomposition of π' when α and δ are unitarily equivalent, *i.e.*, when there exists a unitary $u \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ such that $\alpha = u\delta u^*$. By conjugating π with $u' = \left[\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right]$, we obtain $u'\pi(a)u'^* = \left[\begin{smallmatrix} \alpha(a) & \beta(a)u^* \\ u\gamma(a) & \alpha(a) \end{smallmatrix}\right]$. To ease notation, we set $\beta' \colon a \in A \mapsto \beta(a)u^*$ and $\gamma' \colon a \in A \mapsto u\gamma(a)$. We also denote $\mathcal{H}_1$ by $\mathcal{J}$ and (up to a trivial isomorphism) we write $\mathcal{H} = \mathcal{J} \oplus \mathcal{J}$. Now α, β', and γ' are all linear maps on $\mathcal{J}$. The representation $u'\pi u'^*$ is denoted by θ.

Let $b \in A_1$ and $a \in A_{-1}$. Then $(ba)^2 \in A_1$ and

$$\begin{aligned}
\begin{bmatrix} \alpha\big((ba)^2\big) & 0 \\ 0 & \alpha\big((ba)^2\big) \end{bmatrix} &= \theta\big((ba)^2\big) = \big(\theta(b)\theta(a)\big)^2 \\
&= \left(\begin{bmatrix} \alpha(b) & 0 \\ 0 & \alpha(b) \end{bmatrix} \begin{bmatrix} 0 & \beta'(a) \\ \gamma'(a) & 0 \end{bmatrix} \right)^2 \\
&= \begin{bmatrix} \alpha(b)\beta'(a)\alpha(b)\gamma'(a) & 0 \\ 0 & \alpha(b)\gamma'(a)\alpha(b)\beta'(a) \end{bmatrix},
\end{aligned}$$

and thus for all $a \in A_{-1}$ and $b \in A_1$ we have

$$\alpha(b)\beta'(a)\alpha(b)\gamma'(a) = \alpha(b)\gamma'(a)\alpha(b)\beta'(a).$$

Now, since $\alpha(A_1)$ is SOT-dense in $\mathcal{B}(\mathcal{J})$, we conclude that for all $T \in \mathcal{B}(\mathcal{J})$ we have for all $a \in A_{-1}$

$$T\beta'(a)T\gamma'(a) = T\gamma'(a)T\beta'(a)$$

and thus we have $\beta'(a)T\gamma'(a) = \gamma'(a)T\beta'(a)$ for all $T \in \mathcal{B}(\mathcal{J})$ and $a \in A_{-1}$. By Lemma 2.6, for each $a \in A_{-1}$ there exists $\lambda(a) \in \mathbb{C}$ such that $\lambda(a)\beta'(a) = \gamma'(a)$. On the other hand, let $a, b \in A_{-1}$ be given. Then

$$\begin{aligned}
\lambda(a)\beta'(a) + \lambda(b)\beta'(b) &= \gamma'(a) + \gamma'(b) = \lambda(a+b)\beta'(a+b) \\
&= \lambda(a+b)\beta'(a) + \lambda(a+b)\beta'(b).
\end{aligned}$$

If $\beta'(a)$ and $\beta'(b)$ are linearly independent then $\lambda(a) = \lambda(b) = \lambda(a+b)$ (thus λ is constant if $\beta'(A_{-1})$ is at least two dimensional).

If instead, $\beta'(a) = t\beta'(b)$ for some $t \in \mathbb{C}$, then we get

$$\lambda(ta)\beta'(ta) = \gamma'(ta) = t\gamma'(a) = t\lambda(a)\beta'(a).$$

Hence, if $t \neq 0$ and $\beta'(a) \neq 0$, then $\lambda(ta) = \lambda(a)$.

