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STOCHASTIC TRAJECTORIES OF SPACE VEHICLES WITH GRAVITATIONAL MANOEUVRES

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ABSTRACT

The theoretical aspect of close flybies of space-crafts near planets-gravitational manoeuvres have been studied. It is proved that the space-craft motion with gravitational manoeuvres admits the existence of routing scheme and, consequently shows the existence of quasi-random motion.

INTRODUCTION

The practical application of close flybies of spacecrafts near planets - gravitational manoeuvres - is wide-spread and usual in space research. Gravitational manoeuvres minimize flytime or fuel expense and give diverse opportunities to choose variants of space missions.

This is the applied aspect of the problem. But it is not less interesting from theoretical view point: the trajectories with numerous flybies prove to be an example of stochastic motions in classical deterministic dynamical system. It is this side of the problem that is under consideration in the present paper.

The cause of stochasticity is, of principle, evident: this is instability, leading to the expansion of initial tube of trajectories in the course of time (Fig. 1). The negligible small deviations of elements of initial orbit result in small changes of impact parameter, but the latter cause considerable changes of orbital elements after encounter. As a result, the dynamical system rapidly "forgets" its initial state.

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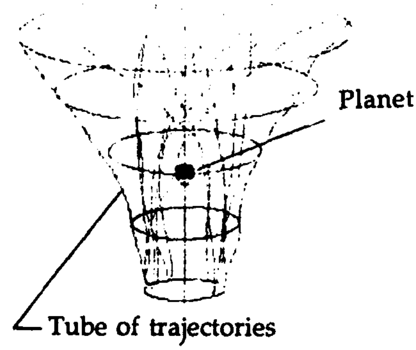

Fig. 1. The cause of stochasticity is unstability leading to the expansion of initial tube of trajectories.

Let us start with giving more rigorous mathematical meaning to the statement "the motion has stochastic properties". Introduce the concept "quasi-random motion", following V.M. Alekseev (1981). Suppose the dynamical system allows the transformation to a system with discrete time $t \in \{t_n\}_{n=-\infty}^{\infty}$ i.e. to the iterations of some phase space transformation. Let us assume, further, that any motion is such as $\forall n \; x_n \equiv x(t_n) \in \ell_n$, $L \equiv \bigcup_i \ell_i$ being the totality of non-overlapping subsets of the phase space. Hence, any motion is to be represented by a sequence of the "letters" of the "alphabet" L:

$$\ldots \ell_1 \ell_2 \ell_3 \ldots$$

The sequence for which corresponding motion exists is called admissible. A part of admissible sequence of the length 2 (i.e.consisted of two "letters") is called admissible transition. One can say that the motion is quasi-random if it's impossible to predict the next letter of the sequence, provided the fragment of admissible sequence of arbitrary length is known.

The another (probably,more understandable) way to describe these ideas is to use the notion of routing scheme. The latter is the oriented graph, its nodes being the "letters" of the "alphabet" L,and the presence of arc (with arrow prescribed) between two nodes means that corresponding transition is admissible.The motion is called quasi-random if it may be represented by routing scheme with two or more arrows starting from each node (Fig.2).

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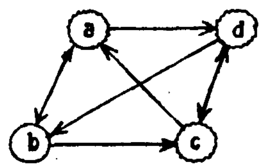

Fig. 2. An example of quasi-random motion represented by routing scheme.

If the part of the admissible sequence is known (e.g., ...abc...), it is impossible to find its unique continuation. The continuation ...abca... is admissible as well as ...abcd....

Our goal is to prove that the problem of spacecraft motion with gravitational manoeuvres admits the existence of routing scheme and, consequently, there exist quasi-random motions in this problem.

THE MODEL

Let us consider the simplest problem of celestial mechanics revealing stochastic properties - the plane circular restricted three-body problem Sun-planet-spacecraft (Fig. 3). Moreover, for the sake of simplicity let us treat now the

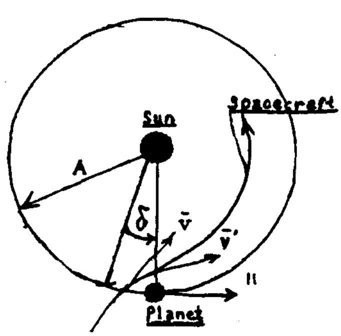

Fig. 3 The plane circular restricted three-body problem Sun-planet-spacecraft.

