B.H. NEUMANN'S QUESTION ON ENSURING COMMUTATIVITY OF FINITE GROUPS

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Dedicated to the memory of Bernhard H. Neumann

This paper is an attempt to provide a partial answer to the following question put forward by Bernhard H. Neumann in 2000: "Let $G$ be a finite group of order $g$ and assume that however a set $M$ of $m$ elements and a set $N$ of $n$ elements of the group is chosen, at least one element of $M$ commutes with at least one element of $N$. What relations between $g$, $m$, $n$ guarantee that $G$ is Abelian?" We find an exponential function $f(m,n)$ such that every such group $G$ is Abelian whenever $|G| > f(m,n)$ and this function can be taken to be polynomial if $G$ is not soluble. We give an upper bound in terms of $m$ and $n$ for the solubility length of $G$, if $G$ is soluble.

1. INTRODUCTION AND RESULTS

This paper is an attempt to provide a partial answer to the following question put forward by Bernhard H. Neumann in [10]: "Let $G$ be a finite group of order $g$ and assume that however a set $M$ of $m$ elements and a set $N$ of $n$ elements of the group is chosen, at least one element of $M$ commutes with at least one element of $N$ (call this condition Comm$(m,n)$). What relations between $g$, $m$, $n$ guarantee that $G$ is Abelian?"

Following Neumann, for given positive integers $m$ and $n$ we say that a group $G$ satisfies the condition Comm$(m,n)$ if and only if for every two subsets $M$ and $N$ of cardinalities $m$ and $n$ respectively, there are elements $x \in M$ and $y \in N$ such that $xy = yx$.

We note that an infinite group $G$ satisfying the condition Comm$(m,n)$ for some $m$ and $n$ is Abelian. This is because every infinite subset of such group contains two commuting elements. Thus by a famous Theorem of Neumann [9], it is centre-by-finite. Therefore $Z(G)$, the centre of $G$ is infinite. Now let $M$ and $N$ be two subsets of $Z(G)$, of sizes $m$ and $n$ respectively. Then for any two elements $x$ and $y$ of $G$, there are elements $z_1 \in M$ and $z_2 \in N$ such that $xz_1y = yx$, so that $xy = yx$; namely $G$ is...
Abelian. Therefore in considering non-Abelian groups satisfying $\text{Comm}(m, n)$ we need only consider finite cases.

We use the usual notations: for example $C_G(a)$ is the centraliser of an element $a$ in a group $G$, $S_n$ is the symmetric group on $n$ letters, $A_n$ is the alternating group on $n$ letters, $D_{2n}$ is the dihedral group of order $2n$, $Q_8$ is the quaternion group of order 8 and $T$ will stand for the group $(x, y \mid x^6 = 1, y^2 = x^3, y^{-1}xy = x^{-1})$. If $G$ satisfies the condition $\text{Comm}(m, n)$, then we say $G$ is a $C(m, n)$-group, or $G \in C(m, n)$.

Throughout $G$ will denote a finite non-Abelian group unless otherwise is stated. We shall show that a $C(m, n)$-group has order bounded by a function of $m$ and $n$ which may not always be chosen to be a polynomial function in terms of $m$ and $n$. Our main results are

**Theorem 1.1.** Let $G$ be a $C(m, n)$-group. Then $|G|$ is bounded by a function of $m$ and $n$.

The solubility length of a soluble $C(m, n)$-group is bounded above in terms of $m$ and $n$. In fact we prove the following.

**Theorem 1.2.** Let $G \in C(m, n)$ be a soluble group of solubility length $d$. Then

$$d \leq \max\{\lfloor \log_2 m \rfloor, \lfloor \log_2 n \rfloor\}$$

We also obtain a solubility criterion for $C(m, n)$-groups in terms of $m$ and $n$, namely

**Theorem 1.3.** Let $G$ be a $C(m, n)$-group and $m + n \leq 58$. Then $G$ is a soluble group.

We give a complete characterisation of $C(m, n)$-groups, where $m + n \leq 10$, in the next theorem.

**Theorem 1.4.** Let $G$ be a $C(m, n)$-group, where $m + n \leq 10$. Then $G$ is isomorphic to one of the following: $S_3$, $D_{2n}$ for $n \in \{3, 4, 5, 6\}$, $Q_8$, $T$ or a non-Abelian group of order 16 whose centre is of order 4.