Thus, if $a, b \in A_{-1}$ and a, b are not in $\ker \beta'$, then $\lambda(a) = \lambda(b)$ (as $\{a, b\}$ is either linearly independent or they are dependant but $\beta'(a)$ and $\beta'(b)$ are not zero). We can make the choice we wish for $\lambda(a)$ when $a \in \ker \beta'$, so naturally we set $\lambda(a) = \lambda(b)$ for any $b \in A_{-1} \backslash \ker \beta'$ (note that $A_{-1} \backslash \ker \beta' \neq \varnothing$ since θ is irreducible and since $\beta'(a) = \gamma'(a^*)^*$ for all $a \in A$). With this choice, we have shown that there exists a $\lambda \in \mathbb{C}$ such that $\lambda\beta'(a) = \gamma'(a)$ for all $a \in A_{-1}$.

Moreover, let $a \in A_{-1}$. Then $\beta'(a^*) = \gamma'(a)^*$ and $\beta'(a)^* = \gamma'(a^*)$ by definition of β' and γ', yet $\gamma'(a) = \lambda\beta'(a)$. So if $a = a^*$, then

$$\begin{aligned}\gamma'(a) &= \lambda\beta'(a) = \lambda\beta'(a^*) = \lambda\gamma'(a)^* = \lambda\big(\lambda\beta'(a)\big)^* \\ &= |\lambda|^2\beta'(a)^* = |\lambda|^2\gamma'(a).\end{aligned}$$

Now, suppose that $\gamma'(a) = 0$ for all $a = a^* \in A_{-1}$. By assumption, γ' is not zero (since then β' would be since $\beta'(a) = \gamma'(a^*)^*$ and then θ would be reducible), so there exists $a \in A_{-1}$ such that $a^* = -a$ and $\gamma'(a) \neq 0$ (since γ' linear and every element in A_{-1} is of the sum of a self-adjoint and anti-selfadjoint element in A_{-1}). But then ia is self-adjoint, and since γ' is linear, $\gamma'(ia) = 0$. This is a contradiction. Hence there exists $a \in A_{-1}$ such that $a = a^*$ and $\gamma'(a) \neq 0$. Therefore, $|\lambda|^2 = 1$. Let η be any square root of λ in $\mathbb{C}$.

Set $\nu = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \eta \end{smallmatrix}\right]$ and $\psi = \nu\theta\nu^*$ so that

$$\psi(a) = \begin{bmatrix} \alpha(a) & \eta\beta'(a) \\ \eta\beta'(a) & \alpha(a) \end{bmatrix}.$$

Let $\nu' = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ so that

$$\nu'\psi(a)\nu'^* = \begin{bmatrix} \alpha(a) + \eta\beta'(a) & 0 \\ 0 & \alpha(a) - \eta\beta'(a) \end{bmatrix}.$$

Letting $\varphi = \alpha + \eta\beta'$ we see that φ is a $*$-representation of A and that π is unitarily equivalent to the representation π_φ defined by $\pi_\varphi(a) = \left[\begin{smallmatrix} \varphi(a) & 0 \\ 0 & \varphi(\sigma(a)) \end{smallmatrix}\right]$ and $\pi_\varphi(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. In particular, $\pi' = \varphi \oplus \varphi \circ \sigma$ is a reducible representation of A.

Note that we could have done the same proof by limiting ourselves to the case where $a \in A_{-1}$ is selfadjoint and by calculating $\pi(a)^*\pi(a)$, using Lemma 2.7 instead of Lemma 2.6. ∎

We easily observe that both types of representations described in Proposition 2.3 and Theorem 2.4 do actually occur.

Example 2.8 Let $A = M_2$ and $\sigma\colon a \in M_2 \mapsto WaW$, where $W = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. All irreducible representations of $M_2 \rtimes_\sigma \mathbb{Z}_2$ are unitarily equivalent to the identity representation of M_2.

Example 2.9 Let $A = C(\mathbb{T})$ and $\sigma\colon f \mapsto f \circ \sigma^*$, where $\sigma^*\colon \omega \in \mathbb{T} \mapsto -\omega$. Then all irreducible representations of $C(\mathbb{T}) \rtimes_\sigma \mathbb{Z}_2$ are given by the construction of Proposition 2.3. Indeed, if π' is the restriction of an irreducible representation π of

$C(\mathbb{T}) \rtimes_\sigma \mathbb{Z}_2$, then π' is irreducible if and only if π' is one-dimensional. In this case, π is one-dimensional and thus corresponds to a fixed point in $\mathbb{T}$ for σ. Since there is no such fixed point, π' is reducible and the direct sum of the evaluations at ω and $-\omega$ for some $\omega \in \mathbb{T}$.

