Canad. J. Math. Vol. **60** (6), 2008 pp. 1387–1405

On n-Dimensional Steinberg Symbols

Fernando Pablos Romo

Abstract. The aim of this work is to provide a new approach for constructing n-dimensional Steinberg symbols on discrete valuation fields from $(n+1)$-cocycles and to study reciprocity laws on curves related to these symbols.

1 Introduction

In 1959 J.-P. Serre [16] introduced the local symbols on curves in order to study the factorization of a rational morphism of complete curves through a generalized Jacobian of Rosenlicht. The additive local symbol (the classic residue on curves) and the multiplicative local symbol (the tame symbol) are examples of these symbols.

For years, an open problem was to provide a "local" definition of the local symbols, because the definition offered by J.-P. Serre involved a "global" condition, *i.e.*, the reciprocity law (the Weil reciprocity law in the multiplicative case and the residue theorem in the additive one).

In 1968 J. Tate [17] gave a definition of the residues of differentials on curves in terms of traces of certain linear operators on infinite-dimensional vector spaces. Furthermore, he proved the residue theorem (the additive reciprocity law) from the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$. With this work, he gave a "local" construction of the additive local symbol (it depends only on the local ring of a closed point in an algebraic curve).

In 1989 Arbarello, De Concini, and Kac [1] solved the problem of the "local" characterization of the multiplicative local symbol by offering a new definition of the tame symbol of an algebraic curve from the commutator of a certain central extension of groups. Analogously to Tate's construction, those authors deduced the reciprocity law for a complete curve from the finiteness of its cohomology when the ground field is algebraically closed. In 2002, I provided an interpretation of their central extension in terms of determinants associated with vector subspaces[11]. The cohomological definition offered, which involves topics of Steinberg symbols, is valid for curves over a perfect field and gives a new method for studying arithmetic symbols. Moreover, the sign of the expression of the tame symbol appears in a natural way because I proved that the tame symbol is a distinguished element in the cohomology class induced by the commutator of the central extension.

In 2004 I also obtained a cohomological characterization of the tame symbol over an arbitrary discrete valuation field, where the sign was again obtained from a distin-

Received by the editors January 16, 2006; revised April 3, 2007.

This work is partially supported by the DGI research contract no. MTM2006-07618 and Castilla y León regional government contract SA071/04.

AMS subject classification: Primary: 19F15; secondary: 19D45, 19C09.

Keywords: Steinberg symbols, reciprocity laws, discrete valuation field, algebraic curves.

guished element in the cohomology class induced by the commutator of a determinantal central extension of groups [10].

Moreover, A. N. Parshin [12] and J. L. Brylinsky and D. A. McLaughlin [3] have studied symbols on surfaces that generalize the definition of J.-P. Serre on curves, and have offered reciprocity laws for them.

Explicitly, in 1985 Parshin introduced a symbol associated with a sequence $p \in C \subset S$, where C is a curve on an algebraic surface S, and p is a closed point of C. If f, g and h are three functions on S, the expression of the symbol is

$$(1.1)\quad \langle f,g,h\rangle_{(p,C)} = (-1)^{\alpha_{(p,C)}}\Big(\frac{f^{\nu_C(g)\cdot\overline{\nu_p}(h)-\nu_C(h)\cdot\overline{\nu_p}(g)}}{g^{\nu_C(f)\cdot\overline{\nu_p}(h)-\nu_C(h)\cdot\overline{\nu_p}(f)}}\cdot h^{\nu_C(f)\cdot\overline{\nu_p}(g)-\nu_C(g)\cdot\overline{\nu_p}(f)}\Big)_{|_C}(p),$$

where

$$\begin{aligned}\alpha_{(p,C)} = {}& \nu_C(f)\cdot\nu_C(g)\cdot\overline{\nu_p}(h) + \nu_C(f)\cdot\nu_C(h)\cdot\overline{\nu_p}(g) + \nu_C(g)\cdot\nu_C(h)\cdot\overline{\nu_p}(f)\\ &+ \nu_C(f)\cdot\overline{\nu_p}(g)\cdot\overline{\nu_p}(h) + \nu_C(g)\cdot\overline{\nu_p}(f)\cdot\overline{\nu_p}(h) + \nu_C(h)\cdot\overline{\nu_p}(f)\cdot\overline{\nu_p}(g)\end{aligned}$$

where ν_C is the discrete valuation induced by C (a codimension one subvariety of S), and $\overline{\nu_p}$ is a discrete valuation induced by the closed point p and a function z on S such that $\nu_C(z) = 1$.

We point out that this explicit expression is not completely due to Parshin: *i.e.,* the higher dimensional tame symbol was defined by Parshin up to the sign, and the full definition, including the sign, was given by I. Fesenko in his thesis in 1986, published in 1988 [4] (for the English translation, see [5]).

In [10] I offered a new interpretation of the Parshin symbol on an algebraic surface as iterated tame symbols in order to deduce a reciprocity law from the finiteness of the cohomology groups $H^0(C,\mathcal{O}_C)$ and $H^1(C,\mathcal{O}_C)$, and, have recently studied the 2-dimensional Steinberg symbols on arbitrary discrete valuation fields, which can be defined from 3-cocycles related to iterated commutators of central extensions of groups [8].

In [1,8,10,11] the authors offered a new method for studying the respective symbols, and in [8,10,11] I obtained a cohomological characterization of them from the commutators of central extensions of groups where the sign appears in a natural way.

This paper is devoted to studying n-dimensional generalizations of Serre's local symbols by extending the method given in [8] from $n = 3$ to $n \in \mathbb{N} - \{1\}$.

If F is a field and G is a commutative group, a map $\psi\colon F^{*(n+1)} \longrightarrow G$ is called an *n-dimensional Steinberg symbol* when:

(i) ψ is multiplicative in each argument:

$$\psi(\dots, f_i\cdot\widetilde{f_i},\dots) = \psi(\dots,f_i,\dots)\cdot\psi(\dots,\widetilde{f_i},\dots)$$

for all $f_i, \widetilde{f_i} \in F^*$.

(ii) $\psi(\dots,f,\dots,1-f,\dots) = 1$ for all $f \neq 1$.

Parshin [13], Brylinsky and McLaughlin [2], and Fesenko and S. Vostokov [4, 5] have studied these symbols on algebraic and analytic varieties, and have offered reciprocity laws for them.

In the above context, the aim of this paper is to offer a general theory for n-dimensional Steinberg symbols on discrete valuation fields and to contribute to a better understanding of these symbols. From the determinantal central extension offered in [10] we construct an $(n+1)$-cocycle that determines an element of the cohomology group $H^{n+1}(F^*, k^*)$ such that there exists a distinguished symbol in the cohomology class of the $(n+1)$-cocycle satisfying a condition that is the generalization to the n-dimensional case of one of the conditions given by J.-P. Serre to define the local symbol in the case $n = 1$. In this construction the sign of the explicit expression of the symbol also appears in a natural way.

We show that this symbol is an n-dimensional Steinberg symbol; that it generalizes the well-known explicit expression of the n-dimensional tame symbol; that it is possible to apply it to several geometric situations (where it is not necessary to assume that the ground field is algebraically closed); and that the reciprocity laws associated with complete curves C are again a direct consequence of the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$.

The organization of the paper is as follows:

Section 2 contains a brief summary of the results of the paper [8]. Section 3 deals with the main results of this work. Namely, if we consider a family of fields $\{F_i\}_{i\in\{0,1,\dots,n\}}$, such that F_i is a discrete valuation field for $i \in \{1, \dots, n\}$, $K(\nu_{F_i})$ is a finite separable extension of F_{i-1}, $\deg(\nu_{F_i}) = \dim_{F_{i-1}} K(\nu_{F_i})$, and $N_{K(\nu_{F_i})/F_{i-1}}$ is the norm of the extension of fields $F_{i-1} \hookrightarrow K(\nu_{F_i})$, and we set $F_0 = k$; then we construct a map: $(\,\cdot\,, \dots, \,\cdot\,)_{\nu_{F_n},\dots,\nu_{F_1}} : F_n^{*(n+1)} \longrightarrow k^*$.

We show that $(\,\cdot\,, \dots, \,\cdot\,)_{\nu_{F_n},\dots,\nu_{F_1}}$ is an n-dimensional Steinberg symbol when $F_{i-1} = K(\nu_{F_i})$ for all $i \neq 1$. We should remark that our construction generalizes the definition of the Parshin–Fesenko–Vostokov symbols and the Brylinsky–McLaughlin symbols, because those symbols satisfy the condition that $F_{i-1} = K(\nu_{F_i})$ for all $i \neq 1$ (Example 3.15).

Finally, Section 4 is devoted to offering a reciprocity law for the symbols defined previously when $F_1 = \Sigma_C$, Σ_C being the function field of a complete, irreducible and non-singular curve over a perfect field. Similar to Tate's proof of the Residue Theorem [17], this reciprocity law is a direct consequence of the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$.

