1. Introduction. Eigenvalue problems for the biharmonic operator (iterated Laplacian) \( L = \Delta \Delta \) will be studied on bounded plane domains. Our purpose is to obtain asymptotic variational formulae for eigenvalues and eigenfunctions under the deformation of removing an \( \epsilon \)-disk and adjoining additional boundary conditions on the new boundary component thereby introduced, valid on a positive interval \( 0 < \epsilon \leq \epsilon_0 \). Eigenvalue problems can be considered in connection with each of the following sets of homogeneous boundary conditions:

1. \( u = \frac{\partial u}{\partial n} = 0 \),
2. \( u = \Delta u = 0 \),
3. \( \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \),
4. \( \Delta u = \frac{\partial \Delta u}{\partial n} = 0 \),

where \( \partial / \partial n \) represents differentiation in the exterior normal direction to the boundary. Representation formulae for Green's functions associated with each of the four problems have been given, for example, by Gould (6) and Weinstock (9).

Boundary conditions 1 pertain to the clamped vibrating plate, which has been widely treated. Detailed references are given in (6). The eigenvalues are proportional to the squares of the natural vibration frequencies of such a plate. Our first result (section 3) gives asymptotic estimates for the perturbed eigenvalues and eigenfunctions of this problem, obtained when additional boundary conditions 1 are imposed on the boundary of a deleted \( \epsilon \)-disk.

An essential feature of the variational problem for boundary conditions 1, which does not occur for the analogous second-order problems (8), is that the perturbed eigenvalues \( \mu_\epsilon \) do not converge in general to the (classical) eigenvalues for the unperturbed domain. Accordingly, it is one of our main purposes to define a (non-classical) eigenvalue problem whose eigenvalues \( \lambda \) can be matched with the eigenvalues \( \mu_\epsilon \) in a one-to-one manner such that \( \mu_\epsilon \to \lambda \) as \( \epsilon \to 0 \) (section 2).

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The eigenvalue problem for $L$ with boundary conditions 2 is the problem for a supported vibrating plate. It is well known and easy to verify that the eigenvalues of problem 2 are the squares of the eigenvalues of the operator $-\Delta$ with null boundary conditions (fixed-edge membrane problem (6, p. 105). Thus asymptotic variational formulae for problem 2 can be obtained from the author's results (8) for second-order operators (section 4).

Likewise the eigenvalues of $L$ with boundary conditions 3 are the squares of the positive eigenvalues of $-\Delta$ with the boundary condition $\frac{\partial u}{\partial n} = 0$ (free-edge membrane problem). Variational formulae for the latter eigenvalues and corresponding eigenfunctions will be obtained in section 5 by methods analogous to those in (8).

The eigenvalue problem with boundary conditions 4 (natural conditions) is the so-called free plate problem. It is known (9) that the positive eigenvalues are identical with those of problem 1. (Every harmonic function is a solution of problem 4 for $\lambda = 0$.)

We shall not consider mixed boundary conditions here: e.g. $u = 0$ on part of the boundary and $\frac{\partial u}{\partial n} = 0$ on the remaining part. Also we shall restrict ourselves to the plane, although our methods, involving functional analysis rather than complex function theory, are capable of generalization to Euclidean space $E^n$ or to Riemannian space.

2. Basic and perturbed domains. Let $M$ be an (open connected) bounded plane domain whose boundary $B$ consists of a finite number of arcs having continuous curvature. Throughout, $\frac{\partial}{\partial n}$ denotes differentiation in the exterior normal direction $n$ to $B$. The basic domain $\mathcal{D}^j$ for $L$ is defined to be the set of all complex-valued functions on $\tilde{M}$ that are of class $C^4[\tilde{M}] \cap C^1(\tilde{M})$ and satisfy boundary conditions $j$ on $B$, $j = 1, 2, 3$. The corresponding basic eigenvalue problem for each $j$ is

