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SOME INEQUALITIES FOR NORM UNITARIES IN BANACH ALGEBRAS

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1. Introduction

Let A be a complex unital Banach algebra. An element $u \in A$ is a *norm unitary* if

$$\|u\| = \|u^{-1}\| = 1.$$

(For the algebra of all bounded operators on a Banach space, the norm unitaries are the invertible isometries.) Given a norm unitary $u \in A$, we have $\mathrm{Sp}(u) \subset \Gamma$, where $\mathrm{Sp}(u)$ denotes the spectrum of u and Γ denotes the unit circle in $\boldsymbol{C}$. If $\mathrm{Sp}(u) \neq \Gamma$ we may suppose, by replacing u by $e^{i\theta}u$, that $-1 \notin \mathrm{Sp}(u)$. Then there exists $h \in A$ such that

$$u = e^{ih}, \quad \mathrm{Sp}(h) \subset]-\pi, \quad \pi[.$$

Since $\|u^n\| = 1$ $(n \in \boldsymbol{Z})$ it follows that $\{e^{i\xi h} : \xi \in \boldsymbol{R}\}$ is norm bounded; thus h is Hermitian equivalent (see (4, §5)), i.e. there exists an equivalent algebra norm $|.|$ on A for which $|e^{i\xi h}| = 1$ $(\xi \in \boldsymbol{R})$. Prompted by questions of L. A. Harris concerning J^*-algebras and Banach Lie groups (see (5)), we wish to determine the best bounds for $\|e^{i\xi h}\|$ without renorming. To be precise we wish to determine the best bound for $\|e^{i\xi h}\|$ subject to $\|e^{\pm ih}\| = 1$, $r(h) \leq \tau$ where $r(h)$ denotes the spectral radius of h and $0 < \tau < \pi$. By using results on entire functions of exponential type we are able to give precise best bounds for some cases in which τ is a rational multiple of π. We show that the extremal algebra subject to $\|e^{\pm ih}\| = 1$, $r(h) \leq \tau$ can be identified with a quotient algebra of the Wiener algebra. By the same technique we are able to refine S. Kaijser's generalisation of a result of N. Th. Varopoulos (6) concerning inequalities between $r(e^{ih} - 1)$ and $\|e^{ih} - 1\|$.

2. Preliminaries on entire function of exponential type

Let g be an entire function and let

$$M(r) = \sup\{|g(z)| : |z| = r\}.$$

We recall that g is of *order* ρ if

$$\limsup_{r\to\infty} \frac{\log \log M(r)}{\log r} = \rho$$

and then g is of *type* τ if

$$\limsup_{r\to\infty} r^{-\rho} \log M(r) = \tau.$$

An entire function g is of *exponential type* τ if g has order less than 1 or if g has order 1 and type equal to or less than τ.

Let $\mathcal{F}(\tau)$ be the set of all entire functions g of exponential type τ for which

$$|g(k)| \le 1 \quad (k \in \boldsymbol{Z})$$

and let

$$\mu(\tau) = \sup\{|g(\xi)| : \xi \in \boldsymbol{R}, \quad g \in \mathcal{F}(\tau)\}.$$

We shall use the following results, most of which are implicit in R. P. Boas (**2**).

Theorem 1. *Let $g \in \mathcal{F}(\tau)$ where $0 < \tau < \pi$.*

(i) *For $\xi \in \boldsymbol{R} \backslash \boldsymbol{Z}$,*

$$g(\xi) = \frac{\sin \pi\xi}{\pi(\pi - \tau)} \sum_{k \in Z} \frac{(-1)^k g(k)}{(k-\xi)^2} \sin[(\pi - \tau)(\xi - k)].$$

(ii) *For $0 < \alpha < 1$, $\beta = 1 - \alpha$,*

$$g(0) \sin \beta\tau + g(1) \sin \alpha\tau = \sum_{k \in Z} c_k g(\alpha + k\pi/\tau)$$

where

$$c_k = (-1)^k \frac{\sin \alpha\tau \sin \beta\tau}{(\alpha + k\pi/\tau)(\beta - k\pi/\tau)}.$$

(iii) $\mu(\tau) \le \sec \frac{1}{2}\tau$.

(iv) $|g(1) - g(0)| \le 2 \sin \frac{1}{2}\tau \sup\{|g(\frac{1}{2} + (k - \frac{1}{2})\pi/\tau)| : k \in \boldsymbol{Z}\}$.