2 Preliminaries on 3-Cocycles and 2-Dimensional Steinberg symbols

Let F be a discrete valuation field, and let $K(\nu_F)$ be its residue class field. The valuation ring associated with ν_F is denoted by $\mathcal{O}_{\nu_F}$, and $\mathfrak{m}_{\nu_F}$ is its maximal ideal. Let us consider a field K, such that $K(\nu_F)$ is a finite separable extension of K, $\deg(\nu_F) = \dim_K K(\nu_F)$, and $N_{K(\nu_F)/K}$ is the norm of the extension of fields $K \hookrightarrow K(\nu_F)$.

Definition 2.1 We shall use the term *tame central extension* to refer to a central extension of groups $1 \to K^* \to \widetilde{F}^* \to F^* \to 1$, such that its commutator is

$$\{f,g\}_{\widetilde{F}^*} = N_{K(\nu_F)/K}\Big[\frac{f^{\nu_F(g)}}{g^{\nu_F(f)}}(\operatorname{mod} \mathfrak{m}_{\nu_F})\Big] \in K^* \text{ for } f,g \in F^*.$$

Remark 2.2. If F is any field with a discrete valuation ν_F whose residue class field is $K(\nu_F)$, for each field K such that $K(\nu_F)$ is a finite separable extension of K there exists a tame central extension associated with (F, ν_F, K) [8]. This extension is, in fact, a determinantal central extension in the sense of [1].

If we consider two tame central extensions of groups,

$$1 \to K^* \to \widetilde{F}^* \to F^* \to 1, \quad 1 \to k^* \to \widetilde{K}^* \to K^* \to 1,$$

by fixing an element $z \in F^*$ such that $\nu_F(z) = 1$, we have a 3-cocycle

$$\{\cdot\,,\,\cdot\,,\,\cdot\}^z_{\nu_F,\nu_K} \colon F^* \times F^* \times F^* \longrightarrow k^*,$$

defined as:

$$\{f_0,f_1,f_2\}^z_{\nu_F,\nu_K} = \prod_{j\in\mathbb{Z}/3}\Big[\{\{f_j,z\}_{\widetilde{F}^*},\{f_{j+1},z\}_{\widetilde{F}^*}\}_{\widetilde{K}^*}\Big]^{-\nu_F(f_{j+2})\cdot\deg(\nu_F)} \in k^*,$$

with $f_i \in F^*$. It is clear that

$$\begin{aligned}\{f_1,f_2,f_3\}^z_{\nu_F,\nu_K} = &\Big[\{\{f_1,z\}_{\widetilde{F}^*},\{f_2,z\}_{\widetilde{F}^*}\}_{\widetilde{K}^*}\Big]^{(-1)^3\nu_F(f_3)\cdot\deg(\nu_F)}\\ &\cdot\Big[\{\{f_1,z\}_{\widetilde{F}^*},\{f_3,z\}_{\widetilde{F}^*}\}_{\widetilde{K}^*}\Big]^{(-1)^2\nu_F(f_2)\cdot\deg(\nu_F)}\\ &\cdot\Big[\{\{f_2,z\}_{\widetilde{F}^*},\{f_3,z\}_{\widetilde{F}^*}\}_{\widetilde{K}^*}\Big]^{(-1)^1\nu_F(f_1)\cdot\deg(\nu_F)}.\end{aligned}$$

Moreover, if $f \in F^*$, we write

$$f(\nu_F,\nu_K) = N_{k(\nu_K)/k}\Big[N_{K(\nu_F)/K}(f(\operatorname{mod} \mathfrak{m}_{\nu_F}))(\operatorname{mod} \mathfrak{m}_{\nu_K})\Big] \in k^*,$$

which is a well-defined map when $\nu_F(f) = 0$, and

$$\nu_K\Big[N_{K(\nu_F)/K}(f(\operatorname{mod} \mathfrak{m}_{\nu_F}))\Big] = 0.$$

Hence, if $f,g,h \in F^*$, and $\nu^z_K(f) = \nu_K(\{f,z\}_{\widetilde{F}^*})$, it follows from the results of [8] that the value of the 3-cocycle $\{f,g,h\}^z_{\nu_F,\nu_K}$ is

$$\Big(\frac{f^{\deg(\nu_F)[\nu_F(g)\cdot\nu^z_K(h)-\nu_F(h)\cdot\nu^z_K(g)]}}{g^{\deg(\nu_F)[\nu_F(f)\cdot\nu^z_K(h)-\nu_F(h)\cdot\nu^z_K(f)]}}\cdot h^{\deg(\nu_F)[\nu_F(f)\cdot\nu^z_K(g)-\nu_F(g)\cdot\nu^z_K(f)]}\Big)(\nu_F,\nu_K) \in k^*.$$

This expression is independent of the choice of z and we can write $\{\cdot\,,\,\cdot\,,\,\cdot\}_{\nu_F,\nu_K}$ to refer to this cocycle. Note that this expression generalizes the Parshin-Fesenko-Vostokov symbol on an algebraic surface (1.1).

Another explicit expression of $\{\,\cdot\,,\,\cdot\,,\,\cdot\,\}_{\nu_F,\nu_K}$ is:

$$\{f,g,h\}_{\nu_F,\nu_K} = (f^{\nu_K(\{h,g\}_{\overline{F}^*})} \cdot g^{\nu_K(\{f,h\}_{\overline{F}^*})} \cdot h^{\nu_K(\{g,f\}_{\overline{F}^*})})^{\deg(\nu_F)}(\nu_F,\nu_K),$$

with $f,g,h \in F^*$. Moreover, for each morphism of commutative groups $\varphi\colon k^* \to G$, the tame central extension of groups associated with (K,ν_K,k) determines another central extension

$$1 \to G \to \widetilde{K}^*_\varphi \to K^* \to 1 \tag{2.1}$$

such that the 3-cocycle, induced by the tame central extension associated with (F,ν_F,K) and the central extension (2.1), is:

$$\{f,g,h\}^\varphi_{\nu_F,\nu_K} = \varphi(\{f,g,h\}_{\nu_F,\nu_K}) \in G.$$

Let us a now consider a discrete valuation field F and a commutative group G.

Definition 2.3 A map $\psi\colon F^*\times F^*\times F^* \to G$ is called a *Steinberg symbol* when

(i) ψ is multiplicative in each argument:

$$\psi(f_1\cdot f_2,g,h) = \psi(f_1,g,h)\cdot\psi(f_2,g,h),$$
$$\psi(f,g_1\cdot g_2,h) = \psi(f,g_1,h)\cdot\psi(f,g_2,h),$$
$$\psi(f,g,h_1\cdot h_2) = \psi(f,g,h_1)\cdot\psi(f,g,h_2),$$

for all $f_i,g_i,h_i \in F^*$.

(ii) $\psi(f,1-f,g) = \psi(f,g,1-f) = \psi(g,f,1-f) = 1$ for all $f \neq 1$.

Remark 2.4. If $K(\nu_F) = K$, according to [8], for each morphism of commutative groups $\varphi\colon k^* \to G$ there exists a unique Steinberg symbol $(f,g,h)^\varphi_{\nu_F,\nu_K}$ in the cohomology class

$$[\{\,\cdot\,,\,\cdot\,,\,\cdot\,\}^\varphi_{\nu_F,\nu_K}] \in H^3(F^*,G)$$

that satisfies the condition

$$(f,g,h)^\varphi_{\nu_F,\nu_K} = \{f,g,h\}^\varphi_{\nu_F,\nu_K} \text{ when } \nu_F(f) = \nu^z_K(f) = 0.$$

Moreover, there exist reciprocity laws on curves associated with the Steinberg symbols $(\,\cdot\,,\,\cdot\,,\,\cdot\,)^\varphi_{\nu_F,\nu_K}$.

The aim of this paper is to provide an approach for constructing n-dimensional Steinberg symbols from $(n+1)$-cocycles that will generalize the above method.

3 n-Dimensional Steinberg Symbols

3.1 Central Extensions and $(n+1)$-Cocycles

Let us consider $n+1$ commutative groups $\{G_i\}_{i\in\{0,1,\dots,n\}}$ with $n \geq 2$. If we have n central extensions of groups

$$1 \to G_{i-1} \to \widetilde{G}_i \to G_i \to 1, \tag{3.1}$$

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

by fixing $(n-1)$ morphisms of groups $\tau_j\colon G_j \to \mathbb{Z}$, for $j \in \{2,\dots,n\}$, and a family of elements $Z_{2,n} = \{z_2,\dots,z_n\}$, with $z_i \in G_i$, we can define recurrently the map $\{\,\cdot\,,\dots,\,\cdot\,\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}\colon G_n^{(n+1)} \longrightarrow G_0$ as

$$(3.2)\quad \{f_1,\dots,f_{n+1}\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n} = \prod_{i=1}^{n+1} [\{\{f_1,z_n\}_{\widetilde{G}_n},\dots,\widehat{\{f_i,z_n\}}_{\widetilde{G}_n},\dots,\{f_{n+1},z_n\}_{\widetilde{G}_n}\}^{\tau_{n-1},Z_{2,n-1}}_{\widetilde{G},n-1}]^{(-1)^i\tau_n(f_i)},$$

where $\{\,\cdot\,,\,\cdot\,\}_{\widetilde{G}_i}$ is the commutator of the central extension (3.1), and $\widehat{\{f_i,z_n\}}$ means that $\{f_i,z_n\}$ is omitted.