2. A PARTIAL ANSWER TO NEUMANN’S QUESTION

A subset of a non-Abelian group $G$ no two of whose distinct elements commute is called non-commuting. A non-commuting subset of maximal size is called a maximal non-commuting set and this maximal size will be denoted by $\omega(G)$. In this section we give a partial answer to Neumann’s question by proving that a $C(m, n)$-group has the order bounded by a function of $m$ and $n$.

**Proof of Theorem 1.1:** Let $Z(G) = \{z_1, z_2, \ldots, z_t\}$, where $t \geq \max\{m, n\}$. Choose any two elements $a$ and $b$ in $G$, and put

$$M = \{az_1, az_2, \ldots, az_m\} \text{ and } N = \{bz_1, bz_2, \ldots, bz_n\}$$
Since $G$ is a $C(m, n)$-group, there exist $a_i \in M$ and $b_j \in N$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, such that $a_ib_j = b_ja_i$. This implies that $ab = ba$, and so $G$ is an Abelian group, which is a contradiction. Thus $|Z(G)| < \max\{m, n\}$. Suppose, for a contradiction, that $\omega = \omega(G) \geq m + n$. Then there are $\omega$ pairwise non-commuting elements $a_1, \ldots, a_{m+n} \in G$. Put

$$M = \{a_1, \ldots, a_m\} \text{ and } N = \{a_{m+1}, \ldots, a_{m+n}\}$$

Since $G$ is a $C(m, n)$-group, there exist $a_i \in M$ and $a_j \in N$ such that $a_ia_j = a_ja_i$, which is a contradiction. Thus $\omega < m + n$. Now the main result of [11] implies that $|G : Z(G)| \leq c^\omega$, where $c$ is a constant. Therefore

$$|G| \leq c^\omega |Z(G)| \leq c^{m+n} \max\{m, n\},$$

which completes the proof.

**Remark 2.1.** Since an extra-special 2-group of order $2^{2k+1}$, has maximal non-commuting sets of size $2k + 1$ (see [4] or [11]), if $f(m, n)$ is the least integer such that $|G| \leq f(m, n)$ for all $C(m, n)$-groups, then $f(m, n)$ cannot be chosen to be a polynomial in terms of $m$ and $n$.

The following is a key lemma to some of our results.

**Lemma 2.2.** Let $G$ be a $C(m, n)$-group and let $N$ be a normal subgroup of $G$ such that $G/N$ is non-Abelian. Then $|N| < \max\{m, n\}$.

**Proof:** Suppose on the contrary that $N = \{a_1, a_2, \ldots, a_t\}$ and $t \geq \max\{m, n\}$. Choose any two elements $x$ and $y$ in $G \setminus N$, and put

$$X = \{xa_1, xa_2, \ldots, xa_m\} \text{ and } Y = \{ya_1, ya_2, \ldots, ya_n\}.$$

Since $G$ is a $C(m, n)$-group, there exist $xa_i$ in $X$ and $ya_j$ in $Y$ such that $[xa_i, ya_j] = 1$. Thus $[x, y] \in N$ and $G/N$ is Abelian, which is a contradiction.

**Corollary 2.3.** Let $G$ be an insoluble $C(m, n)$-group. Then

$$|G| \leq 4^4(m + n)^8 \cdot \max\{m, n\}.$$

**Proof:** Let $S$ be the largest soluble normal subgroup of $G$. Then $G/S$ has no non-trivial normal Abelian subgroup and by [12, Theorem 1.3], $|G/S| < (n(G))^4$, where $n(G)$ is the size of the largest conjugacy class in $G$. Now by [11] we have $n(G) \leq 4\omega(G)^2$. Then by the proof of Theorem 1.1, $\omega(G) < m + n$ and by Lemma 2.2, $|S| < \max\{m, n\}$, which completes the proof.
3. SOLUBLE GROUPS SATISFYING THE CONDITION \(\text{Comm}(m, n)\)

In this section we prove Theorems 1.2 and 1.3. First we need some preliminary lemmas.