Example 2.10 Both types of representations occur if we replace σ^* in Example 2.9 by $\sigma^{**}: \omega \in \mathbb{T} \mapsto \overline{\omega}$. With the notations of Example 2.9, π' is irreducible if and only if it is the evaluation at one of the fixed points 1 or -1. In this case, $\pi(W) = \pm 1$. Otherwise, π' is reducible and the direct sum (up to unitary conjugation) of the evaluations at ω and $\overline{\omega}$ for $\omega \in \mathbb{T} \backslash \{-1, 1\}$.

We can deduce one more interesting piece of information on the structure of irreducible representations of $A \rtimes_\sigma \mathbb{Z}_2$ from the proof of Theorem 2.4.

Corollary 2.11 *Let π be an irreducible representation of A. Then there exists a unitary u such that $u^2 = 1$ and $u\pi u^* = \pi \circ \sigma$ if and only if the restriction π'' of π to the fixed point C^*-algebra A_1 is the sum of two unitarily non-equivalent (irreducible) representations of A_1.*

2.3 Representation Theory of $A \rtimes_\sigma \mathbb{Z}$ with $\sigma^2 = \mathrm{Id}$

We wish to point out that the previous description of the representation theory of the crossed-product $A \rtimes_\sigma \mathbb{Z}_2$ can be used to derive just as well the representation theory of $A \rtimes_\sigma \mathbb{Z}$, as described in the following proposition. The C^*-crossed-product $A \rtimes_\sigma \mathbb{Z}$ is the universal C^*-algebra generated by A and a unitary $W_\mathbb{Z}$ with the relations $W_\mathbb{Z} a W_\mathbb{Z}^* = \sigma(a)$ for all $a \in A$ [5].

Proposition 2.12 *Let σ be an order-two $*$-automorphism of a unital C^*-algebra A. Let π_2 be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}_2$ on some Hilbert space $\mathcal{H}$. Let $\lambda \in \mathbb{T}$. Denote by W the canonical unitary in $A \rtimes_\sigma \mathbb{Z}_2$ and $W_\mathbb{Z}$ the canonical unitary in $A \rtimes_\sigma \mathbb{Z}$. Set π on A by $\pi(a) = \pi_2(a)$ for all $a \in A$ and set $\pi(W_\mathbb{Z}) = \lambda \pi_2(W)$. Then π extends uniquely to a representation of $A \rtimes_\sigma \mathbb{Z}$. Moreover, all irreducible representations of $A \rtimes_\sigma \mathbb{Z}$ are obtained this way.*

Proof It is obvious that π thus constructed from π_2 is an irreducible representation of $A \rtimes_\sigma \mathbb{Z}$. Now let π be an irreducible representation of $A \rtimes_\sigma \mathbb{Z}$. Since π is irreducible and $\pi(W_\mathbb{Z})^2$ commutes with $\pi(A)$ (since $\sigma^2 = 1$), we conclude that $\pi(W_\mathbb{Z})^2 = \lambda^2$ for some $\lambda \in \mathbb{T}$. Let $U = \lambda^{-1}\pi(W_\mathbb{Z})$. Then U is an order-two unitary. Define $\pi_2(a) = \pi(a)$ for all $a \in A$ and $\pi_2(W) = U$: by universality of $A \rtimes_\sigma \mathbb{Z}_2$, the map π_2 extends to a representation of $A \rtimes_\sigma \mathbb{Z}_2$. It is irreducible since π is. This proves our proposition. ∎

3 Application to C^*-Crossed-Products of $C^*(\mathbb{F}_2)$

This section concerns itself with two examples of an action on the free group $\mathbb{F}_2$ on two generators. This paper deals with representation theory, so we present here examples which can be handled using representation theory more or less directly.