Bearing in mind that the commutator of a central extension is a bimultiplicative map, one has that $\{\,\cdot\,,\dots,\,\cdot\,\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}$ is multiplicative in each argument:

$$\{\dots,f_1\cdot f_2,\dots\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n} = \{\dots,f_1,\dots\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}\cdot\{\dots,f_2,\dots\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n},$$

for every $f_1, f_2 \in G_n$. Moreover, since the commutator of a central extension is skew-symmetric, the map $\{\,\cdot\,,\dots,\,\cdot\,\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}$ satisfies the property:

$$\{\dots,f,\dots,f,\dots\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n} = 1$$

for all $f \in G_n$. Furthermore, from the definition of $\{\,\cdot\,,\dots,\,\cdot\,\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}$, one also has that:

$$\{f_{\sigma(1)},\dots,f_{\sigma(n+1)}\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n} = [\{f_1,\dots,f_{n+1}\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}]^{\operatorname{sign}\sigma}$$

for any permutation $\sigma \in S_{n+1}$.

Lemma 3.1 *The map $\{\,\cdot\,,\dots,\,\cdot\,\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}$ is a $(n+1)$-cocycle.*

Proof Since $\{\,\cdot\,,\dots,\,\cdot\,\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}$ is bimultiplicative in each argument, one has that

$$\begin{aligned}\{f_2,\dots,f_{n+2}\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}\cdot[\{f_1,\dots,f_{n+1}\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}]^{(-1)^{n+2}} \\ = \prod_{i=1}^{n+1}[\{f_1,\dots,f_i\cdot f_{i+1},\dots f_{n+1}\}^{\tau_n,Z_{2,n}}_{\widetilde{G},n}]^{(-1)^{i+1}}\end{aligned}$$

with $f_1,\dots,f_{n+2} \in G_n$, which is the definition of a $(n+1)$-cocycle [6]. ■

3.2 $(n+1)$-Cocycles on Discrete Valuation Fields

Let F be a discrete valuation field and let $K(\nu_F)$ be its residue class field. The valuation ring associated with ν_F is denoted by $\mathcal{O}_{\nu_F}$, and $\mathfrak{m}_{\nu_F}$ is its maximal ideal.

Let us consider a family of fields $\{F_i\}_{i\in\{0,1,\dots,n\}}$, such that F_i is a discrete valuation field for $i \in \{1,\dots,n\}$; $K(\nu_{F_i})$ is a finite separable extension of F_{i-1}, $\deg(\nu_{F_i}) =$

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

$\dim_{F_{i-1}} K(\nu_{F_i})$, and $N_{K(\nu_{F_i})/F_{i-1}}$ is the norm of the extension of fields $F_{i-1} \hookrightarrow K(\nu_{F_i})$. We set $F_0 = k$. If we now consider n tame central extensions of groups (Definition 2.1),

$$1 \to F_{i-1}^* \to \widetilde{F}_i^* \to F_i^* \to 1,$$

fixing elements $Z_{2,n} = \{z_2, \dots, z_n\}$, with $z_i \in F_i^*$ and $\nu_{F_i}(z_i) = 1$, and considering the group morphisms

$$\tau_j := \operatorname{ind}_{K(\nu_{F_j})}^{F_{j-1}} \colon F_j^* \longrightarrow \mathbb{Z}$$
$$f \longmapsto \nu_{F_f}(f) \cdot \deg(\nu_{F_j}),$$

by using the method of subsection 3.1 we obtain an induced $(n+1)$-cocycle

$$\{\,\cdot, \dots, \cdot\,\}_{\widetilde{F_n^*}, \dots, \widetilde{F_1^*}}^{\operatorname{ind}_{K(\nu_{F_n})}^{F_{n-1}}, Z_{2,n}} \colon F_n^{*(n+1)} \longrightarrow k^*.$$

To simplify, we denote this $(n+1)$-cocycle by $\{\,\cdot, \dots, \cdot\,\}_{\nu_{F_n}, \dots, \nu_{F_1}}^{Z_{2,n}}$.

Let us now set

$$\bar{\nu}_{n-i}^{Z_{n-i+1,n}}(f) = \nu_{F_{n-i}}\big(\{\dots\{\{f, z_n\}_{\widetilde{F_n^*}}, z_{n-1}\}_{\widetilde{F_{n-1}^*}}, \dots, z_{n-i+1}\}_{\widetilde{F_{n-i+1}^*}}\big) \in \mathbb{Z},$$

for $f \in F_n^*$. Note that $Z_{n,n} = z_n$.

Different choices for the parameters $Z_{n-i+1,n} = \{z_n, \dots, z_{n-i+1}\}$ induce transformations

$$\bar{\nu}_{n-i}^{Z'_{n-i+1,n}} = \bar{\nu}_{n-i}^{Z_{n-i+1,n}} + \lambda_1 \bar{\nu}_{n-i+1}^{Z_{n-i+2,n}} + \dots + \lambda_{i-1} \bar{\nu}_{n-1}^{z_n} + \lambda_i \nu_{F_n},$$

with $\lambda_j \in \mathbb{Z}$, $Z'_{n-i+1,n}$ being $\{z'_n, \dots, z'_{n-i+1}\}$.

Hence, if $f_1, \dots, f_{n+1} \in F_n^*$, we can consider the $(n+1, n)$-matrix

$$\widetilde{A}(f_1, \dots, f_{n+1}) = \begin{pmatrix} \nu_{F_n}(f_1) & \bar{\nu}_{n-1}^{z_n}(f_1) & \dots & \bar{\nu}_1^{Z_{2,n}}(f_1) \\ \vdots & \vdots & & \vdots \\ \nu_{F_n}(f_{n+1}) & \bar{\nu}_{n-1}^{z_n}(f_{n+1}) & \dots & \bar{\nu}_1^{Z_{2,n}}(f_{n+1}) \end{pmatrix}.$$

If we denote by $\widetilde{A}_i(f_1, \dots, f_{n+1})$, the matrix obtained from $\widetilde{A}(f_1, \dots, f_{n+1})$ by eliminating its i-th row, we have that the determinant $\det \widetilde{A}_i(f_1, \dots, f_{n+1})$ will be independent of the choices of the elements $z_2, \dots, z_n$. Moreover, we can denote by $\widetilde{A}_{i,j}^s(f_1, \dots, f_{n+1})$ the matrix obtained from $\widetilde{A}(f_1, \dots, f_{n+1})$ by eliminating its i-th and j-th rows and its s-th column.

Proposition 3.2 *If $f_1 \dots, f_{n+1} \in F_n^*$, the $(n+1)$-cocycle $\{f_1, \dots, f_{n+1}\}_{\nu_{F_n}, \dots, \nu_{F_1}}^{Z_{2,n}}$ has the value:*

$$N_{K(\nu_{F_1})/k}\Big(\dots N_{K(\nu_{F_n})/F_{n-1}}\big(\Big[\prod_{i=1}^{n+1} f_i^{(-1)^{i+1} d(n,\dots,2) \det \widetilde{A}_i(f_1,\dots,f_{n+1})}\Big](\nu_{F_n})\big) \dots\big)(\nu_{F_1})\Big),$$

where $d(m, \dots, m-i) = \deg(\nu_{F_m}) \dots \deg(\nu_{F_{m-i}})$, and (ν_{F_i}) names an equivalent class (modulo $\mathfrak{m}_{\nu_{F_i}}$) that determines an element of $K(\nu_{F_i})$.

Proof Proceeding by induction on n, when $n = 2$ this cocycle is the 3-cocycle $\{\cdot, \dots, \cdot\}^{z}_{\nu_F, \nu_K}$ (§2) and its explicit expression coincides with the required result.

In the general case, the statement is deduced from formula (3.2), bearing in mind the properties of the determinants. ■

Corollary 3.3 *The* $(n+1)$*-cocycle* $\{\cdot, \dots, \cdot\}^{Z_{2,n}}_{\nu_{F_n}, \dots, \nu_{F_1}}$ *is independent of the choices of* $z_2, \dots, z_n$*.*

Accordingly, we henceforth write $\{\cdot, \dots, \cdot\}_{\nu_{F_n}, \dots, \nu_{F_1}}$ to denote this $(n+1)$-cocycle.

Furthermore, bearing in mind again the properties of the determinants, we have the following from Proposition 3.2.

Corollary 3.4 *Keeping the above notation,*

$$\nu_{F_1}(\{f_1, \dots, \widehat{f_i}, \dots, f_{n+1}\}_{\nu_{F_n}, \dots, \nu_{F_2}}) = (-1)^{n-1} d(n, \dots, 3) \det \widetilde{A}_i(f_1, \dots, f_{n+1}).$$

Corollary 3.5 *If* $f_1, \dots, f_n \in F_n^*$*, then an explicit expression of the* $(n+1)$*-cocycle* $\{\cdot, \dots, \cdot\}_{\nu_{F_n}, \dots, \nu_{F_1}}$ *is*

$$\{f_1, \dots, f_{n+1}\}_{\nu_{F_n}, \dots, \nu_{F_1}} = N_{K(\nu_{F_1})/k}\Big(\dots N_{K(\nu_{F_n})/F_{n-1}}\Big(\Big[\prod_{i=1}^{n+1} f_i^{\gamma_i}\Big](\nu_{F_n})\Big) \dots \Big)(\nu_{F_1})\Big),$$

with $\gamma_i = (-1)^{i+n} \deg(\nu_{F_2}) \nu_{F_1}(\{f_1, \dots, \widehat{f_i}, \dots, f_{n+1}\}_{\nu_{F_n}, \dots, \nu_{F_2}})$.