**Lemma 3.1.** Let \(G\) be a \(C(m, n)\)-group. If \(a_1, a_2, \ldots, a_n\) are \(n\) distinct elements of \(G\), then \(|G \setminus \bigcup_{i=1}^{n} C_G(a_i)| < m\).

**Proof:** Suppose, for a contradiction, that there exist \(m\) distinct elements \(b_1, b_2, \ldots, b_m\) in \(G \setminus \bigcup_{i=1}^{n} C_G(a_i)\). Since \(G\) is a \(C(m, n)\)-group, there exist elements \(a_i, b_j\) such that \(a_i b_j = b_j a_i\) and so \(b_j \in C_G(a_i)\), which is a contradiction. \(\square\)

**Lemma 3.2.** If \(G\) is a \(C(m, n)\)-group, then \(m + n \geq 6\).

**Proof:** Suppose, for a contradiction, that \(m + n < 6\). We distinguish two cases:

**Case 1:** \(n = 1\). Then \(|G| \leq 6\) and so \(G \cong S_3\), since \(G\) is non-Abelian. If \(a \in S_3\) is of order 3, then Lemma 3.1 gives \(3 = |G \setminus C_G(a)| < m\). It follows that \(m = 4\). But \(S_3\) is not a \(C(1, 4)\)-group.

**Case 2:** \(n = 2\). Since \(G\) is non-Abelian, there exists an element \(a\) in \(G \setminus Z(G)\) such that \(a^2 \neq 1\); for let \(g^2 = 1\) for all \(g \in G \setminus Z(G)\). Then \((g z)^2 = 1\) for all \(z \in Z(G)\) and \(g \in G \setminus Z(G)\). It follows that \(1 = g^2 z^2 = z^2\) and so we have \(z^2 = 1\) for all \(z \in Z(G)\). Hence \(g^2 = 1\) for all \(g \in G\) which implies that \(G\) is Abelian, a contradiction.

Now since \(a \neq a^{-1}\), it follows from Lemma 3.1 that

\[|G \setminus (C_G(a) \cup C_G(a^{-1}))| \leq m - 1 \leq 2.\]

Since \(C_G(a) = C_G(a^{-1})\), we have that \(|G| \leq |C_G(a)| + 2\). As \(a \in G \setminus Z(G)\), it follows that \(|C_G(a)| \leq |G|/2\) and so \(|G| \leq |G|/2 + 2\). Hence \(|G| \leq 4\), so \(G\) is Abelian. This contradiction completes the proof. \(\square\)

**Lemma 3.3.** Let \(G\) be a \(C(m, n)\)-group and let \(N\) be a non-trivial normal subgroup of \(G\). Then \(G/N\) is a \(C(m-r, n-t)\)-group, for all positive integers \(r, t\) such that \(2r \leq m\) and \(2t \leq n\).

**Proof:** Suppose, for a contradiction, that \(G/N\) is not a \(C(m-r, n-t)\)-group. Thus there exist two subsets

\[X = \{x_1 N, \ldots, x_{m-r} N\}\] and \[Y = \{y_1 N, \ldots, y_{n-t} N\}\]

such that \([x_i, y_j] \notin N\) for all \(i, j\). Let \(a\) be a non-trivial element of \(N\) and consider

\[X_1 = \{a x_1, a x_{m-r}, x_1, \ldots, x_r\}\] and \[Y_1 = \{a y_1, \ldots, a y_{n-t}, y_1, \ldots, y_t\}.\]
It is clear that \(|X| = m\) and \(|Y| = n\) and no element of \(X_1\) commutes with no element of 
\(Y_1\), which completes the proof.

**Proof of Theorem 1.2:** We argue by induction on \(d\). By hypothesis \(G\) is non-Abelian, thus it follows from Lemma 3.2 that either \(m \geq 3\) or \(n \geq 3\). Thus for \(d = 2\), the result holds, since \([\log_2 3] = 2\). So assume that \(d \geq 3\) and the result holds for \(d - 1\). Now \(G/G^{(d-1)}\) has solubility length \(d - 1\). Let \(k\) and \(\ell\) be positive integers such that \(2^k < m \leq 2^{k+1}\) and \(2^\ell < n \leq 2^{\ell+1}\). Thus by Lemma 3.3, \(G/G^{(d-1)}\) satisfies \(Comm(2^k, 2^\ell)\). Thus by the induction hypothesis \(d - 1 \leq \max\{k, \ell\}\) and so \(d \leq \max\{[\log_2 m], [\log_2 n]\}\), as required.