More precisely, given the universal C^*-algebra $C^*(\mathbb{F}_2)$ generated by two unitaries U and V, there are three obvious and natural automorphisms of order 2 to consider: α defined by $\alpha(U) = U^*$ and $\alpha(V) = V^*$, as well as β defined by $\beta(U) = -U$ and $\beta(V) = -V$, and finally γ defined by $\gamma(U) = V$ and $\gamma(V) = U$. A companion paper [1] to this one by the same authors deals with the interesting structure of the fixed point C^*-algebra for γ, and thus the study of the related C^*-crossed-product of $C^*(\mathbb{F}_2)$ by γ is done in [1] as well. The study of α and β is undertaken in this section.

The following propositions will help us compute the K-theory of these crossed-products by bringing the problem back to simple type crossed-products on abelian C^*-algebras, to which it will be easy to apply Theorem 2.4.

Proposition 3.1 *Let A_1 and A_2 be two unital C^*-algebras, and let α_1 and α_2 be two actions of a discrete group G on A_1 and A_2, respectively. Let α be the unique action of G on $A_1 *_{\mathbb{C}} A_2$ extending α_1 and α_2. Then*

$$(A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G = (A_1 \rtimes_{\alpha_1} G) *_{C^*(G)} (A_2 \rtimes_{\alpha_2} G),$$

where the free product is amalgated over the natural copies of $C^*(G)$ in $A_1 \rtimes_{\alpha_1} G$ and $A_2 \rtimes_{\alpha_2} G$, respectively.

Proof This result follows from universality. Since G is discrete, there is a natural embedding $i_k \colon C^*(G) \to A_k \rtimes_{\alpha_k} G$ for $k = 1, 2$. Now, given a commuting diagram

(3.1)
$$\begin{tikzcd} C^*(G) \arrow[r, "i_1"] \arrow[d, "i_2"] & A_1 \rtimes_{\alpha_1} G \arrow[d, "j_1"] \\ A_2 \rtimes_{\alpha_2} G \arrow[r, "j_2"] & B, \end{tikzcd}$$

by universality of the amalgated free product, there exists a unique surjection $\varphi_B \colon (A_1 \rtimes_{\alpha_1} G) *_{C^*(G)} (A_2 \rtimes_{\alpha_2} G) \to B$ such that, if we use the notations

$$\begin{tikzcd} C^*(G) \arrow[r, "i_1"] \arrow[d, "i_2"] & A_1 \rtimes_{\alpha_1} G \arrow[d, "\varphi_1"] \\ A_2 \rtimes_{\alpha_2} G \arrow[r, "\varphi_2"] & (A_1 \rtimes_{\alpha_1} G) *_{C^*(G)} (A_2 \rtimes_{\alpha_2} G), \end{tikzcd}$$

then $\varphi_B \circ \varphi_k = j_k$ for $k = 1, 2$. Of course, up to a $*$-isomorphism, there is a unique such universal object. Let us prove that $(A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G$ is this universal object, which will prove the proposition.

First, let $g \in G$ and let $U^g \in C^*(G)$, $U_1^g = i_1(U^g) \in A_1 \rtimes_{\alpha_1} G$ and $U_2^g = i_2(U^g) \in A_2 \rtimes_{\alpha_2} G$ and $U_3^g \in (A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G$ be the naturally associated unitaries. Now we

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observe that $(A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G$ fits in the commutative diagram

$$\begin{array}{ccc} C^*(G) & \xrightarrow{i_1} & A_1 \rtimes_{\alpha_1} G \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle \theta_1} \\ A_2 \rtimes_{\alpha_2} G & \xrightarrow{\theta_2} & (A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G, \end{array} \tag{3.2}$$

where $\theta_k(a) = a$ and $\theta_k(U_k^g) = U_3^g$ for $a \in A_k$ and $k = 1, 2$. Indeed, one checks immediately that, for $k = 1, 2$, the map θ_k satisfies $\theta_k(U_k^g)\theta_k(a)\theta_k(U_k^g)^* = \alpha_k(a) = \theta_k(\alpha_k(a))$, and then we can extend θ_k by universality of $A \rtimes_{\alpha_k} G$. The commutativity of the diagram is obvious.