Proof. (i) In (**2**, (10.2.9)) let $\delta \to \pi - \tau$ and apply the dominated convergence theorem.

(ii) Apply the method of (**2**, §11.2). In the notation of Boas, take

$$\lambda(\xi) = e^{-i\alpha\xi} \sin \beta\tau + e^{i\beta\xi} \sin \alpha\tau.$$

Note that $\lambda(\tau) = \lambda(-\tau)$ and $\lambda(\xi) = \sum_{k \in Z} c_k e^{ik\pi\xi/\tau}$ with c_k as in the statement of the theorem. Since

$$L[g(z)] = g(z - \alpha) \sin \beta\tau + g(z + \beta) \sin \alpha\tau$$

we conclude that

$$g(\xi - \alpha) \sin \beta\tau + g(\xi + \beta) \sin \alpha\tau = \sum_{k \in Z} c_k g(\xi + k\pi/\tau).$$

Put $\xi = \alpha$.

(iii) Given $g \in \mathcal{F}(\tau)$, $k \in \boldsymbol{Z}$, we note that $z \to g(z + k)$ is also in $\mathcal{F}(\tau)$. It follows that

$$\mu(\tau) = \sup\{|g(\xi)| : 0 \le \xi \le 1, \ g \in \mathcal{F}(\tau)\}.$$

By Montel's theorem and the compactness of $[0, 1]$, there exists $g_0 \in \mathcal{F}(\tau)$ and $0 < \alpha < 1$ such that $\mu(\tau) = |g_0(\alpha)|$. By applying part (ii) with $g(z) = \cos \tau(z - \frac{1}{2})$ we find that

$$\cos \tfrac{1}{2}\tau(\alpha-\beta)\Big[|c_0| - \sum_{k\neq 0} |c_k|\Big] = \cos \tfrac{1}{2}\tau[\sin \alpha\tau + \sin \beta\tau].$$

By applying part (ii) with g_0 we find that

$$\begin{aligned}\sin \alpha\tau + \sin \beta\tau &\geq |g_0(1) \sin \alpha\tau + g_0(0) \sin \beta\tau| \\ &= \Big|\sum_{k\in Z} c_k g_0(\alpha + k\pi/\tau)\Big| \\ &\geq |c_0|\,|g_0(\alpha)| - \mu(\tau) \sum_{k\neq 0} |c_k| \\ &= \mu(\tau)\Big[|c_0| - \sum_{k\neq 0} |c_k|\Big].\end{aligned}$$

It follows that

$$\mu(\tau) \leq \sec \tfrac{1}{2}\tau \cos \tau(\alpha - \tfrac{1}{2}) \leq \sec \tfrac{1}{2}\tau.$$

(iv) This follows from the details of the proof of (**2**, (11.4.2)).

For a discussion of estimates for $\mu(\tau)$, see (**2**, p. 203). In particular it should be noted that there is some overlap between the results in this paper and the work of S. N. Bernstein (**1**), although rather different arguments are employed by Bernstein. The exact value of $\mu(\tau)$ is given below for the cases $\tau = \pi/m$, $\pi(1 - 1/m)$ where $m \in \boldsymbol{N}$, $m \geq 2$.

3. Estimates for $\|e^{i\xi h}\|$

Theorem 2. *Let $h \in A$ with $\|e^{\pm ih}\| = 1$ and $r(h) \leq \tau$.*
(i) *If $\tau = \pi/m$ with $m \in \boldsymbol{N}$, $m \geq 2$, then $\|e^{i\xi h}\| \leq \phi_m(\xi)$ $(0 \leq \xi \leq 1)$ where*

$$\phi_m(\xi) = \sec \tfrac{1}{2}\pi/m \cos(\xi - \tfrac{1}{2})\pi/m.$$

(ii) *If $\tau = \pi(1 - 1/m)$ with $m \in \boldsymbol{N}$, $m \geq 2$, then $\|e^{i\xi h}\| \leq \psi_m(\xi)$ $(0 \leq \xi \leq 1)$ where*

$$\psi_m(\xi) = \frac{\sin \pi\xi}{m} \sum_{k=1}^{m} \operatorname{cosec}(k - \xi)\pi/m.$$

The above inequalities are best possible.