$$Lx = \lambda x, \quad x \in \mathcal{D}^j.$$  

The variation of the eigenvalues and eigenfunctions will be obtained when the domain $\mathcal{D}^j$ is perturbed to a domain $\mathcal{D}^j_\epsilon$, defined as follows. Let $p_0$ be a fixed (but arbitrary) point in $M$. Let $|p - p_0|$ denote the Euclidean distance between $p$ and $p_0$. Let $\gamma$ be the circle $|p - p_0| = \varepsilon$, and let $N_\varepsilon$ be the disk $|p - p_0| < \varepsilon$. Let $M_\varepsilon = M \setminus N_\varepsilon$. The parameter $\varepsilon$ measures the smallness of the deleted disk, $0 < \varepsilon \leq \varepsilon_0$. Then the perturbed domain $\mathcal{D}^j_\epsilon$ is defined to be the set of all complex-valued functions on $\tilde{M}_\varepsilon$ that are of class $C^4[M_\varepsilon] \cap C^1(\tilde{M}_\varepsilon)$ and satisfy the boundary conditions $j$ on $B \cup \gamma$. The perturbed eigenvalue problem for each $j$ is

$$Ly = \mu y, \quad y \in \mathcal{D}^j_\epsilon.$$  

Functions in $\mathcal{D}^j_\epsilon$ are defined only on $\tilde{M}_\varepsilon$, but they will be extended to $M$ by defining them to be zero inside $\gamma$.

It will be shown that the eigenvalues $\mu_\varepsilon$ corresponding to the perturbed
domain \(\mathcal{D}\) do not converge in general to the eigenvalues for \(\mathcal{D}\), but rather to the eigenvalues of the following non-classical problem:

\[
Lx = \lambda x, \quad x \in \mathcal{D},
\]

where \(\mathcal{D}\) denotes the set of all complex-valued functions \(u\) in \(M\) that are of class \(C^4[M \setminus \{p_0\}] \cap C^1[M]\), satisfy boundary conditions 1 on \(B\), and satisfy \(u(p_0) = 0\) with \(\Delta u(p)\) having (at worst) the singularity \(\log |p - p_0|\) at \(p_0\).

If \(A(p, q)\) denotes the standard biharmonic Green’s function for \(\mathcal{D}\) (6, p. 101; 9, p. 9; 4), we define the “Green’s function” for \(\mathcal{D}\) to be

\[
A(p, q; p_0) = A(p, q) - A(p, p_0)A(q, p_0)/A(p_0, p_0),
\]

which has the following properties:

(i) \(A(p, q; p_0) = \partial A(p, q; p_0)/\partial n_p = 0\) if \(p \in B\);

(ii) \(A(p_0, q; p_0) = 0\);

(iii) as a function of either argument \(p\) or \(q\), \(A(p, q; p_0)\) is of class \(C^4[M \setminus \{p_0\}] \cap C^1[M]\);

(iv) \(\Delta_p A(p, q; p_0) = A(q, p; p_0)\) \((p, q \in M)\);

(v) \(\Delta_p A(p, q; p_0)\) has logarithmic singularities at \(p = p_0\) and at \(p = q\).

For arbitrary \(f \in C^4[M \setminus \{p_0\}] \cap C^1[M]\), we obtain from the Rayleigh–Green identity and the above properties that

\[
f(p) = \frac{A(p, p_0) f(p_0)}{A(p_0, p_0)} + \int_M A(p, q; p_0)Lf(q) \, dq
\]

\[
+ \int_B \left[ f(q) \frac{\partial \Delta A(p, q; p_0)}{\partial n_q} - \frac{\partial f}{\partial n_q} \Delta A(p, q; p_0) \right] dq.
\]

Suppose \(f = x \in \mathcal{D}\) and \(Lx = \lambda x\). Then \(f = \partial f/\partial n = 0\) on \(B\) and \(f(p_0) = 0\). Hence \(x\) satisfies the integral equation

\[
x\cdot(\cdot) = \lambda \int_M A(p, q; p_0)x(q) \, dq
\]

with symmetric \(L^2\) kernel.