Now, let us be given a C^*-algebra B fitting in the commutative diagram (3.1). Let $a \in A_k$ $(k = 1, 2)$. Then set $\psi(a) = j_k(a)$. Note that $\psi(1) = j_1(1) = j_2(1) = j_k \circ i_k(1)$ as i_k is unital for $k = 1, 2$. Hence, ψ extends to $A_1 *_{\mathbb{C}} A_2$ by universality of $A_1 *_{\mathbb{C}} A_2$. Now, with the notations of (3.2), we have $\theta_1(U_1^g) = \theta_2(U_2^g) = U_3^g$ by construction. We set $\psi(U_3^g) = j_1(U_1^g) = j_1 \circ i_1(U^g)$. As the diagram (3.1) is commutative, $\psi(U_3^g) = j_1 \circ i_2(U^g)$. Moreover, $\psi(U_3^g)\psi(a)\psi(U_3^g)^* = j_k(U_k^g a U_k^{g*}) = j_k(\alpha_k(a))$ for all $a \in A_k$ with $k = 1, 2$ by construction of ψ. It is easy to deduce that $\psi(U_3^g)\psi(a)\psi(U_3^g)^* = \psi \circ \alpha(a)$ for all $a \in A_1 *_{\mathbb{C}} A_2$. Hence, by universality of the crossed-product, the map ψ extends to $(A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G$ into B. Moreover, by construction $\psi \circ \theta_1 = j_1$ and $\psi \circ \theta_2 = j_2$. Thus, $(A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G$ is universal for the diagram (3.1), so $(A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G = (A_1 \rtimes_{\alpha_1} G) *_{C^*(G)} (A_2 \rtimes_{\alpha_2} G)$. ■

Proposition 3.2 *Let A_1 and A_2 be two unital C^*-algebras with two respective one-dimensional representations ε_1 and ε_2. Let α_1 and α_2 be two actions of a discrete group G on A_1 and A_2 respectively such that $\varepsilon_1 \circ \alpha_1 = \varepsilon_1$ and $\varepsilon_2 \circ \alpha_2 = \varepsilon_2$. Let α be the unique action of G on $A_1 *_{\mathbb{C}} A_2$ extending α_1 and α_2. Let i_k be the natural injection of $C^*(G)$ into $A_k \rtimes_{\alpha_k} G$ for $k = 1, 2$. Then $K_*\big((A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G\big)$ equals*

$$\big(K_*(A_1 \rtimes_{\alpha_1} G) \oplus K_*(A_2 \rtimes_{\alpha_2} G)\big) \big/ \ker(i_1^* \oplus (-i_2^*)),$$

where for any $*$-morphism $\varphi\colon A \to B$ between two C^*-algebras A and B we denote by $K_\varepsilon(\varphi)$ the lift of φ to the K-groups by functoriality (where $\varepsilon \in \{0, 1\}$).

Proof Let $k \in \{1, 2\}$. Denote by V_k^g the canonical unitary in $A_k \rtimes_{\alpha_k} G$ for $g \in G$ such that $V_k^g a (V_k^g)^* = \alpha_k(a)$ for all $a \in A_k$. Identify $\varepsilon_k(a)$ with $\varepsilon_k(a)1 \in C^*(G)$ for all $a \in A_k$. Then by universality of the crossed-product $A_k \rtimes_{\alpha_k} G$ and since $U_g \varepsilon_k(a) U_g^* = \varepsilon_k(a) = \varepsilon_k \circ \alpha_k(a)$ for all $a \in A_k$ (the latter equality is by hypothesis on ε_k), the map ε_k extends to $A_k \rtimes_{\alpha_k} G$ uniquely with $\varepsilon_k(V_k^g) = U^g$ for all $g \in G$, where U_g is the canonical unitary associated to $g \in G$ in $C^*(G)$. Note that ε_k thus extended is valued in $C^*(G)$.