To prove Theorem 1.3 we need the following lemma.

If \(G\) is a finite group, then for each prime divisor \(p\) of \(|G|\), we denote by \(\nu_p(G)\) the number of Sylow \(p\)-subgroups of \(G\).

**Lemma 3.4.** Let \(G\) be a \(C(m, n)\)-group and \(p\) be a prime number dividing \(|G|\) such that every two distinct Sylow \(p\)-subgroups of \(G\) have trivial intersection. Then \(\nu_p(G) \leq m + n - 1\).

**Proof:** It follows from the proof of Theorem 1.1, that \(\omega(G) < m + n\). Now \([7, \text{Lemma } 3]\) completes the proof.

**Proof of Theorem 1.3:** Suppose, on the contrary, that there exists a non-soluble finite group \(G \in C(m, n)\) of the least possible order, where \(m + n \leq 58\). If there exists a non-trivial proper normal subgroup \(N\) of \(G\), then both \(N\) and \(G/N\) are in \(C(m, n)\) and so they are soluble. It follows that \(G\) is soluble, which is a contradiction. Therefore \(G\) is a minimal simple \(C(m, n)\)-group. By Thompson's classification of minimal simple groups \([13]\), \(G\) is isomorphic to one of the following simple groups:

- \(A_5\) the alternating group of degree 5,
- \(PSL(2, 2^p)\), where \(p\) is an odd prime,
- \(PSL(2, 3^p)\), where \(p\) is an odd prime,
- \(PSL(2, p)\), where \(5 < p\) is prime and \(p \equiv 2 \pmod{5}\),
- \(PSL(3, 3)\), and
- \(Sz(2^p)\), \(p\) an odd prime.

We first prove that \(A_5\) is not a \(C(m, n)\)-group, where \(m + n \leq 58\). Let \(P_1, \ldots, P_5, Q_1, \ldots, Q_{10}, R_1, \ldots, R_6\) be Sylow \(p\)-subgroups of \(A_5\), for \(p = 2, 3, 5\), respectively. It is easy to see that \(A_5\) is the union of these Sylow subgroups and no two distinct non-trivial elements of coprime orders in \(A_5\) commute (see \([3]\)). Since every non-trivial element in 
\[
\bigcup_{i=1}^6 R_i \cup Q_1 \cup Q_2 \text{ does not commute with one in } \left( \bigcup_{i=1}^5 P_i \cup \bigcup_{i=3}^{10} Q_i \right) \setminus \{a\}
\]

(where $a$ is an arbitrary non-trivial element of $Q_{10}$), $A_5$ is not a $C(28, 30)$-group and since every non-trivial element in
\[
\left( \bigcup_{i=1}^{6} R_i \cup Q_1 \cup Q_2 \cup Q_3 \right) \setminus \{b\}
\]
(where $b$ is an arbitrary non-trivial element of $Q_1$) does not commute with one in
\[
\bigcup_{i=1}^{5} P_i \cup \bigcup_{i=4}^{10} Q_i,
\]
$A_5$ is not a $C(29, 29)$-group. Now suppose that $n \leq 27$. Then $n = 4k + \ell$ for some integers $k$ and $\ell$, where $0 \leq k \leq 6$ and $0 \leq \ell \leq 3$. Let $a$ be an arbitrary non-trivial element of $Q_{10}$ and define
\[
A_n = \begin{cases} 
\bigcup_{i=1}^{k} R_i & \text{if } \ell = 0 \\
\left( \bigcup_{i=1}^{k} R_i \cup Q_{10} \right) \setminus \{a\} & \text{if } \ell = 1 \\
\bigcup_{i=1}^{k} R_i \cup Q_1 & \text{if } \ell = 2 \\
\bigcup_{i=1}^{k} R_i \cup P_1 & \text{if } \ell = 3
\end{cases}
\]
and
\[
B_n = \begin{cases} 
\left( \bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=1}^{5} P_i \cup \bigcup_{i=1}^{10} Q_i \right) \setminus \{a\} & \text{if } \ell = 0 \\
\left( \bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=1}^{5} P_i \cup \bigcup_{i=1}^{9} Q_i \right) & \text{if } \ell = 1 \\
\left( \bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=1}^{5} P_i \cup \bigcup_{i=1}^{10} Q_i \right) \setminus \{a\} & \text{if } \ell = 2 \\
\left( \bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=1}^{5} P_i \cup \bigcup_{i=1}^{10} Q_i \right) \setminus \{a\} & \text{if } \ell = 3
\end{cases}
\]
Then no non-trivial element of $A_n$ commutes with one of $B_n$. It then follows that $A_5$ is not a $C(n, m)$-group, where $n + m \leq 58$.