Remark 3.6. For each morphism of commutative groups $\varphi \colon k^* \to G$, the tame central extension of groups associated with (F_1, ν_{F_1}, k) determines another central extension

$$1 \to G \to (\widetilde{F}_1^*)_\varphi \to F_1^* \to 1, \tag{3.3}$$

such that the $(n+1)$-cocycle, induced by the tame central extensions associated with $(F_i, \nu_{F_i}, F_{i-1})$ and the central extension (3.3), is

$$\{f_1, \dots, f_{n+1}\}^{\varphi}_{\nu_{F_n}, \dots, \nu_{F_1}} = \varphi(\{f_1, \dots, f_{n+1}\}_{\nu_{F_n}, \dots, \nu_{F_1}}).$$

3.3 n-dimensional Steinberg Symbols

This subsection is devoted to constructing symbols from the $(n+1)$-cocycles $\{\cdot, \dots, \cdot\}^{\varphi}_{\nu_{F_n}, \dots, \nu_{F_1}}$ referred to previously.

With the above notations, let us now consider a map: $\psi \colon F_n^{*(n+1)} \longrightarrow G$.

We say that ψ satisfies the Steinberg relations when:

(i) ψ is multiplicative in each argument:

$$\psi(\dots, f_i \cdot \widetilde{f_i}, \dots) = \psi(\dots, f_i, \dots) \cdot \psi(\dots, \widetilde{f_i}, \dots)$$

for all $f_i, \widetilde{f_i} \in F_n^*$;

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

(ii) $\psi(\dots, f, \dots, 1-f, \dots) = 1$ for all $f \neq 1$.

Definition 3.7 A map $\psi\colon F_n^{*(n+1)} \longrightarrow G$ is called an *n-dimensional Steinberg symbol* when it satisfies the Steinberg relations.

Remark 3.8. It follows from its definition that an n-dimensional Steinberg symbol also satisfies the relations

- $\psi(f_{\sigma(1)}, \dots, f_{\sigma(n+1)}) = \psi(f_1, \dots, f_{n+1})^{\operatorname{sign}\sigma}$ for any permutation $\sigma \in S_{n+1}$,
- $\psi(f, f, f_3, \dots, f_{n+1}) = \psi(f, -1, f_3, \dots, f_{n+1})$ and $\psi(f, -f, f_3, \dots, f_{n+1}) = 1$.

Moreover, each n-dimensional Steinberg symbol ψ factors through $(n+1)$ Milnor's K-group $K_{n+1}(F)$, *i.e.*,:

$$\begin{tikzcd} F_n^{*(n+1)} \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & G \\ & K_{n+1}(F) \arrow[ur, "\phi"'] & \end{tikzcd},$$

where ϕ is a morphism of groups.

The $(n+1)$-cocycle $\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_1}}$ satisfies the first n-dimensional Steinberg relation. However, $\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_1}}$ is not an n-dimensional Steinberg symbol because it follows from Proposition 3.2 that:

$$\{f, -f, f_3 \dots, f_{n+1}\}_{v_{F_n}, \dots, v_{F_1}} = (-1)^{d(n, \dots, 1) \det \widetilde{A}_1(f, f, f_3 \dots, f_{n+1})}.$$

We shall now give a cohomological definition of an n-dimensional Steinberg symbol from the $(n+1)$-cocycle $\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_1}}$. According to the definition of the $(n+1)$-th cohomology group, $H^{n+1}(F_n^*, k^*) = Z^{n+1}(F_n^*, k^*)/B^{n+1}(F_n^*, k^*)$ (see [6], p. 53), one has that $\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_1}}$ determines a cohomology class

$$\big[\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_1}}\big] \in H^{n+1}(F_n^*, k^*).$$

We should recall that $\bar{c} \in Z^{n+1}(F_n^*, k^*)$ is an $(n+1)$-coboundary *i.e.*,

$$\bar{c} \in B^{n+1}(F_n^*, k^*)$$

if there exists a function on n variables ϕ on F_n^* to k^* such that:

$$\begin{aligned} \bar{c}(x_1, \dots, x_{n+1}) &= (\delta\phi)(x_1, \dots, x_{n+1}) \\ &= \phi(x_2, \dots, x_{n+1}) \cdot \phi(x_1, \dots, x_n)^{(-1)^{n+1}} \cdot \prod_{i=1}^{n} \phi(x_1, \dots, x_i \cdot x_{i+1}, \dots, x_{n+1})^{(-1)^i} \end{aligned}$$

for all $x_1, \dots, x_{n+1} \in F_n^*$.

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

Lemma 3.9 *With the previous notations, if $n > 1$ and setting $\widetilde{A}(f_1, \dots, f_{n+1}) = (a_{ij})$, the function $c_\lambda \colon F_n^{*(n+1)} \longrightarrow k^*$, defined as: $c_\lambda(f_1, \dots, f_{n+1}) = (-1)^{\lambda \cdot \beta(f_1, \dots, f_{n+1})}$, with $\lambda \in \mathbb{Z}$ and $\beta(f_1, \dots, f_{n+1}) = \sum_i \sum_{j<k} a_{ji} a_{ki} \det \widetilde{A}^i_{j,k}(f_1, \dots, f_{n+1})$, is an $(n+1)$-coboundary for all $\lambda \in \mathbb{Z}$.*

Proof If $A(x_1, \dots, x_n)$ is a (n, n)-matrix, we denote by $A_h^k(x_1, \dots, x_n)$ the matrix obtained from it by eliminating its h-th row and its k-th column.

With the above notation, we write:

$$\widetilde{A}(\widetilde{f}_1, \dots, \widetilde{f}_n) = \begin{pmatrix} v_{F_n}(\widetilde{f}_1) & \bar{v}_{n-1}^{z_n}(\widetilde{f}_1) & \dots & \bar{v}_1^{Z_{2,n}}(\widetilde{f}_1) \\ \vdots & \vdots & & \vdots \\ v_{F_n}(\widetilde{f}_n) & \bar{v}_{n-1}^{z_n}(\widetilde{f}_n) & \dots & \bar{v}_1^{Z_{2,n}}(\widetilde{f}_n) \end{pmatrix},$$

with $\widetilde{f}_1, \dots, \widetilde{f}_n \in F_n^*$. Note that $\widetilde{A}(\widetilde{f}_1, \dots, \widetilde{f}_n)$ is an (n, n)-matrix with entries in $\mathbb{Z}$.

Setting $\phi(\widetilde{f}_1, \dots, \widetilde{f}_n)$ to

$$\sum_i \Big(\Big[\sum_{j \text{ odd}} \frac{a_{ji}(a_{ji}-1)}{2} \det \widetilde{A}^i_j(\widetilde{f}_1, \dots, \widetilde{f}_n) \Big] + \Big[\sum_{\substack{j<k, \\ j \text{ odd}}} a_{ji} a_{ki} \det \widetilde{A}^i_k(\widetilde{f}_1, \dots, \widetilde{f}_n) \Big] \Big),$$

if we consider the function on n variables $\widetilde{\phi}(\widetilde{f}_1, \dots, \widetilde{f}_n) = (-1)^{\phi(\widetilde{f}_1, \dots, \widetilde{f}_n)}$, an easy computation shows that

$$(\delta \widetilde{\phi})(f_1, \dots, f_{n+1}) = (-1)^{\sum_i \sum_{j<k} a_{ji} a_{ki} \det \widetilde{A}^i_{j,k}(f_1, \dots, f_{n+1})},$$

and the statement is proved. ∎

Proposition 3.10 *If $n > 1$, there exists a unique $(n+1)$-cocycle $(\,\cdot\,, \dots, \,\cdot\,)_{v_{F_n}, \dots, v_{F_1}}$ in the cohomology class $[\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_1}}] \in H^{n+1}(F_n^*, k^*)$ satisfying the following conditions:*

(i) $(\,\cdot\,, \dots, \,\cdot\,)_{v_{F_n}, \dots, v_{F_1}}$ *is multiplicative in each argument.*
(ii) $(f_{\sigma(1)}, \dots, f_{\sigma(n+1)})_{v_{F_n}, \dots, v_{F_1}} = [(f_1, \dots, f_{n+1})_{v_{F_n}, \dots, v_{F_1}}]^{\operatorname{sign} \sigma}$ *for any permutation* $\sigma \in S_{n+1}$.
(iii) $(f, -f, f_3, \dots, f_{n+1})_{v_{F_n}, \dots, v_{F_1}} = 1$ *for all* $f, f_3, \dots, f_{n+1} \in F_n^*$.
(iv) $(f_1, f_2, f_3, \dots, f_{n+1})_{v_{F_n}, \dots, v_{F_1}} = \{f_1, f_2, f_3, \dots, f_{n+1}\}_{v_{F_n}, \dots, v_{F_1}}$ *if*

$$v_{F_n}(f_1) = \bar{v}_{n-1}^{z_n}(f_1) = \dots = \bar{v}_1(f_1) = 0.$$

Proof Since $(\,\cdot\,, \dots, \,\cdot\,)_{v_{F_n}, \dots, v_{F_1}} \in [\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_1}}]$, one has that

$$(f_1, \dots, f_{n+1})_{v_{F_n}, \dots, v_{F_1}} = c(f_1, \dots, f_{n+1}) \cdot \{f_1, \dots, f_{n+1}\}_{v_{F_n}, \dots, v_{F_1}},$$

where c is an $(n+1)$-coboundary.