Using the retractions $\varepsilon_k\colon A_k \rtimes_{\alpha_k} G \longrightarrow C^*(G)$ for $k \in \{1, 2\}$, we can apply [2] and thus the sequence:

$$\begin{aligned} 0 \longrightarrow K_*(C^*(G)) &\xrightarrow{K_*(i_1) \oplus K_*(i_2)} K_*(A_1 \rtimes_{\alpha_1} G) \oplus K_*(A_2 \rtimes_{\alpha_2} G) \\ &\longrightarrow K_*\big((A_1 \rtimes_{\alpha_1} G) *_{C^*(G)} (A_2 \rtimes_{\alpha_2} G)\big) \longrightarrow 0 \end{aligned}$$

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is exact. This calculates the K-groups of $(A_1 \rtimes_{\alpha_1} G) *_{C^*(G)} (A_2 \rtimes_{\alpha_2} G)$ which, by Proposition 3.1 is the crossed-product $(A_1 *_{\mathbb{C}} A_2) \rtimes_\alpha G$. ■

Remark 3.3 In particular, if A_1 is abelian then the existence of ε_1 is equivalent to the existence of a fixed point for the action α_1.

Of course, $C^*(\mathbb{Z}_2) = C(\mathbb{Z}_2) = \mathbb{C}^2$. Thus $K_1(C^*(\mathbb{Z}_2)) = 0$ while $K_0(C^*(\mathbb{Z}_2)) = \mathbb{Z}^2$ is generated by the spectral projection of the universal unitary W such that $W^2 = 1$.

Now we use Proposition 3.2 to compute the K-theory of two examples. The key in each case is to calculate explicitly the type I crossed-products $C(\mathbb{T}) \rtimes \mathbb{Z}_2$. We propose to do so using Theorem 2.4.

Proposition 3.4 *Let β be the $*$-automorphism of $C^*(\mathbb{F}_2)$ defined by $\beta(U) = -U$ and $\beta(V) = -V$. Then*

$$K_0\big(C^*(\mathbb{F}_2) \rtimes_\beta \mathbb{Z}_2\big) = \mathbb{Z} \quad \textit{and} \quad K_1\big(C^*(\mathbb{F}_2) \rtimes_\beta \mathbb{Z}_2\big) = \mathbb{Z}^2.$$

Proof Let z be the map $\omega \in \mathbb{T} \mapsto \omega$. Write $\beta = \beta_1 * \beta_1$ where $\beta_1(z) = -z$. The crossed-product $C(\mathbb{T}) \rtimes_{\beta_1} \mathbb{Z}_2$ is $\mathcal{C} = C(\mathbb{T}, M_2) = C(\mathbb{T}) \otimes M_2$. Indeed, if we set $\psi(f)(x) \mapsto \left[\begin{smallmatrix} f(x) & 0 \\ 0 & f(-x) \end{smallmatrix}\right]$ and $\psi(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, then ψ extends naturally to a $*$-morphism from $C(\mathbb{T}) \rtimes_{\beta_1} \mathbb{Z}_2$ into $\mathcal{C}$. Moreover, the range of ψ is the C^*-algebra spanned by $\psi(z)$ and $\psi(W)$, which is easily checked to be $\mathcal{C}$ by the Stone–Weierstrass theorem, so ψ is surjective. It is injective as well; let $a \in \ker \psi$. If π is an irreducible $*$-representation of $C(\mathbb{T}) \rtimes_{\beta_1} \mathbb{Z}_2$, then by Theorem 2.4, π is (up to unitary equivalence) acting on M_2 by $\pi(f) = \left[\begin{smallmatrix} f(x) & 0 \\ 0 & f(-x) \end{smallmatrix}\right]$ and $\pi(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, for some fixed $x \in \mathbb{T}$. Thus, if ρ_x is the evaluation at x in $\mathcal{C}$, then $\rho \circ \psi = \pi$ and thus $\pi(a) = 0$. Thus $a = 0$ as π arbitrary, and thus ψ is a $*$-isomorphism.