Proof. Given $f \in A'$, $\|f\| \leq 1$, let $g(z) = f(e^{izh})$ $(z \in \boldsymbol{C})$. Then g is entire and $|g(z)| \leq \|e^{izh}\|$. Since $r(h) \leq \tau$ it follows that g is of exponential type τ, and since $\|e^{inh}\| = 1$ $(n \in \boldsymbol{Z})$ we have $g \in \mathscr{F}(\tau)$. To obtain estimates for $\|e^{i\xi h}\|$ it is now sufficient, by the Hahn-Banach theorem, to obtain the corresponding estimates for $|g(\xi)|$ with $g \in \mathscr{F}(\tau)$.

(i) Let $\tau = \pi/m$, $m \in \boldsymbol{N}$, $m \geq 2$. Given $0 < \xi < 1$ let

$$\mu_\xi = \sup\{|g(\xi)| : g \in \mathscr{F}(\tau)\}.$$

As in the proof of Theorem 1 (iii) there exists $g_0 \in \mathscr{F}(\tau)$ with $|g_0(\xi)| = \mu_\xi$ and the same argument gives

$$\mu_\xi \leqslant \sec \tfrac{1}{2}\tau \cos \tau(\xi - \tfrac{1}{2}).$$

(ii) Let $\tau = \pi(1 - 1/m)$, $m \in \boldsymbol{N}$, $m \geqslant 2$. Note that ψ_m is of exponential type τ. For $k = 1, 2, \ldots, m$ and $0 \leqslant \xi \leqslant 1$, we have $\psi_m(k) = (-1)^{k-1}$, $\sin(\pi - \tau)(\xi - k) < 0$, and hence

$$(-1)^k \psi_m(k) \sin(\xi - k)\pi/m = |\sin(\xi - k)\pi/m|.$$

This last equation holds for all k since each side is unchanged when k is replaced by $k + m$. Given $g \in \mathscr{F}(\tau)$, Theorem 1(i) now gives for $0 < \xi < 1$,

$$\begin{aligned} |g(\xi)| &\leqslant \frac{m \sin \pi\xi}{\pi^2} \sum_{k \in Z} \frac{|\sin(\xi - k)\pi/m|}{(k - \xi)^2} \\ &= \frac{m \sin \pi\xi}{\pi^2} \sum_{k \in Z} \frac{(-1)^k \psi_m(k)}{(k - \xi)^2} \sin(\xi - k)\pi/m \\ &= \psi_m(\xi). \end{aligned}$$

Finally we show that these inequalities are best possible, even for finite dimensional algebras. We write

$$\omega_s(t) = e^{ist} \quad (t \in \boldsymbol{R}).$$

For the case $\tau = \pi/m$, let E_m be the linear span of $\omega_{\pi/m}$, $\omega_{-\pi/m}$ with norm defined by

$$\|g\| = \sup\{|g(k)| : k \in \boldsymbol{Z}\}.$$

(Theorem 1(i) shows that $g = 0$ whenever $\|g\| = 0$.) Let D be the bounded linear operator on E_m given by

$$Dg = -ig' \quad (g \in E_m).$$

Then $\mathrm{Sp}(D) = \{\pm\pi/m\}$. Taylor's theorem gives

$$e^{i\xi D} g(t) = g(\xi + t)$$

and hence $\|e^{\pm iD}\| = 1$. Consideration of the function

$$g(t) = \cos(t - \tfrac{1}{2})\pi/m$$

shows that $\|e^{i\xi D}\| \geqslant \phi_m(\xi)$ $(0 \leqslant \xi \leqslant 1)$, as required. For the case $\tau = \pi(1 - 1/m)$ let F_m be the linear span of

$$\{\omega_{k\pi/m} : k = 0, \pm 1, \pm 2, \ldots, \pm(m-1)\}.$$

With $\|.\|$ and D as above we obtain $\|e^{\pm iD}\| = 1$, $r(D) = \pi(1 - 1/m)$. Since $\psi_m \in F_m$ we also obtain $\|e^{i\xi D}\| \geqslant \psi_m(\xi)$ $(0 \leqslant \xi \leqslant 1)$, as required.