Conversely, any solution of this integral equation satisfies boundary conditions 1 on \(B\), \(x(p_0) = 0\), and

\[
\Delta x(p) \sim [\Delta A(p, p_0)/A(p_0, p_0)] \int_M A(q, p_0)x(q) \, dq
\]

as \(p \to p_0\). Thus \(\Delta x(p)\) has the required singularity at \(p = p_0\). Differentiation of (2.4) by a well-known procedure leads to the differential equation \(\Delta \Delta x(p) = \lambda x(p)\), \(p \neq p_0\).

Since integral operators with symmetric \(L^2\) kernels are known to be symmetric, completely continuous transformations on Hilbert space, problem (2.3) has a complete orthonormal sequence of eigenfunctions and corresponding eigenvalues \(\lambda_n\) such that \(\lambda_n \to \infty\) as \(n \to \infty\). Since functions in \(\mathcal{D}\) can be continued to \(M\) such that they belong to \(\mathcal{D}\), it follows from the standard

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maximum–minimum principle for eigenvalues that $\lambda_n \leq \mu_n$ for each $n = 1, 2, \ldots$ and each $\epsilon$ (2).

3. Asymptotic estimates for problem 1. The purpose is to obtain asymptotic estimates for the difference between the eigenvalues of (2.2) (when $j = 1$) and (2.3). The following notation will be used:

\[ \phi(\epsilon) = \epsilon \log(1/\epsilon), \quad 0 < \epsilon < 1. \]

Let $\mathfrak{S}$, $\mathfrak{S}_\epsilon$ be the Hilbert spaces $L^2[M]$, $L^2[M_\epsilon]$ respectively. The inner product in each space is given by

\[ (u, v) = \int_M u\overline{v} \, dp \]

when it is agreed that functions defined in $M_\epsilon$ are extended to $M$ by defining them to be zero inside $\gamma$. Let $\mathcal{D}_\epsilon$ denote the space of single-valued harmonic functions in $M_\epsilon$, and let $\mathcal{Q}_\epsilon$ denote the projection mapping from $\mathfrak{S}_\epsilon$ into the subspace $\mathcal{D}_\epsilon$.

Let $A_\epsilon(p, q)$ be the Green's functions for $L$ on $\mathcal{D}_\epsilon$, i.e. corresponding to boundary conditions 1 on $B \cup \gamma$ (6, p. 101). Let $A$, $A_\epsilon$ be the linear integral operators whose kernels are the respective Green's functions $A(p, q; p_0)$, $A_\epsilon(p, q)$. These operators are known to be symmetric, completely continuous transformations on $\mathfrak{S}$, $\mathfrak{S}_\epsilon$ respectively, whose eigenvalues $\alpha_i$, $\beta_i$ are reciprocals of the eigenvalues $\lambda_i$, $\mu_i$ of (2.3), (2.2), $i = 1, 2, \ldots$.

Let $\alpha$ be an arbitrary eigenvalue of $A$, of multiplicity $m$ ($m \geq 1$), and let $\mathfrak{A}_\alpha$ be the corresponding $(m$-dimensional)$)$ eigenspace. Let $\mathfrak{A}_\alpha^\epsilon$ be the space consisting of the functions of $\mathfrak{A}_\alpha$ restricted to $M_\epsilon$. We may assume an $\epsilon$-neighbourhood $(0, \epsilon_0]$ to have been selected initially on which

\[ \dim \mathfrak{A}_\alpha = \dim \mathfrak{A}_\alpha^\epsilon = m. \]

**Lemma** There exist positive constants $\epsilon_1 < 1$ and $c$ such that

\[ ||A_\epsilon x - \alpha x|| \leq c\phi(\epsilon) ||x|| \]

for every $x \in \mathfrak{A}_\alpha^\epsilon$ and every $\epsilon$ satisfying $0 < \epsilon \leq \epsilon_1$.