We shall compute the value of c bearing in mind the conditions of the Proposition.

If we consider the morphism of groups:

$$\widetilde{\nu}_n := \nu_{F_n} \times \widetilde{\nu}_{n-1}^{\mathbb{Z}_n} \times \cdots \times \widetilde{\nu}_1^{\mathbb{Z}_{2,n}} \colon F_n^* \longrightarrow \mathbb{Z}^{(n)},$$

we have a commutative diagram of morphisms of groups:

$$\begin{tikzcd} F_n^{*\,(n+1)} \arrow[d, "\prod_{i=1}^{n+1}(\widetilde{\nu}_n)"'] \arrow[dr, "c"] & \\ \prod_{i=1}^{n+1}\mathbb{Z}^{(n)} \arrow[r, "\widetilde{c}"] & k^*, \end{tikzcd}$$

where $\widetilde{c}$ is a $(n+1)$-coboundary satisfying the properties:

- $\widetilde{c}(\dots, x_i + \widetilde{x}_i, \dots) = \widetilde{c}(\dots, x_i, \dots) \cdot \widetilde{c}(\dots, \widetilde{x}_i, \dots)$;
- $\widetilde{c}(x_{\sigma(1)}, \dots, x_{\sigma(n+1)}) = \widetilde{c}(x_1, \dots, x_{(n+1)})^{\operatorname{sign}\sigma}$ for any permutation σ,

for all $x_i, \widetilde{x}_i \in \mathbb{Z}^{(n)}$.

If $x_1 = (x_{1j}), \dots, x_{n+1} = (x_{(n+1)j}) \in \mathbb{Z}^{(n)}$, we can consider the $(n+1, n)$-matrix

$$\widetilde{A}(x_1, \dots, x_{n+1}) = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & & \vdots \\ x_{(n+1)1} & x_{(n+1)2} & \dots & x_{(n+1)n} \end{pmatrix}.$$

A computation shows that

$$\widetilde{c}(x_1, \dots, x_{n+1}) = (-1)^{d(n,\dots,1)\cdot B(x_1,\dots,x_{n+1})},$$

where

$$B(x_1, \dots, x_{n+1}) = \sum_i \sum_{j<k} \alpha_{ji}\alpha_{ki} \det \widetilde{A}^i_{j,k}(x_1, \dots, x_{n+1})$$

with $\widetilde{A}(x_1, \dots, x_{n+1}) = (\alpha_{ij})$.

Hence, it follows from Lemma 3.9 that for all $f_i \in F_n^*$, the only $(n+1)$-cocycle in the cohomology class

$$\left[\{\,\cdot\,, \dots, \,\cdot\,\}_{\nu_{F_n},\dots,\nu_{F_1}}\right] \in H^{n+1}(F_n^*, k^*)$$

satisfying the conditions (i)–(iv) is

$$(f_1, \dots, f_{n+1})_{\nu_{F_n},\dots,\nu_{F_1}} = c(f_1, \dots, f_{n+1}) \cdot \{f_1, \dots, f_{n+1}\}_{\nu_{F_n},\dots,\nu_{F_1}},$$

c being the $(n+1)$-coboundary

$$c(f_1, \dots, f_{n+1}) = (-1)^{d(n,\dots,1)\cdot\beta(f_1,\dots,f_{n+1})},$$

and

$$\beta(f_1, \dots, f_{n+1}) = \sum_i \sum_{j<k} a_{ji}a_{ki} \det \widetilde{A}^i_{j,k}(f_1, \dots, f_{n+1})$$

with $\widetilde{A}(f_1, \dots, f_{n+1}) = (a_{ij})$. Note that this expression is independent of the choices made. ∎

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

Proposition 3.11 *If $F_{i-1} = K(\nu_{F_i})$ for all $i \neq 1$, one has that*

$$(f, 1-f, f_3, \dots, f_{n+1})_{\nu_{F_n},\dots,\nu_{F_1}} = 1$$

for all $f, f_3, \dots, f_{n+1} \in F_n^*$ with $f \neq 1$. Therefore, $(\,\cdot\,, \dots, \,\cdot\,)_{\nu_{F_n},\dots,\nu_{F_1}}$ is an n-dimensional Steinberg symbol.

Proof When $\nu_{F_n}(f) < 0$ or $\nu_{F_n}(f) > 0$, the claim is confirmed by translating to the n-dimensional situation the arguments of the proof of J. Milnor in the case of the tame symbol [7]. Moreover, in these cases the statement is true for an arbitrary family of valuation fields $\{F_i\}$.

For the remaining case, when $\nu_{F_n}(f) = \nu_{F_n}(1-f) = 0$, it follows from the definition of the $n+1$-cocycle $\{\,\cdot\,, \dots, \,\cdot\,\}_{\nu_{F_n},\dots,\nu_{F_1}}$ (3.2) that

$$\{f, 1-f, f_3, \dots, f_{n+1}\}_{\nu_{F_n},\dots,\nu_{F_1}} = \prod_{i\geq 3}\Big[\big\{\{f, z_n\}_{\widetilde{F_n^*}}, 1-\{f, z_n\}_{\widetilde{F_n^*}} \dots, \{f_{n+1}, z_n\}_{\widetilde{F_n^*}}\big\}^{z_n,\dots,z_3}_{\nu_{F_n},\dots,\nu_{F_2}}\Big]^{(-1)^i \nu_{F_n}(f_i)}.$$

Since the statement of the proposition is true for $n = 2$ [8], we can proceed by induction on n. Thus, we have that

$$\big\{\{f, z_n\}_{\widetilde{F_n^*}}, 1-\{f, z_n\}_{\widetilde{F_n^*}}, \dots, \widehat{\{f_i, z_n\}}_{\widetilde{F_n^*}} \dots, \{f_{n+1}, z_n\}_{\widetilde{F_n^*}}\big\}_{\nu_{F_n},\dots,\nu_{F_2}} = (-1)^{\sum_{s\neq 1}\sum_{j<k; j\neq i\neq k; i\geq 3} a_{js}a_{ks} \det \widetilde{A}^{1,s}_{i,j,k}(f,1-f,f_3,\dots,f_{n+1})},$$

where $\widetilde{A}^{1,s}_{i,j,k}(f, 1-f, f_3, \dots, f_{n+1})$ is the matrix obtained from $\widetilde{A}(f, 1-f, f_3, \dots, f_{n+1})$ by eliminating its i-th, j-th and k-th rows and its first and s-th columns.

One can therefore see that

$$\{f, 1-f, f_3, \dots, f_{n+1}\}_{\nu_{F_n},\dots,\nu_{F_1}} = (-1)^{\deg(\nu_{F_1})\sum\limits_{s\neq 1; j<k} a_{js}a_{ks} \det \widetilde{A}^{s}_{j,k}(f,1-f,f_3,\dots,f_{n+1})},$$

and bearing in mind that

$$\sum_{j<k, j\geq 3} a_{j1}a_{k1} \det \widetilde{A}^{1}_{j,k}(f, 1-f, f_3, \dots, f_{n+1}) = 0$$

because, according to the above considerations, the first and second rows of the matrix $\widetilde{A}(f, 1-f, f_3, \dots, f_{n+1})$ are equal or the second row is null, we conclude that

$$\{f, 1-f, f_3, \dots, f_{n+1}\}_{\nu_{F_n},\dots,\nu_{F_1}} = (-1)^{\deg(\nu_{F_1})\beta(f,1-f,f_3,\dots,f_{n+1})}.$$

Accordingly, one also sees that $(f, 1-f, f_3, \dots, f_{n+1})^{z_n,\dots,z_2}_{\nu_{F_n},\dots,\nu_{F_1}} = 1$ in this latter case, and the claim is proved. ∎

Theorem 3.12 *When $F_{i-1} = K(v_{F_i})$ for all $i \neq 1$, $(\cdot, \dots, \cdot)_{v_{F_n},\dots,v_{F_1}}$ is the unique Steinberg symbol in the cohomology class $[\{\cdot, \dots, \cdot\}^{z_n,\dots,z_2}_{v_{F_n},\dots,v_{F_1}}] \in H^{n+1}({F_n}^*, k^*)$ satisfying the condition $(f_1, \dots, f_{n+1})_{v_{F_n},\dots,v_{F_1}} = \{f_1, \dots, f_{n+1}\}_{v_{F_n},\dots,v_{F_1}}$ when $v_{F_n}(f_1) = \bar{v}^{z_n}_{n-1}(f_1) = \cdots = \bar{v}^{Z_{2,n}}_1(f_1) = 0$.*