Let W denote the Wiener algebra of all continuous functions x on $[-\pi, \pi]$ with absolutely convergent Fourier series

$$x(t) = \sum_{k \in Z} c_k e^{ikt} \quad (-\pi \leqslant t \leqslant \pi)$$

and with the norm

$$\|x\| = \sum_{k \in Z} |c_k|.$$

Given $0 < \tau < \pi$ let $\ker[-\tau, \tau]$ be the closed ideal of W given by

$$\ker[-\tau, \tau] = \{x \in W : x(t) = 0 \quad (-\tau \leqslant t \leqslant \tau)\}.$$

Let $(A_\tau, \|.\|_\tau)$ denote the quotient algebra $W/\ker[-\tau, \tau]$. It is elementary that A_τ may be identified with the algebra of continuous functions x on $[-\tau, \tau]$ which have representations

$$x(t) = \sum_{k \in Z} c_k e^{ikt} \quad (-\tau \leqslant t \leqslant \tau)$$

and then

$$\|x\|_\tau = \inf\left\{\sum_{k \in Z} |c_k| : \text{all such representations}\right\}.$$

We note that $\eta \in A_\tau$ where $\eta(t) = t(-\tau \leqslant t \leqslant \tau)$. It is now a standard argument to show that A_τ is the extremal algebra with respect to the constraints $\|e^{\pm ih}\| = 1$, $r(h) \leqslant \tau$. In particular we have the following result.

Theorem 3. *Let $h \in A$ with $\|e^{\pm ih}\| = 1$, $r(h) \leqslant \tau$, where $0 < \tau < \pi$. Then for $0 \leqslant \xi \leqslant 1$*

$$\|e^{i\xi h}\| \leqslant \|e^{i\xi\eta}\|_\tau = \sup\{|g(\xi)| : g \in \mathcal{F}(\tau)\}.$$

Proof. Let $\{c_k\}$ be any absolutely summable sequence with

$$e^{i\xi t} = \sum_{k \in Z} c_k e^{ikt} \quad (-\tau \leqslant t \leqslant \tau).$$

For any entire function g of exponential type τ bounded on $\boldsymbol{R}$ the method of (2, §11.2) gives

$$g(\xi) = \sum_{k \in Z} c_k g(k).$$

In particular, given $f \in A'$ let

$$g(z) = f(e^{izh}) \quad (z \in \boldsymbol{C})$$

and we obtain

$$f(e^{i\xi h}) = \sum_{k \in Z} c_k f(e^{ikh}) = f\left(\sum_{k \in Z} c_k e^{ikh}\right).$$

The Hahn-Banach theorem gives

$$e^{i\xi h} = \sum_{k \in Z} c_k e^{ikh}$$

and so

$$\|e^{i\xi h}\| \leqslant \sum_{k \in Z} |c_k|.$$

Since this holds for all representations of $e^{i\xi t}$ we conclude that $\|e^{i\xi h}\| \leqslant \|e^{i\xi\eta}\|_\tau$.

The method of (3) shows that A'_τ may be identified with the space of all entire functions g of exponential type τ bounded on $\boldsymbol{R}$ with norm given by

$$\|g\| = \sup\{|g(k)| : k \in \boldsymbol{Z}\}.$$

Under the identification we obtain

$$\|e^{i\xi\eta}\|_\tau = \sup\{|g(\xi)| : g \in \mathscr{F}(\tau)\}.$$

Remarks. (1) We note that for $m \in \boldsymbol{N}$, $m \geq 2$

$$\mu(\pi/m) = \sec(\tfrac{1}{2}\pi/m) = 1 + O(m^{-2})$$

$$\mu(\pi(1-1/m)) = \frac{1}{m}\sum_{k=1}^{m} \operatorname{cosec}(k-\tfrac{1}{2})\pi/m = O(\log m).$$

(2) Given $\|e^{\pm ih}\| = 1$, $r(h) \leq \tau$, we need not have h Hermitian no matter how small τ is. By taking a direct sum of the spaces E_m of Theorem 2 we may obtain a Banach space X for which there is no $\epsilon > 0$ such that every invertible isometry L with $\|I - L\| < \epsilon$ is of the form $L = e^{iH}$ with H Hermitian. This is essentially the example in (5) in which the group of invertible isometries of X is not a Banach Lie group.

(3) Given $\|e^{\pm ih}\| = 1$, $r(h) < \pi$ there is no uniform bound on $\|e^{(1/2)ih}\|$.

(4) When $\tau = \frac{1}{2}\pi$ it is easily verified that the operator D on E_2 is isometrically equivalent to the matrix

$$\begin{pmatrix} 0 & \frac{1}{2}\pi \\ \frac{1}{2}\pi & 0 \end{pmatrix}$$

acting on $\boldsymbol{C}^2$ with the sup norm.