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approximate model of this problem (although the latter is already a model problem). Namely, we shall use the method of action spheres with zero radii (or, briefly, point-like action spheres method). It can be summarized as follows. The spacecraft motion is supposed to be Keplerian heliocentric everywhere except for points of encounter with planet, where instant change of velocity vector occurs. This change is reduced to the rotation of velocity vector to the angle ψ (between the asymtotes of planetocentric hyperbola). Any flyby, therefore, looks like a "collision" of spacecraft with planet.

Let heliocentric gravitational constant be equal to unity whereas planetocentric one equals to μ. The planet moves on the circular orbit of radius A with linear velocity $u = 1/\sqrt{A}$. Denote heliocentric velocity vector of spacecraft as $\bar{v}$, planetocentric one as $\bar{w}$.

It is easy to deduce formulas expressing the change of spacecraft orbit as a result of a collision with planet:

$$v'^2 = v^2 + \frac{4uz}{1+z^2}[z(u - v^{(n)}) - v^{(r)}] \quad (1)$$

$$v^{(n)'} = (v'^2 + u^2 - w^2)/2u \quad (2)$$

$$v^{(r)'^2} = v'^2 - v^{(n)'^2} \quad (3)$$

Here the prime indicates to values immediately after collision, whereas its absence corresponds to values directly before it. The quantity $z \equiv \tan(\psi/2)$ may be referred to as flyby parameter. It is connected with the impact parameter ρ by known relation

$$z = \mu/w^2 \rho \quad (4)$$

One can obtain an approximate expression for ρ by means of linearization of spacecraft motion in the vicinity of the planet. With technical details omitted, the final result is

$$\rho \approx A|\delta| \, |v^{(r)}| w^{-1} , \quad (5)$$

δ being the angular distance from spacecraft to planet while the unperturbed spacecraft is crossing the planet orbit.

Let us recall, finally, the well-known Keplerian relations:

$$v^2 = 2/A - 1/a, \quad (6)$$

$$v^{(n)2} = a(1-e^2)/A^2, \quad (7)$$

with a and e standing for major semiaxis and eccentricity of spacecraft orbit, respectively.

The set of formulas (1)-(7) completely describes the transformation of Keplerian heliocentric orbit of spacecraft as a result of a flyby:

$$(a,e) \rightarrow (a',e').$$

STOCHASTIC PROPERTIES

In many well-known stochastic dynamical systems the repetition of collisions is guaranteed automatically (e.g. in billiards). It is not the case here: we ought to take care of such a repetition. Namely, one have to choose the parameters of each flyby such as to reach next flyby in the nearest future.

This condition is satisfied particularly for resonant orbits, for which the next collision occurs at the same point as previous one after p completes revolutions of planet and q completes revolutions of spacecraft:

$$(a/A)^{3/2} = p/q, \quad p,q \in \mathbb{N} \tag{8}$$

The non-resonant orbits (also containing the orbits of collision) were also investigated (Sokolov, 1990) but are not discussed here.

Restricting ourselves by the consideration of resonant orbits we have:

$$\delta_{i+1} = \delta_i + 2\pi A^{-3/2}[q_i(2/A - v_i^2)^{-3/2} - p_i A^{3/2}] \tag{9}$$

$$z_{i+1} = \frac{\mu}{A|\delta_{i+1}|v_i^{(r)} (v_i^2 + u^2 - 2u\, v_i^{(n)})^{1/2}} \tag{10}$$

$$v_{i+1}^2 = v_i^2 + \frac{4u\, z_{i+1}}{1+z_{i+1}^2}[z_{i+1}(u - v_i^{(n)}) - v_i^{(r)}] \tag{11}$$

$$v_i^{(n)} = (v_i^2 + u^2 - w^2)/2u, \quad v_i^{(r)^2} = v_i^2 - v_i^{(n)^2} \tag{12}$$

Here $i = 1,2,3,\ldots$ stands for the number of collision, $v_i = \{v_i^{(n)}, v_i^{(r)}\}$ is the velocity after i-th collision (the same as the velocity directly before i+1-th collision, because the orbits are resonant).