If $G$ is isomorphic to $PSL(2, 2^p)$ or $PSL(2, 3^p)$, where $p$ is an odd prime, then by [1, Lemma 4.4], $\omega(G) > 64$, which is a contradiction. If $G \cong PSL(3, 3)$, then $|G| = 2^4 \times 3^3 \times 13$ so that $\nu_{13}(H) = 144 > 57$, which is not possible by Lemma 3.4. If $G \cong PSL(2, p)$ and $p > 7$ ($p$ is a prime number), then [1, Lemma 4.4] implies that $\omega(G) \geq 133$, a contradiction. If $G \cong PSL(2, 7)$, then by [1, Proposition 3.21] and a similar argument as for $A_5$ we conclude that $G$ is not a $C(m, n)$-group. If $G \cong Sz(2^p)$, then $|G| = 2^{2p} \times (2^p - 1) \times (2^{2p} + 1)$ and $\nu_2(G) = 2^{2p} + 1 \geq 65$ (see Theorem 3.10 (and its proof) of [8, Chapter XI]).

We note that the bound 58 in Theorem 1.3 is the best possible. In fact we have
Theorem 3.5. The alternating group $A_5$ is the only non-Abelian finite simple $C(m, n)$-group, for some positive integers $m$ and $n$ such that $m + n = 59$.

Proof: First we note that, since every centraliser of $A_5$ has order at least 3, $A_5$ is a $C(1, 58)$-group. For uniqueness, suppose, on the contrary, that there exists a non-Abelian finite simple group not isomorphic to $A_5$ and of least possible order which is a $C(m, n)$-group, for some positive integers $m$ and $n$ with $m + n = 59$. Then by [5, Proposition 3], $G$ is isomorphic to one of the following groups:

- $PSL(2, 2p)$, $p = 4$ or a prime;
- $PSL(2, 3p)$, $PSL(2, 5p)$, $p$ a prime;
- $PSL(2, p)$, $p$ a prime and $7 < p$;
- $PSL(2, 3)$;
- $PSL(2, 5)$;
- $PSU(3, 4)$ (the projective special unitary group of degree 3 over the finite field of order $4^3$) or
- $Sz(2p)$, $p$ an odd prime.

Now an argument similar to the one in the proof of Theorem 1.3 gives a contradiction in each case.

4. Groups satisfying the condition $Comm(m, n)$ for some small positive integers $m$ and $n$

In this section we characterise $C(m, n)$-groups for some particular $m$ and $n$ and hence prove Theorem 1.4. First we need some preliminary lemmas.

Lemma 4.1. Let $G$ be a $C(m, n)$-group. Let $x$ be a non-central element of finite order such that $\varphi(|x|) > n$, where $\varphi$ is the Euler $\varphi$-function. Then $|G \setminus C_G(x)| < m$.

Proof: Suppose that

$$\{k \in \mathbb{N} : 1 \leq k \leq |x| \text{ and } \gcd(k, |x|) = 1\} = \{d_1, d_2, \ldots, d_{\varphi(|x|)}\}.$$  

Since $x^{d_i} \neq x^{d_j}$ for all $i \neq j$, by Lemma 3.1

$$|G \setminus \bigcup_{i=1}^{d_{\varphi(|x|)}} C_G(x^{d_i})| < m.$$  

Also we have $C_G(x) = C_G(x^{d_i})$ for all $1 \leq i \leq d_{\varphi(|x|)}$. Hence $|G \setminus C_G(x)| < m$.