(5) It would be useful to have a periodic extension of $e^{(1/2)it}$ from $(-\tau, \tau]$ to $[-\pi, \pi]$ whose Fourier coefficients give $\|e^{(1/2)i\eta}\|_\tau$. For the case $\tau = \pi(1-1/m)$ it may be verified that the required extension is provided by

$$\frac{m}{\pi}(\pi - t)e^{(1/2)it} \quad (\pi - \pi/m \leq t \leq \pi + \pi/m).$$

For the case $\tau = \pi/m$ the extension is again of the form $\chi(t)e^{(1/2)it}$ where χ is piecewise linear. It would be of interest to resolve the case $\tau = r\pi$ where r is rational, $0 < r < 1$.

4. Estimates for $\|e^{ih} - 1\|$

In the course of proving that Kronecker sets are Helson sets, Varopoulos (6) established for the Wiener algebra that if $\|u_n\| = \|u_n^{-1}\| = 1$ and $r(u_n - 1) \to 0$ then $\|u_n - 1\| \to 0$. S. Kaijser informed us that the argument in (6) would give for an arbitrary Banach algebra essentially the following result.

"If $\|u\| = \|u^{-1}\| = 1$, $r(u-1) \leq 2\sin\pi/4m$ where $m \in \boldsymbol{N}$, then $\|u - 1\| \leq \pi/me^{(1/2)\pi/m}$."

We give below a sharper version of this (unpublished) result of Kaijser.

Theorem 4. *Let* $\|u\| = \|u^{-1}\| = 1$, $r(u-1) \leq 2\sin\frac{1}{2}\tau$ *where* $0 < \tau < \pi$.

(i) $\|u - 1\| \leq 2\tan\frac{1}{2}\tau$.

(ii) *The inequality is best possible if* $\tau = \pi/m$, $m \in \boldsymbol{N}$, *m even.*
(iii) *If* $\tau = \pi/m$, $m \in \boldsymbol{N}$, *m odd, then* $\|u-1\| \leq 2\sin\frac{1}{2}\tau$.

Proof. (i) The conditions on u give $u = e^{ih}$, $r(h) \leq \tau$. Let $f \in A'$ with $\|f\| \leq 1$ and let

$$g(z) = f(e^{izh}) \quad (z \in \boldsymbol{C}).$$

Theorem 1 (iii) and (iv) gives

$$|g(1) - g(0)| \leq 2\sin\tfrac{1}{2}\tau \sec\tfrac{1}{2}\tau = 2\tan\tfrac{1}{2}\tau$$

and the Hahn-Banach theorem gives $\|u-1\| \leq 2\tan\frac{1}{2}\tau$.

(ii) Let $\tau = \pi/m$, $m \in \boldsymbol{N}$, m even. Let E_m, D be as in Theorem 2. Then

$$(e^{iD} - I)g(0) = g(1) - g(0)$$

and the function $g(t) = \sec\frac{1}{2}\pi/m \sin(t - \frac{1}{2})\pi/m$ gives

$$\|e^{iD} - I\| \geq 2\tan\tfrac{1}{2}\pi/m.$$

(iii) For the case m odd, Theorem 1(iv) gives

$$|g(1) - g(0)| \leq 2\sin\tfrac{1}{2}\tau \sup\{|g(k)| : k \in \boldsymbol{Z}\} \leq 2\sin\tfrac{1}{2}\tau$$

and hence $\|u-1\| \leq 2\sin\frac{1}{2}\tau$.

Corollary. If $\|u\| = \|u^{-1}\| = 1$, $r(u-1) = 2\sin\frac{1}{2}\pi/m$ where $m \in \boldsymbol{N}$, m odd, then $\|u-1\| = r(u-1)$.

REFERENCES

(1) S. N. Bernstein, The extension of properties of trigonometric polynomials to entire functions of finite degree, *Izv. Akad. Nauk*, **12** (1948), 421–444 (Russian).

(2) R. P. Boas, *Entire functions* (Academic Press, 1954).

(3) B. Bollobás, The numerical range in Banach algebras and complex functions of exponential type, *Bull. London Math. Soc.* 3 (1971), 27–33.

(4) F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras* (London Math. Soc. Lecture Note Series, No. 2, Cambridge University Press, 1971).

(5) L. A. Harris and W. Kaup, Linear algebraic groups in infinite dimensions, *Illinois J. Math.* **21** (1977), 666–674.

(6) N. Th. Varopoulos, Sur les ensembles parfaits et les series trigonométriques, *C.R. Acad. Sci. Paris* **260** (1965), 3831–3834.

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