Proof The statement follows immediately from the results proved in Proposition 3.10 and Proposition 3.11. ∎

Corollary 3.13 *With the notations of Remark 3.6, if $F_{i-1} = K(v_{F_i})$ for all $i \neq 1$, one has that there exists a unique Steinberg symbol $(\cdot, \dots, \cdot)^{\varphi}_{v_{F_n},\dots,v_{F_1}}$ in the cohomology class $[\{\cdot, \dots, \cdot\}^{\varphi}_{v_{F_n},\dots,v_{F_1}}] \in H^{n+1}(F^*, G)$ satisfying the condition $(f_1, \dots, f_{n+1})^{\varphi}_{v_{F_n},\dots,v_{F_1}} = \{f_1, \dots, f_{n+1}\}^{\varphi}_{v_{F_n},\dots,v_{F_1}}$ when $v_{F_n}(f_1) = \bar{v}^{z_n}_{n-1}(f_1) = \cdots = \bar{v}^{Z_{2,n}}_1(f_1) = 0$.*

Proof A direct consequence of the previous theorem and the definition of the $(n+1)$-cocycle $\{\cdot, \dots, \cdot\}^{\varphi}_{v_{F_n},\dots,v_{F_1}}$ is that

$$(f_1, \dots, f_{n+1})^{\varphi}_{v_{F_n},\dots,v_{F_1}} = c_{\varphi}(f_1, \dots, f_{n+1}) \cdot \{f_1, \dots, f_{n+1}\}^{z_n,\dots,z_2}_{v_{F_n},\dots,v_{F_1},\varphi}$$

for all $f_1, \dots, f_{n+1} \in F^*$, where $c_{\varphi}(f_1, \dots, f_{n+1}) = (h_{-1})^{\deg(v_{F_1})\cdot\beta(f_1,\dots,f_{n+1})}$ with $h_{-1} = \varphi(-1)$. ∎

Example 3.14 Let F_n be an n-dimensional local field: that is, a complete discrete valuation field whose residue field F_{n-1} is an $(n-1)$-local field, 0-dimensional local fields being perfect fields of positive characteristic. Thus, with the above notations, in this case one has that $K(v_{F_i}) = F_{i-1}$ for every $i \in \{1, \dots, n\}$, and setting $k = F_0$, the respective tame central extensions induce a symbol

$$(\cdot, \dots, \cdot)_{v_{F_n},\dots,v_{F_1}} : F_n^{*\,(n+1)} \longrightarrow k^*.$$

In particular, if k is a perfect field of positive characteristic, and we consider the discrete valuation fields $F_i = k((u_1))\dots((u_i))$, where v_{F_i} is the valuation induced by the parameter $z_i = u_i$, and $f_1, \dots, f_{n+1} \in k((u_1))\dots((u_n))^*$, we have that the value of the symbol $(f_1, \dots, f_{n+1})_{v_{F_n},\dots,v_{F_1}}$ is

$$(-1)^{\beta_{v_{F_n},\dots,v_{F_1}}(f_1,\dots,f_{n+1})} \Big(\prod_{h=1}^{n+1} f_h^{(-1)^{h+1}\det \widetilde{A}_h(f_1,\dots,f_{n+1})} \Big)_{|_{u_i=0}} \in k^*,$$

where

$$\beta_{v_{F_n},\dots,v_{F_1}}(f_1, \dots, f_{n+1}) = \sum_i \sum_{j<k} a_{ji} a_{ki} \det \widetilde{A}^i_{j,k}(f_1, \dots, f_{n+1}),$$

with $\widetilde{A}(f_1, \dots, f_{n+1}) = (a_{ij})$.

Example 3.15 Let X be an n-dimensional smooth, proper, and geometrically irreducible scheme over a perfect field k, and let us consider a complete flag $x \subset$

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

$X_1 \subset \cdots \subset X_n = X$ of smooth, proper, and geometrically irreducible schemes with $\dim(X_i) = i$.

If Σ_{X_i} is the function field of X_i, the codimension-one closed subscheme X_{i-1} defines a discrete valuation $\nu_{X_{i-1}} \colon \Sigma^*_{X_i} \to \mathbb{Z}$, whose residue class field is $\Sigma_{X_{i-1}}$ (the function field of X_{i-1}).

Moreover, since X_1 is an irreducible and non-singular algebraic curve, the point $x \in X_1$ defines another discrete valuation $\nu_x \colon \Sigma^*_C \to \mathbb{Z}$, whose residue class field is $k(x)$.

Hence, setting $F_i = \Sigma_{X_i}$, $\nu_{F_i} = \nu_{X_{i-1}}$, $F_0 = k$, and fixing elements $z_i \in \Sigma_{X_i}$ such that $\nu_{X_{i-1}}(z_i) = 1$, for all $f_1, \dots, f_{n-1} \in \Sigma^*_X$, one has that

$$(f_1, \dots, f_{n+1})_{\nu_{X_{n-1}},\dots,\nu_x} = (-1)^{\gamma_{\nu_{X_{n-1}},\dots,\nu_x}(f_1,\dots,f_{n+1})} \cdot \{f_1, \dots, f_{n+1}\}_{\nu_{X_{n-1}},\dots,\nu_x},$$

where

$$\gamma_{\nu_{X_{n-1}},\dots,\nu_x}(f_1, \dots, f_{n+1}) = \deg(x) \cdot \beta_{\nu_{X_{n-1}},\dots,\nu_x}(f_1, \dots, f_{n+1}),$$

$\beta_{\nu_{X_{i-1}},\dots,\nu_x}(f_1, \dots, f_{n+1})$ being the integer number referred to in Example 3.14, and

$$\{f_1, \dots, f_{n+1}\}_{\nu_{X_{n-1}},\dots,\nu_x} = N_{k(x)/k}\Big[\Big(\prod_{h=1}^{n+1} f_h^{(-1)^{h+1}\det \widetilde{A}_h(f_1,\dots,f_{n+1})}\Big)_{|X_1}(x)\Big] \in k^*,$$

where $\deg(x) = \dim_k k(x)$, and $N_{k(x)/k}$ is the norm of the finite extension $k \hookrightarrow k(x)$.

When k is an algebraically closed field, the symbol $(\,\cdot\,, \dots, \,\cdot\,)_{\nu_{X_{n-1}},\dots,\nu_x}$ is the Brylinsky–McLaughlin symbol [2] associated with the sequence of varieties $x \subset X_1 \subset \cdots \subset X_n = X$, and if $n = 2$, one has the Parshin symbol associated with the sequence of varieties $x \in C \subset S$ [12]. Identically, when $n = 1$ for a perfect field, we recover the tame symbol of an algebraic curve [11].

Furthermore, if k is a finite field that contains the m-th roots of unity, with $\#k = q$, one has the morphism of groups

$$\phi_m \colon k^* \longrightarrow \mu_m, \qquad \lambda \longmapsto \lambda^{\frac{q-1}{m}},$$

which induces a symbol

$$(f_1, \dots, f_{n+1})^{\phi_m}_{\nu_{X_{n-1}},\dots,\nu_x} = (-1)^{\delta_{\nu_{X_{n-1}},\dots,\nu_x}(f_1,\dots,f_{n+1})} \cdot \{f_1, \dots, f_{n+1}\}^{\phi_m}_{\nu_{X_{n-1}},\dots,\nu_x},$$

$\{f_1, \dots, f_{n+1}\}^{\phi_m}_{\nu_{X_{n-1}},\dots,\nu_x}$ being equal to

$$N_{k(x)/k}\Big[\Big(\prod_{h=1}^{n+1} f_h^{(-1)^{h+1}\det \widetilde{A}_h(f_1,\dots,f_{n+1})}\Big)_{|X_1}(x)\Big]^{\frac{q-1}{m}},$$

and $\delta_{\nu_{X_{n-1}},\dots,\nu_x} = \frac{q-1}{m}\gamma_{\nu_{X_{n-1}},\dots,\nu_x}$ for all $f_1, \dots, f_{n+1} \in \Sigma^*_X$.

In this case, when $n = 1$ we obtain the Hilbert norm residue symbol of an algebraic curve [15].

Note that the symbols of this example satisfy the hypothesis of Theorem 3.12 and Corollary 3.13.

4 Reciprocity Law on Curves

This section is devoted to studying a reciprocity law for the symbols defined above, using a method similar to Tate's proof of the Residue Theorem [17], when F_1 is the function field of an algebraic curve.