**Proof.** It is sufficient to prove (3.2) in the case $||x|| = 1$. For such $x \in \mathfrak{A}_\alpha^\epsilon$, the function $f = A_\epsilon x - \alpha x$ is a solution of the Dirichlet problem $Lf = 0$ in $M_\epsilon$, $f = \partial f/\partial n = 0$ on $B$, $f = x$ on $\gamma$, $\partial f/\partial n = \partial x/\partial n$ on $\gamma$ (5). It then follows from the Rayleigh–Green representation formula that

\[ f(p) = \int [x(q) \partial A_\epsilon(p, q) / \partial n_q - \Delta_\epsilon A_\epsilon(p, q) \partial x / \partial n_q] \, ds_q. \]

It is known (6, p. 101; 9, p. 17) that

\[ \Delta_\epsilon A_\epsilon(p, q) = -G_\epsilon(p, q) + \int_{M_\epsilon} a_\epsilon(q, r) G_\epsilon(p, r) \, dr, \]

where $G_\epsilon(p, r)$ is the standard harmonic Green’s function for $M_\epsilon$ corresponding to null boundary conditions, and $a_\epsilon(q, r)$ is the reproducing kernel in the space.
§ of harmonic functions (1; 6). The integral term on the right side of (3.4) is therefore the projection of $G_\epsilon(p, q)$ into $\mathfrak{H}$. It follows from (3.3) and (3.4) that $f(p)$ may be decomposed into the following three terms:

$$f_1(p) = \int_\gamma \left[ G_\epsilon(p, q) \frac{\partial x}{\partial n_q} - x(q) \frac{\partial G_\epsilon(p, q)}{\partial n_q} \right] ds_\epsilon;$$

$$f_2(p) = \int_{M_\epsilon} \int_\gamma x(q) G_\epsilon(p, r) \frac{\partial a_\epsilon(q, r)}{\partial n_q} ds_\epsilon dr;$$

$$f_3(p) = - \int_{M_\epsilon} \int_\gamma G_\epsilon(p, r) a_\epsilon(q, r) \frac{\partial x}{\partial n_q} ds_\epsilon dr.$$

The existence of the boundary integrals follows upon transformation by Green’s formula (see below). Since a standard theorem on harmonic functions gives $f_1(p) = O(\epsilon \log \epsilon)$ uniformly in $M_\epsilon$, it will be sufficient in the proof of (3.2) to establish that $f_2(p) = O(\epsilon)$ and $f_3(p) = O(\epsilon \log \epsilon)$ uniformly in $M_\epsilon$.

Since $U(q) = x(q)/\epsilon$ is a bounded function of $q$ and $\epsilon$ for $q \in \gamma$, $0 < \epsilon \leq \epsilon_0$, we can construct a function $u \in C^2[M]$ independent of $\epsilon$ satisfying the boundary conditions $u = \partial u/\partial n = 0$ on $B$, $u = U$ on $\gamma$, and $\partial u/\partial n = 0$ on $\gamma$. Since $a_\epsilon(q, r)$ is harmonic in $M_\epsilon$, we obtain from Green’s symmetric formula

$$f_2(p) = - \epsilon \int_{M_\epsilon} \int_\gamma \Delta u(q) a_\epsilon(q, r) G_\epsilon(p, r) dq dr.$$

By the reproducing property of $a_\epsilon(q, r)$,

$$f_2(p) = - \epsilon \int_{M_\epsilon} Q_\epsilon[\Delta u(r)] G_\epsilon(p, r) dr,$$

where $Q_\epsilon$ denotes the projection operator onto $\mathfrak{H}_\epsilon$. Then the Schwartz inequality yields

$$|f_2(p)| \leq \epsilon \| \Delta u \| \| G_\epsilon(p, r) \|.$$  \hspace{1cm} (3.6)