Lemma 4.2. Let $G$ be a finite nilpotent $C(m, n)$-group. Then $\prod_{p | |G|} p < \max\{m, n\}$.
PROOF: The group $G$ is the direct product of its Sylow subgroups. So $G = \prod_{p \mid |G|} P$, where $P$ is the Sylow $p$-subgroup. Then $Z(G) = \prod_{p \mid |G|} Z(P)$ and $\max\{m, n\} \geq |Z(G)| \geq \prod_{p \mid |G|} p$, by the proof of Theorem 1.1.

LEMMA 4.3. If $G$ is a $C(m, n)$-group, then for any prime divisor $p$ of $|G|$, $p \leq \max\{m, n\}$.

PROOF: Suppose that $p$ is a prime divisor of $|G|$. Let $a$ be an element of order $p$ in $G$. For any $x$ in $G$ put $X = \{xa, xa^2, \ldots, xa^m\}$ and $Y = \{a, a^2, \ldots, a^n\}$. Then, by the hypothesis, there exist $xa^i \in X$ and $a^j \in Y$ such that $xa^i a^j = a^j xa^i$. Since $\gcd(j, p) = 1$, we have $[x, a] = 1$. Thus $a \in Z(G)$, so that $p \mid |Z(G)|$ and by the proof of Theorem 1.1, $p < |Z(G)| < \max\{m, n\}$.

LEMMA 4.4. Let $G$ be a non-Abelian finite group such that $|G/Z(G)| = 4$. Then $G$ is not a $C(z, 2z)$-group, where $z = |Z(G)|$.

PROOF: Since $G$ is non-Abelian, $G/Z(G) \cong C_2 \times C_2$. Thus there exist elements $a, b \in G$ such that $G = Z(G) \cup aZ(G) \cup bZ(G) \cup abZ(G)$. Therefore $\langle aZ(G), bZ(G) \rangle$ is an elementary Abelian 2-group of order 4. Thus $G = \langle a, b \rangle Z(G)$ and so $ab \neq ba$, since $G$ is not Abelian. Now consider the subsets $M = aZ(G) \cup bZ(G)$ and $N = abZ(G)$. We have $xy \neq yx$ for all $x \in M$ and $y \in N$, since $ab \neq ba$. This shows that $G$ is not a $C(z, 2z)$-group.

REMARK 4.5.

(1) Let $G$ be a $C(m, n)$-group. Then it is easy to see that $G$ is not a $C\left(t, \frac{|G \setminus C_G(a)|}{|Z(G)|}\right)$-group, where $a$ is any element of $G$ with $t \leq |C_G(a) \setminus Z(G)|$.

(2) If $G$ is a $C(m, n)$-group, then for any two natural numbers $m'$ and $n'$ such that $m \leq m'$ and $n \leq n'$, $G$ is also a $C(m', n')$-group.

COROLLARY 4.6. Let $G$ be a $C(1, n)$-group, where $5 \leq n \leq 9$. Then $G \cong S_3$, $D_8$, $Q_8$, $D_{10}$, $T$, $D_{12}$ or a non-Abelian group of order 16 whose centre is of order 4.

PROOF: By Remark 4.5(2), it is enough to consider only the case $n = 9$. Suppose that $a$ is any non-central element of $G$. By Lemma 3.1 we have $|G \setminus C_G(a)| \leq 8$ and so $|G| \leq 16$. If $|G| = 12$, then $G \cong A_4$, $D_{12}$ or $T$. The alternating group $A_4$ has an element whose centraliser has order 3. Thus by Remark 4.5(1), $A_4$ is not a $C(1, 9)$-group. If $G \cong D_{12}$ or $G \cong T$, then the order of the centraliser of any element in $G$ is at least 4. Thus $G$ is a $C(1, 9)$-group. If $|G| = 14$, then $G \cong D_{14}$ and there exists $x \in D_{14}$ such that $|C_G(x)| = 2$. By Remark 4.5(1), $D_{14}$ is not a $C(1, 9)$-group. Finally if $|G| = 16$, ...
then \( |Z(G)| = 2 \) or \( 4 \). If \( |Z(G)| = 4 \), then for all \( a \in G \), \( |C_G(a)| \geq 8 \). Thus \( G \) is a \( C(1,9) \)-group. If \( |Z(G)| = 2 \), then there exists an element \( a \) in \( G \) such that \( |C_G(a)| = 4 \), so that by Remark 4.5(1), \( G \) is not a \( C(1,9) \)-group.