Let us now consider an irreducible and non-singular curve C over a perfect field k and a closed point $x \in C$. If $F_1 = \Sigma_C$, $\nu_{F_1} = \nu_x$, $k(x)$ is the residue class field of a closed point $x \in C$, $deg(x) = \dim_k k(x)$, and $N_{k(x)/k}$ is the norm of the finite extension $k \hookrightarrow k(x)$, for a family of discrete valuation fields $\{F_i\}_{i\in\{0,1,\dots,n\}}$ with the conditions of subsection 3.2, and for a morphism of commutative groups $\varphi\colon k^* \longrightarrow G$, we have a symbol:

$$(\,\cdot\,,\dots,\,\cdot\,)^{\varphi}_{\nu_{F_n},\dots,\nu_{F_2},\nu_x}\colon F_n^{*\,(n+1)} \longrightarrow G.$$

Keeping the notations of the previous section, and fixing elements $z_i \in F_i^*$ with $\nu_{F_i}(z_i) = 1$ for $i \neq 1$, for each $f_1,\dots,f_{n+1} \in F_n^*$ the explicit expression of this symbol is:

$$(f_1,\dots,f_{n+1})^{\varphi}_{\nu_{F_n},\dots,\nu_{F_2},\nu_x} = (h_{-1})^{\gamma_{\nu_{F_n},\dots,\nu_{F_2},\nu_x}(f_1,\dots,f_{n+1})} \cdot \{f_1,\dots,f_{n+1}\}^{\varphi}_{\nu_{F_n},\dots,\nu_{F_2},\nu_x},$$

with $h_{-1} = \varphi(-1)$, and

$$\gamma_{\nu_{F_n},\dots,\nu_{F_2},\nu_x}(f_1,\dots,f_{n+1}) = \sum_i \sum_{j<k} a_{ji}a_{ki} \det \widetilde{A}(\nu_x)^{i}_{j,k}(f_1,\dots,f_{n+1}),$$

where $\widetilde{A}(\nu_x)^{i}_{j,k}(f_1,\dots,f_{n+1})$ is the $(n-1,n-1)$-matrix obtained, by eliminating its j-th and i-th rows and its i-th column, from the $(n+1,n)$-matrix:

$$\widetilde{A}(\nu_x)(f_1,\dots,f_{n+1}) = \begin{pmatrix} \nu_{F_n}(f_1) & \bar{\nu}^{z_n}_{n-1}(f_1) & \dots & \bar{\nu}^{Z_{2,n}}_x(f_1) \\ \vdots & \vdots & & \vdots \\ \nu_{F_n}(f_{n+1}) & \bar{\nu}^{z_n}_{n-1}(f_{n+1}) & \dots & \bar{\nu}^{Z_{2,n}}_x(f_{n+1}) \end{pmatrix}.$$

We recall from [9, 11] the properties of the commutator $\{\,\cdot\,,\,\cdot\,\}^{K_x}_{A_x,\varphi}$ of the central extension of groups

$$1 \to G \to \widetilde{\Sigma^*_C}^{K_x}_{A_x,\varphi} \to \Sigma^*_C \to 1$$

induced by φ on the tame central extension associated with the discrete valuation ν_x on Σ^*_C. To simplify, we denote this commutator by $\{\,\cdot\,,\,\cdot\,\}^{\varphi}_{A_x}$, and, if $h_{-1} = \varphi(-1)$, it has the following properties:

(i) Given $f, g \in \Sigma^*_C$, if we consider the central extension of groups associated with two closed points $x, y \in C$:

$$(4.1) \qquad 1 \to G \longrightarrow (\widetilde{\Sigma^*_C})^{\varphi}_{A_x\oplus A_y} \longrightarrow \Sigma^*_C \to 1,$$

which is determined by commensurable subspaces to $A_x \oplus A_y$ in $K_x \oplus K_y$, then we have that

$$\{f,g\}^{\varphi}_{A_x\oplus A_y} = (h_{-1})^{\deg(x)\deg(y)[\nu_x(f)\nu_y(g)+\nu_x(g)\nu_y(f)]} \cdot \{f,g\}^{\varphi}_{A_x} \cdot \{f,g\}^{\varphi}_{A_y}.$$

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

(ii) If C is a complete curve, and $X = \{x_1, \dots x_k\}$ is a finite subset of closed points of C such that it contains all zeros and poles of $f, g \in \Sigma_C^*$, then there exists a central extension of groups

$$1 \to G \longrightarrow (\widetilde{\Sigma_C^*})^{\varphi}_{A_{x_1} \oplus \cdots \oplus A_{x_k}} \longrightarrow \Sigma_C^* \to 1\,, \tag{4.2}$$

whose commutator satisfies the condition:

$$\{f, g\}^{\varphi}_{A_{x_1} \oplus \cdots \oplus A_{x_k}} = \prod_{i=1}^{k} (h_{-1})^{\deg(x_i) v_{x_i}(f) v_{x_i}(g)} \cdot \{f, g\}^{\varphi}_{A_{x_i}}.$$

(iii) If C is a complete curve, by using the theory of adeles, and with a similar method to Tate's proof of the Residue Theorem, one has that

$$\prod_{x \in C} \{f, g\}^{\varphi}_{A_x} = (h_{-1})^{\sum_{x \in C} \deg(x) v_x(f) v_x(g)}.$$

This result is a direct consequence of the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$.

Let us now assume that $\{F_i\}_{i \in \{0,1,\dots,n\}}$ is a family of discrete valuation fields, such that $F_1 = \Sigma_C$, C being an irreducible and non-singular curve over a perfect field k. Let us also fix a morphism of groups $\varphi \colon k^* \to G$.

Lemma 4.1 *For each pair of closed points $x, y \in C$, one has a $(n+1)$-cocycle*

$$\{\,\cdot\,, \dots, \,\cdot\,\}_{v_{F_n}, \dots, v_{F_2}, (\widetilde{\Sigma_C^*})^{\varphi}_{A_x \oplus A_y}} \in Z^{n+1}(F_n^*, G),$$

induced, using the method of subsection 3.1*, by the central extension* (4.1) *and the tame central extensions associated with the discrete valuation fields $\{F_i\}_{i \in \{2,\dots,n\}}$, such that*

$$\{f_1, \dots, f_{n+1}\}_{v_{F_n}, \dots, v_{F_2}, (\widetilde{\Sigma_C^*})^{\varphi}_{A_x \oplus A_y}} = (h_{-1})^{\nu_{v_{F_n}, \dots, v_{F_2}, \{x,y\}}(f_1, \dots, f_{n+1})} \cdot$$
$$\cdot \{f_1, \dots, f_{n+1}\}^{\varphi}_{v_{F_n}, \dots, v_{F_2}, v_x} \cdot \{f_1, \dots, f_{n+1}\}^{\varphi}_{v_{F_n}, \dots, v_{F_2}, v_y},$$

for all $f_1, \dots, f_{n+1} \in F_n^*$, with

$$\nu_{v_{F_n}, \dots, v_{F_2}, \{x,y\}}(f_1, \dots, f_{n+1}) =$$
$$\deg(x) \deg(y) d(n, \dots, 2) \cdot \sum_{j<k} \delta^{Z_{2,n}}_{x,y}(f_j, f_k) \det \widetilde{B}_{j,k}(f_1, \dots, f_{n+1}),$$

where $d(n, \dots, 2) = \deg(v_{F_n}) \cdots \deg(v_{F_2})$,

$$\delta^{Z_{2,n}}_{x,y}(f_j, f_k) = \bar{v}_x^{Z_{2,n}}(f_j) \cdot \bar{v}_y^{Z_{2,n}}(f_k) + \bar{v}_y^{Z_{2,n}}(f_j) \cdot \bar{v}_x^{Z_{2,n}}(f_k)],$$

and $\widetilde{B}_{j,k}(f_1, \dots, f_{n+1})$ being the $(n-1, n-1)$-matrix obtained, by eliminating the j-th and k-th rows, from the $(n+1, n-1)$-matrix:

$$\widetilde{B}(f_1, \dots, f_{n+1}) = \begin{pmatrix} v_{F_n}(f_1) & \bar{v}_{n-1}^{Z_n}(f_1) & \dots & \bar{v}_2^{Z_{3,n}}(f_1) \\ \vdots & \vdots & & \vdots \\ v_{F_n}(f_{n+1}) & \bar{v}_{n-1}^{Z_n}(f_{n+1}) & \dots & \bar{v}_2^{Z_{3,n}}(f_{n+1}) \end{pmatrix}.$$

Proof Using the definition of the $(n+1)$-cocycle $\{\cdot,\dots,\cdot\}_{\nu_{F_n},\dots,\nu_{F_2},(\widetilde{\Sigma_C^*})^{\varphi}_{A_x\oplus A_y}}$ (formula (3.2)), the claim is a direct consequence of the properties of the commutator $\{\cdot,\cdot\}^{\varphi}_{A_x}$. ■