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It can be easily shown that for each set of pairs (p_i, q_i), $i = 1,2,3,\ldots$, the system of equations (9)-(12) has an exact solution, provided μ is sufficiently small. In order to show it, let us fix the value of planetocentric velocity w (w is an invariant, because its values before and after each collision coincide). Choose an arbitrary sequence of resonant heliocentric velocity values v_i (corresponding to the values of p_i, q_i chosen from the interval (u-w, u+w), restricting ourselves by not too large values of p_i, q_i. Evaluate normal and radial components $v_i^{(n)}$ and $v_i^{(r)}$ by means of (12). The quadratic equation (11) determines z_i (one can prove the existence of real roots). The next step is to calculate δ_i with the aid of (10). Finally, we use formula (9) either to obtain δ_1 (when i = 1) or to correct the value of v_i (when i= 2,3,...). The corrections to $v_i^{(n)}$ and $v_i^{(r)}$ are to be found now from (12). These operations are to be repeated until the required accuracy is achieved (Fig. 4). This iterative procedure, performed for all transitions, results in the set of values v_i. Note that the iterations converge as geometric progression with factor $\sim\mu$, independently of the path in the routing scheme. It can be clarified by the presence of small parameter μ in (10) which provides the "compression". Indeed, if the correction to z is of order μ^k, the correction to δ (and v) is of order μ^{k+1}.

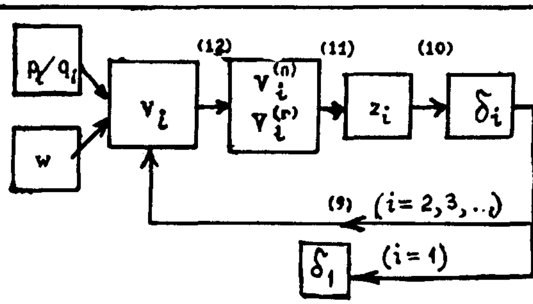

Fig. 4 The iterative procedure of V_{pq} calculation.

Therefore, the existence of the trajectory corresponding to each sequence of (p_i, q_i) is proved. Thus, there exists routing scheme - the infinite oriented graph - with at least two arcs originating in each node (Fig. 5). We can conclude therefore that the motion is quasi-random according to our definition. All routes are permitted -all pairs of nodes are linked by arrows. For each route within this scheme there exists a corresponding trajectory in the simplified three-body problem, e.g., the path 1/2 →2/3 → 1/1 →... on the Fig. 5 may have an arbitrary continuation. The spacecraft rambles through the nodes of scheme.

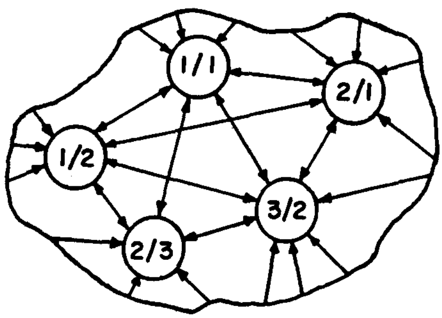

Fig. 5 Central part of the routing scheme. Each node contains the pair of values (p, q)

DISCUSSION

Thus, in the given specific problem of celestial mechanics an example of quasi-random motion has been constructed. It should be noted that we have considered not the "true" (actual) three-body problem but "distorted", "spoiled" by the using of two operations - the point-like action spheres method and linearization of spacecraft motion while evaluating the impact parameter. So the motions under consideration do not exist in reality. Nevertheless, they approximate actual solutions with sufficient accuracy - the fact that has been proved by the comparison with exact analytical results, in particular, with Tisserand criterion. It is especially important that quasi-random nature of the motions "survives" in the actual three-body problem. Let us quote Vladimir Arnold: "Unlike the stability, the instability is stable".

We shall come now to the other consequences of our treatment. It is clear that the measure phase space subset where the stochasticity is actually revealed is very small. On the contrary, the velocity of accuracy loss caused by consequent spacecraft flybies is considerable, amounting to 3-4 decimal orders per one flyby. As a consequence, it is absolutely useless to analyze an individual stochastic trajectory by numerical integration if the time interval is not small.

Further, it seems to be evident that the technique used in the paper may be generalized to more complicated cases of N-body problem. For example, one can consider four-body problem in which two planets like as Mercury and Venus are involved. It doesn't make difficulties also to study 3-dimensional (non-planar) problem. The list of possible generalizations is obviously to be continued.

The applied aspect of the problem mentioned above is also of great interest. It includes the investigation of the regions of attainability, the analysis of the possibilities to fall down to the Sun, or to escape from the Solar system, or to leave the ecliptic plane.

All these generalizations and aspects of the problem are considered in detail in a series of papers (Sokolov, 1990; Sokolov, Titov, 1990). These papers contain also the description of the interactive computer system constructing stochastic trajectories. It makes it possible to "grow" trees of trajectories by continuing appropriate branches and stopping bad ones, due to accepted criteria. Numerical examples of trajectories constructed using the described system are also presented.

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