**COROLLARY 4.7.** Let \( G \) be a \( C(2,n) \)-group, where \( 4 \leq n \leq 8 \). Then \( G \cong S_3, Q_8, D_8 \) or \( D_{20} \).

**PROOF:** By Remark 4.5(2), it is enough to consider only the case \( n = 8 \). Since \( G \) is non-Abelian, there exists an element \( a \) in \( G \setminus Z(G) \) such that \( a^2 \neq 1 \). By Lemma 3.1, \( |G \setminus C_G(a)| \leq 7 \), from which it follows that \( |G| \leq 14 \). If \( |G| = 12 \), then \( G \) contains centralisers of order 4. Thus by Remark 4.5(1), \( G \) is not a \( C(2,8) \)-group. If \( |G| = 14 \), then \( G \cong D_{14} \), and it is not a \( C(2,8) \)-group since \( D_{14} \) contains centralisers of order 2.

**LEMMA 4.8.** Let \( G \) be a \( C(3,n) \)-group, where \( 3 \leq n \leq 7 \). Then \( G \cong S_3, D_8, Q_8 \) or \( D_{20} \).

**PROOF:** By Remark 4.5(2), it is enough to consider only the case \( n = 7 \). Since \( G \) is non-Abelian, there exists non-central element \( a \) in \( G \) such that \( a^2 \neq 1 \). Let \( b \in G \setminus Z(G) \) be such that \( b \neq a, a^{-1} \). Then by Lemma 3.1,

\[
|G \setminus C_G(a) \cup C_G(a^{-1}) \cup C_G(b)| \leq 6
\]

Hence \( |G \setminus C_G(a) \cup C_G(b)| \leq 6 \). Clearly \( |G| \in \{8,10,12,14,16,20\} \). If \( |G| = 12 \), then \( G \cong A_4, D_{12} \) or \( T \). As before \( A_4 \) is not a \( C(3,7) \)-group. For \( D_{12} \), the subsets \( M = \{b,ba^3,ba^5\} \) and \( N = \{a, a^2, a^4, a^5, ba, ba^2, ba^4\} \) show that \( D_{12} \) is not a \( C(3,7) \)-group. For \( T \), the subsets \( M = \{y,yx^3,xy^2\} \) and \( N = \{x,x^2,x^4,x^5,yx,xy,yx^2\} \) show that \( T \) is not a \( C(3,7) \)-group. For \( |G| = 14 \), \( G \cong D_{14} \) and there exists an element \( x \in D_{14} \) such that \( |C_{D_{14}}(x)| = 7 \), showing that \( D_{14} \) is not a \( C(3,7) \)-group. If \( |G| = 16 \), then \( G \) has centralisers of order 8. By Remark 4.5(1), \( G \) is not a \( C(3,7) \)-group. Every non-Abelian group of order 20, has centralisers of order 4, and by Remark 4.5(1), is not a \( C(3,7) \)-group.

**LEMMA 4.9.** If \( G \) is a \( C(4,6) \)-group and \( Z(G) \neq 1 \), then \( G \cong Q_8 \) or \( D_8 \).

**PROOF:** By Lemma 3.3, \( G/(Z(G)) \) is an Abelian group and by Lemma 4.2, \( \prod_{p | |G|} p \leq 5 \). Thus \( G \) is a \( p \)-group for \( p \in \{2,3,5\} \). If \( G \) is a 5-group, then there exists an element \( a \) in \( G \setminus Z(G) \) whose order is 5. Thus

\[
|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(a^4)| \leq 5.
\]

Hence \( |G \setminus C_G(a^2)| \leq 5 \) and therefore \( |G| \leq 10 \), which is a contradiction. If \( G \) is a 3-group, then by the proof of Theorem 1.1, \( Z(G) = \langle z \rangle \), and there exists an element \( a \) in \( G \setminus Z(G) \) such that

\[
|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(az) \cup C_G(az^2)| \leq 5
\]
Hence $|G \setminus C_G(a)| \leq 5$ and so $|G| \leq 10$, which is not possible. Therefore $G$ is a 2-group and by the proof of Theorem 1.1, $|Z(G)| = 2$ or 4. Let $|Z(G)| = 2$ and $Z(G) = \langle z \rangle$. Then there exists an element $a$ in $G \setminus Z(G)$ of order 4. Now we distinguish two cases:

**CASE 1:** $a^2 \notin Z(G)$. In this case

$$|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(a^4)| \leq 5.$$ 

Hence $|G \setminus C_G(a^2)| \leq 5$, so that $|G| \leq 10$, which cannot happen.