Of course, $K_*(M_2(C(\mathbb{T}))) = K_*(\mathbb{T})$ so $K_0(C(\mathbb{T}) \rtimes_{\beta_1} \mathbb{Z}_2) = \mathbb{Z}$ and $K_1(C(\mathbb{T}) \rtimes_{\beta_1} \mathbb{Z}_2) = \mathbb{Z}$. Moreover, K_1 is generated by z while K_0 is simply generated by the identity of $C(\mathbb{T})$. The map $i_k \colon C^*(\mathbb{Z}_2) \to \mathcal{C}$ maps the generator of $C^*(\mathbb{Z}_2)$ to w, and thus i_k^* maps the two spectral projections of w to 1. Hence, $i_k^0 \colon \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $i_k(0,1) = i_k(1,0) = 1$. Thus by Proposition 3.1, we have $K_0\big(C^*(\mathbb{F}_2) \rtimes \mathbb{Z}_2\big) = \mathbb{Z}$ and $K_1\big(C^*(\mathbb{F}_2) \rtimes \mathbb{Z}_2\big) = \mathbb{Z}^2$. ■

Proposition 3.5 *Let α be the $*$-automorphism of $C^*(\mathbb{F}_2) = C^*(U, V)$ defined by $\alpha(U) = U^*$ and $\alpha(V) = V^*$. Then*

$$K_0\big(C^*(\mathbb{F}_2) \rtimes_\alpha \mathbb{Z}_2\big) = \mathbb{Z}^4 \quad \textit{and} \quad K_0\big(C^*(\mathbb{F}_2) \rtimes_\alpha \mathbb{Z}_2\big) = 0.$$

Proof Write $\alpha = a_1 * \alpha_1$ where $\alpha_1(z) = \overline{z}$ where z is the map $\omega \in \mathbb{T} \mapsto \omega$. Now, the crossed-product $C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2$ is the C^*-algebra

$$\mathcal{B} = \{h \in C([-1,1], M_2) : h(1), h(-1) \text{ diagonal}\}.$$

Indeed, define $\psi(f)(t)$ for all $f \in C(\mathbb{T})$ and $t \in [0,1]$ by

$$\frac{1}{2}\begin{bmatrix} f(t,-y) + f(t,y) & f(t,-y) - f(t,y) \\ f(t,-y) - f(t,y) & f(t,-y) + f(t,y) \end{bmatrix},$$

where $y = \sqrt{1-t^2}$. Set $\psi(W) = w = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Then $w^2 = 1$ and $w\psi(f)w = \psi(\alpha_1(f))$, so ψ extends to a unique $*$-morphism from $C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2$ into $\mathcal{B}$. By the Stone–Weierstrass theorem, one can check that ψ is indeed onto. Last, let π be an irreducible $*$-representation of $C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2$. If the restriction π' of π to $C(\mathbb{T})$ is irreducible, then π' is one-dimensional and there exists $x \in \mathbb{T}$ such that $\pi'(f) = \pi(f) = f(x)$ for all $f \in C(\mathbb{T})$. By Theorem 2.4 since π' is irreducible, π is also one-dimensional and $\pi(W)$ is a scalar unitary (hence it is 1 or -1 since $W^2 = 1$), so it commutes with $\pi(f)$ for all f. Since $(WfW)(x) = f(\bar{x})$ we conclude that $x = 1$ or $x = -1$. Either way let ρ_x be the evaluation at x in $\mathcal{B}$. Then $\rho_x(h)$ is diagonal by definition of $\mathcal{B}$ for all $h \in \mathcal{B}$. Let $\rho_{x,+1}$ be the one-dimensional representation defined by the upper-left corner of ρ_x and let $\rho_{x,-1}$ be the one-dimensional representation defined by the lower-right corner of ρ_x. Note that either way, $\rho_{x,+}(\psi(f)) = \rho_{x,-}(\psi(f)) = f(x)$ for all $f \in C(\mathbb{T})$. On the other hand, $\rho_{x,\varepsilon}(\psi(W)) = \varepsilon$. Hence, we have proved that $\rho_{x,W} \circ \psi = \pi$. Thus if $a \in C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2$ and $\psi(a) = 0$, then $\pi(a) = 0$.