Proposition 4.2 *Given $f_1,\dots,f_{n+1}\in F_n^*$, if C is a complete curve, and $X=\{x_1,\dots,x_s\}$ is a finite subset of closed points of C such that it contains all zeros and poles of*

$$\{\dots\{\{f_i,z_n\}_{\widetilde{F_n^*}},z_{n-1}\}_{\widetilde{F_{n-1}^*}},\dots,z_2\}_{\widetilde{F_2^*}}\in\Sigma_C^*,$$

then the $(n+1)$-cocycle

$$\{\cdot,\dots,\cdot\}_{\nu_{F_n},\dots,\nu_{F_2},(\widetilde{\Sigma_C^*})^{\varphi}_{A_{x_1}\oplus\dots\oplus A_{x_s}}}\in Z^{n+1}(F_n^*,G),$$

induced by the central extension (4.2) and the tame central extensions associated with the discrete valuation fields $\{F_i\}_{i\in\{2,\dots,n\}}$ satisfies the condition that

$$\{f_1,\dots,f_{n+1}\}_{\nu_{F_n},\dots,\nu_{F_2},(\widetilde{\Sigma_C^*})^{\varphi}_{A_{x_1}\oplus\dots\oplus A_{x_s}}}=\\ (h_{-1})^{\nu_{\nu_{F_n},\dots,\nu_{F_2},X}(f_1,\dots,f_{n+1})}\cdot\prod_{i=1}^{s}\{f_1,\dots,f_{n+1}\}^{\varphi}_{\nu_{F_n},\dots,\nu_{F_2},\nu_{x_i}},$$

with

$$\nu_{\nu_{F_n},\dots,\nu_{F_2},X}(f_1,\dots,f_{n+1})=\\ d(n,\dots,2)\cdot\sum_{\substack{x_i\in X\\ j<k}}\deg(x_i)\cdot\bar{\nu}_{x_i}^{Z_{2,n}}(f_j)\cdot\bar{\nu}_{x_i}^{Z_{2,n}}(f_k)\cdot\det\widetilde{B}_{j,k}(f_1,\dots,f_{n+1}).$$

Proof Using induction over $\#X$ and bearing in mind that $\deg(x)^2\equiv\deg(x)$ mod. 2, the formula holds by Lemma 4.1 and the property of complete curves,

$$\sum_{p\in C}\deg(p)\nu_p(\phi)=\sum_{p\in X}\deg(p)\nu_p(\phi)=0,$$

for all $\phi\in\Sigma_C^*$ such that X contains all zeros and poles of ϕ. ■

Theorem 4.3 **(Reciprocity Law)** *If $\{F_i\}_{i\in\{0,1,\dots,n\}}$ is a family of discrete valuation fields such that $F_1=\Sigma_C$, C being a complete, irreducible and non-singular curve over a perfect field k, and $f_1,\dots,f_{n+1}\in F_n^*$, for each morphism of commutative groups $\varphi\colon k^*\to G$, one has that*

$$\prod_{x\in C}(f_1,\dots,f_{n+1})^{\varphi}_{\nu_{F_n},\dots,\nu_{F_2},\nu_x}=1\,.$$

https://doi.org/10.4153/CJM-2008-058-3 Published online by Cambridge University Press

Proof Using the theory of adeles and with a similar method to Tate's proof of the Residue Theorem [17], from the characterization of the commutator $\{\cdot,\cdot\}^{\varphi}_{\prod_{x\in C} A_x}$, one has that

$$\prod_{x\in C}\{f_1,\dots,f_{n+1}\}^{\varphi}_{v_{F_n},\dots,v_{F_2},v_x} = (h_{-1})^{\nu_{v_{F_n},\dots,v_{F_2},C}(f_1,\dots,f_{n+1})}, \tag{4.3}$$

with

$$\nu_{v_{F_n},\dots,v_{F_2},C}(f_1,\dots,f_{n+1}) = d(n,\dots,2)\cdot\sum_{\substack{x\in C\\ j<k}}\deg(x)\cdot\bar{v}_x^{Z_{2,n}}(f_j)\cdot\bar{v}_x^{Z_{2,n}}(f_k)\cdot\det\widetilde{B}_{j,k}(f_1,\dots,f_{n+1}).$$

Thus, since C is complete, then $\sum_{x\in C}\deg(x)\cdot\det\widetilde{A}(v_x)^{i}_{j,k}(f_1,\dots,f_{n+1}) = 0$ for all $i\neq n$, and the claim follows immediately from expression (4.3) because

$$\nu_{v_{F_n},\dots,v_{F_2},C}(f_1,\dots,f_{n+1}) = \sum_{x\in C}\gamma_{v_{F_n},\dots,v_{F_2},v_x}(f_1,\dots,f_{n+1}). \qquad \blacksquare$$

Remark 4.4. If X is an n-dimensional smooth, proper, and geometrically irreducible scheme over an algebraically closed field k, by applying the result of Theorem 4.3 to the Brylinsky–Mc Laughlin symbol $(\cdot,\dots,\cdot)_{v_{X_{n-1}},\dots,v_x}$ [2] associated with a sequence of varieties $x\subset C\subset\cdots\subset X_{n-1}\subset X$, where C is a complete, irreducible and non-singular curve, we see that the formula

$$\prod_{x\in C}(f_1,\dots,f_{n+1})_{v_{X_{n-1}},\dots,v_x} = 1 \text{ for all } f_1,\dots,f_{n+1}\in\Sigma_X^*$$

is a direct consequence of the finiteness of the cohomology groups $H^0(C,\mathcal{O}_C)$ and $H^1(C,\mathcal{O}_C)$.

Remark 4.5. If C is again a complete, irreducible and non-singular curve over a finite field k that contains the m-th roots of unity, with $\#k = q$, for each sequence of varieties $x\subset C\subset\cdots\subset X_{n-1}\subset X$ over k, the morphism of groups

$$\phi_m\colon k^*\longrightarrow\mu_m,\quad \lambda\longmapsto\lambda^{\frac{q-1}{m}},$$

induces a symbol

$$(\cdot,\dots,\cdot)^{\phi_m}_{v_{X_{n-1}},\dots,v_x}\colon\Sigma_X^{*\,(n+1)}\longrightarrow\mu_m \qquad \text{(see Example 3.15)},$$

which satisfies the law

$$\prod_{x\in C}(f_1,\dots,f_{n+1})^{\phi_m}_{v_{X_{n-1}},\dots,v_x} = 1 \text{ for all } f_1,\dots,f_{n+1}\in\Sigma_X^*.$$

Remark 4.6. A remaining problem is to obtain reciprocity laws on algebraic curves using a method similar to above, when $F_i = \Sigma_C$ with $i \neq 1$.

Moreover, fixing a partial sequence $x \subset X_1 \subset \cdots \subset \widehat{X}_k \subset \cdots \subset X$, where $\widehat{X}_k$ means that X_k is omitted, another remaining question is to prove, with a similar method to Tate's proof of the Residue Theorem, the existence of reciprocity laws where the product is taken over all irreducible subvarieties X_k lying in the chain

$$x \subset X_1 \subset \cdots \subset X_k \subset \cdots \subset X.$$

These reciprocity laws were offered by J. L. Brylinsky and D. A. McLaughlin [2].

References

[1] E. Arbarello, C. de Concini, and V. G. Kac, *The infinite wedge representation and the reciprocity law for algebraic curves.* Proc. of Sympos. Pure Math, 1989, 49, Part 1, 171–190.

[2] J. L. Brylinsky and D. A. McLaughlin, *Multidimensional reciprocity laws.* J. Reine Angew. Math. **481**(1996), 125–147.

[3] ———, *The geometry of two-dimensional symbols. K*-Theory **10**(1996), no. 3, 215–237.

[4] I. B. Fesenko, *A generalized symbol in multidimensional local fields.* In: Rings and modules. Limit theorems of probability theory, 2, (Russian), Leningrad. Univ., **214**(1988), 88–92.

[5] I. B. Fesenko and S. Vostokov, *On torsion in higher Milnor functors for multidimensional local fields.* Amer. Math. Soc. Transl. 2, **154**(1992), 25–35.

[6] S. Eilenberg and S. McLane, *Cohomology theory in abstract groups.* I. Ann. Math. **48**(1947), 51–78.

[7] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies 72, Princeton University Press, Princeton, N.J., University of Tokyo Press, Tokyo, 1971.

[8] F. Pablos Romo, *3-Cocycles, symbols and reciprocity laws on curves.* J. Pure Appl. Algebra **205**(2006), no. 1, 94–116.

[9] ———, *A note on Steinberg symbols on algebraic curves.* Comm. Algebra **31**(2003), no.2, 981–990.

[10] ———, *Algebraic construction of the tame symbol and the Parshin symbol on a surface.* J. Algebra **274**(2004), no. 1, 335–346.

[11] ———, *On the tame symbol of an algebraic curve.* Comm. Algebra **30**(2002), no. 9, 4349–4368.

[12] A. N.Parshin, *Local class field theory.* Proc. Steklov Inst. Math. **3**(1985), 157–185.

[13] ———, *Galois cohomology and the Brauer group of local fields.* Proc. Steklov Inst. Math., **4**(1991), 191–201.

[14] A. N. Parshin and I. R. Shafarevich, *Number Theory II*, In: Encyclopaedia of Mathematical Sciences, 62, Springer-Verlag, Berlin, 1992.

[15] H. L. Schmidt, *Über das Reziprozitätsgesatz in relativ-zyklischen algebraischen Funktionkörpern mit endlichem Konstantenkörper.* Math. Z. **40**(1936), 94–109.

[16] J. P. Serre, *Groupes algébriques et corps de classes*, Publications de l'institut de mathématique de l'université de Nancago, VII, Hermann, Paris, 1959.

[17] J. Tate, *Residues of differentials on curves.* Ann. Scient. Éc. Norm. Sup.(4) **1**(1968), 149–159.

Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, España
e-mail: fpablos@usal.es