**CASE 2:** $a^2 \in Z(G)$. In this case there exists an element $b$ in $G \setminus \langle a \rangle$ such that

$$|G \setminus C_G(a) \cup C_G(a^{-1}) \cup C_G(b) \cup C_G(ba^2)| \leq 5 \quad \text{and} \quad |G \setminus C_G(a) \cup C_G(b)| \leq 5.$$ 

Clearly $|G| = 8$. Now suppose that $|Z(G)| = 4$. Say, $Z(G) = \{1, z_1, z_2, z_3\}$. There exists an element $a$ in $G \setminus Z(G)$ of order 4 such that $a^2 \neq z_1$, and

$$|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(az_1)| \leq 5.$$ 

Therefore $|G| \leq 10$, which is not possible again.

**LEMMA 4.10.** Let $G$ be a $C(4, n)$-group, where $4 \leq n \leq 6$. Then $G \cong S_3$, $Q_8$, $D_8$, or $D_{10}$.

**PROOF:** By Remark 4.5(2), it is enough to consider only the case $n = 6$. Let $a \in G \setminus Z(G)$. By Lemma 4.1, $|a| \in \{2, 3, 4, 5, 6, 8, 10, 12\}$. Let $Z(G) = 1$. We distinguish three cases:

**CASE 1.** $|a| > 5$. In this case $|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(a^4)| \leq 5$ and $|G| \leq 10$.

**CASE 2.** $|a| = 4$. For $b$ in $G \setminus \langle a \rangle$, we have $|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(b)| \leq 5$, from which it follows that $|G| \in \{8, 10, 12\}$. For $|G| = 12$, then $G \cong A_4$, but the subsets

$$M = \{(12)(34), (13)(24), (14)(23), (123)\} \quad \text{and} \quad N = \{(124), (142), (134), (143), (234), (243)\}$$

show that $A_4$ is not a $C(4, 6)$-group.

**CASE 3.** $|a| \in \{2, 3\}$. In this case there exist elements $a$ and $b$ in $G$ of order 2 and 3, respectively.

**CASE 3(i).** Suppose that there exists an element $c$ in $G \setminus \langle b \rangle$ of order 3. Then

$$|G \setminus C_G(b) \cup C_G(b^{-1}) \cup C_G(c) \cup C_G(c^{-1})| \leq 5,$$

from which it follows that $|G| = 10$, 12 or 14 so that $G \cong D_{10}$, $A_4$ or $D_{14}$. The group $D_{14}$ has centraliser of order 7, and by Remark 4.5(1), it is not a $C(4, 6)$-group.
CASE 3(ii). Every $c$ in $G \setminus \langle b \rangle$ has order two. Let $a_1, a_2, a_3, a_4, a_5$ and $a_6$ be elements of order two. Then

$$G = C_G(a_1) \cup C_G(a_2) \cup C_G(a_3) \cup C_G(a_4) \cup C_G(a_5) \cup C_G(a_6) \cup C_G(b).$$

Now by [2, Theorem B], $|G| \leq 81$. But $|G| = 2^k \cdot 3$ and hence $|G| \in \{6, 12, 24, 48\}$. Since $A_4$ and $S_4$ are the only centreless groups of order 12 and 24 respectively which are not $C(4,6)$-groups, $|G| \neq 12$ or 24.

Finally any centreless group of order 48, has more than two elements of order 3, so that $|G| \neq 48$. Now if $Z(G) \neq 1$, then by Lemma 4.9, $G \cong Q_8$ or $D_8$, and the proof is complete.

**Lemma 4.11.** If $G$ is a $C(5, 5)$-group, then $G \cong S_3$, $Q_8$, $D_8$ or $D_{10}$.

**Proof:** A similar proof to that of Lemma 4.9, gives the result.

**Proof of Theorem 1.4:** It follows easily from Lemmas 4.6–4.11.

**References**