If instead, π restricted to $C(\mathbb{T})$ is reducible, then by Theorem 2.4 π is unitarily equivalent to a representation π' acting on M_2 defined as follows: there exists $x \in \mathbb{T}$ such that $\pi'(f) = \left[\begin{smallmatrix} f(x) & 0 \\ 0 & f(\bar{x}) \end{smallmatrix}\right]$ for all $f \in C(\mathbb{T})$ and $\pi'(W) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Up to conjugating by the unitary $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$, we see that if we set $\rho(h) = h(t)$ for all $h \in \mathcal{B}$, where t is defined by $x = (t, \sqrt{1-t^2})$, then $\rho \circ \psi = \pi'$ and thus, if $a \in \ker\psi$, then $\pi'(a) = 0$, so $\pi(a) = 0$. In conclusion, if $a \in \ker\psi$, then $\pi(a) = 0$ for all (irreducible) $*$-representations of $C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2$ and thus $a = 0$, so ψ is a $*$-isomorphism.

The K-theory of $\mathcal{B}$ is easy to calculate. We start with the exact sequence $0 \to C_0((-1,1), M_2) \xrightarrow{i} \mathcal{B} \xrightarrow{q} \mathbb{C}^4 \to 0$ where i the inclusion and q the quotient map, also defined by $q(a) = a(1) \oplus a(-1)$ for $a \in \mathcal{B}$ and identifying the diagonal matrices in M_2 with $\mathbb{C}^2$. We also used the notation $C_0(X)$ for the space of continuous functions on a locally compact space X vanishing at infinity. The associated six-term exact sequence is then simply

$$\begin{tikzcd}
K_0(C(-1,1)) = 0 \arrow[r, "K_0(i)"] & K_0(\mathcal{B}) \arrow[r, "K_0(q)"] & \mathbb{Z}^4 \arrow[d, "\delta"] \\
0 \arrow[u] & K_1(\mathcal{B}) \arrow[l, "K_1(q)"'] & \mathbb{Z} = K_1(C_0(-1,1)). \arrow[l, "K_1(i)"']
\end{tikzcd}$$

The generator of the K_1 group of $C_0(-1,1) \otimes M_2$ is the unitary $u_1 \colon t \in (-1,1) \mapsto \exp(i\pi t)1_2$, where 1_2 is the unit of M_2. However, u_1 is trivial in $K_1(\mathcal{B})$ via the obvious homotopy $(u_\lambda)_{\lambda\in[0,1]}$ with $u_\lambda \colon t \in [-1,1] \mapsto \exp(\pi i \lambda t)$ (note that u_λ for $\lambda \in (0,1)$ is not in the unitalization of $C_0(-1,1)$ since $u_\lambda(-1) \neq u_\lambda(1)$). Thus $K_1(i) = 0$, $K_1(\mathcal{B}) = 0$ and the range of δ is $\mathbb{Z}$ by exactness. Hence, again by exactness, $\ker\delta$ is a copy of $\mathbb{Z}^3$ inside of $\mathbb{Z}^4 = K_0(\mathbb{C}^4)$.

Let $p = \frac{1}{2}(W+1)$ and $p' = \frac{1}{2}(1+Wz)$ (note that $WzWz = \bar{z}z = 1$, so p' is a projection). We calculate easily that $K_0(q)(p) = (1,0,1,0)$, while $K_0(q)(p') = (1,0,0,1)$.

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The subgroup of $\mathbb{Z}^4$ generated by $(1,0,1,0)$, $(1,1,1,1)$, and $(1,0,0,1)$ is isomorphic to $\mathbb{Z}^3$. By exactness, it must be $\mathbb{Z}^3$. Since $K_0(i) = 0$, the map $K_0(1)$ is an injection and thus $K_0\big(C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2\big) = \mathbb{Z}^3$ generated by the spectral projections of w and of Wz, and $K_1(C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2) = 0$.

Moreover, $i_k \colon C^*(\mathbb{Z}_2) \to C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_2$ maps the generator of $C^*(\mathbb{Z}_2)$ to w, so the range of i_k^* is the subgroup generated by $[p]$ and $[1]$. Thus, by Proposition 3.1, $K_0(C^*(\mathbb{F}_2) \rtimes_{\alpha_1} \mathbb{Z}_2) = \mathbb{Z}^4$ and $K_1(C^*(\mathbb{F}_2) \rtimes_{\alpha_1} \mathbb{Z}_2) = 0$. ■

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