In order to prove that $\| G_\epsilon(p, r) \|$ is a bounded function of $p$ and $\epsilon$, we observe that

$$H_\epsilon(p, r) = G_\epsilon(p, r) - G(p, r) + G(p_0, r) h_\epsilon(p)$$

is a regular harmonic function of $p \in M_\epsilon$, where $h_\epsilon$ is the harmonic measure with boundary values 1 on $\gamma$ and 0 on $B$, and $G(p, r)$ is the harmonic Green’s function for $M$. Since $H_\epsilon(p, r)$ has the boundary values 0 on $B$ and $O(\epsilon)$ on $\gamma$, it follows from the maximum principle (3) that $H_\epsilon(p, r)$ is uniformly $O(\epsilon)$ in $M_\epsilon$. Since $\|h_\epsilon\| = O(1/\log \epsilon)$, it follows that $\| G_\epsilon(p, r) \|$ is bounded for $p \in M_\epsilon$, $0 < \epsilon \leq \epsilon_0$. The inequality (3.6) then shows that $f_2(p) = O(\epsilon)$ in $M_\epsilon$.

To prove that $f_3(p) = O(\epsilon \log \epsilon)$, consider the function defined by

$$g(p, p_0) = -(2\pi/\log \epsilon)[G(p, p_0) + \psi(p)],$$

where $\psi$ is the biharmonic function in $M$ with boundary values such that $g = \partial g/\partial n = 0$ on $B$. Let

$$h(p) = \epsilon \log \epsilon [A(p, p_0)/A(p_0, p_0) - g(p, p_0)].$$
Then $Lh = 0$, $h = \partial h / \partial n = 0$ on $B$, $h = O(\epsilon)$ on $\gamma$, and $\partial h / \partial n = 1 + O(\epsilon \log \epsilon)$ on $\gamma$. Hence we obtain analogues of the representation (3.3) and the decomposition (3.5) of $h$ into $h_1 + h_2 + h_3$. We find that $h_1 = O(\epsilon \log \epsilon)$ and $h_2 = O(\epsilon)$ in the same manner as above. Furthermore,

$$h_3(p) = -\int_{m} \gamma a_s(q, r) G_s(p, r) \partial h / \partial n_s \, ds \, dr$$

by an argument similar to that used above for $f_2(p)$. If $x_0'$ denotes the limit of $x' / \partial n$ as $q \to p_0$, then since $x \in \Omega$, (3.5) yields

$$f_3(p) = -\int_{m} \gamma [x_0' + O(\epsilon \log \epsilon)] a_s(q, r) G_s(p, r) \, ds \, dr = x_0' h_3(p) + O(\epsilon \log \epsilon) = O(\epsilon \log \epsilon).$$

This completes the proof of the lemma.

**Theorem 1.** Corresponding to each eigenvalue $\lambda$ of the basic problem (2.3), of multiplicity $m$, there exist positive numbers $\epsilon_1$ and $c_1$ such that exactly $m$ eigenvalues $\mu_i$ of the perturbed problem (2.2) lie in the interval $[\lambda, \lambda + c_1 \phi(\epsilon)]$ provided $0 < \epsilon \leq \epsilon_1$. Furthermore, there exist orthonormal eigenfunctions $x_i$ associated with $\lambda$ and corresponding orthonormal $y_i$ associated with the $\mu_i$ such that the uniform estimates

$$y_i(p) = x_i(p) + O(\phi),$$

$$Ay_i(p) = Ax_i(p) + O(\phi) \log |p - p_0|$$

are valid, $p \in \Omega$, $0 < \epsilon \leq \epsilon_1$, $i = 1, 2, \ldots, m$.

**Proof.** Let $P(\delta)$ be the projection operator onto the subspace $\Theta_{\delta}^{\epsilon}$ of $\Theta_{\epsilon}$ spanned by all the eigenfunctions of $A_\epsilon$ whose corresponding eigenvalues lie in the interval $(\alpha - \delta, \alpha)$, $\delta > 0$. Then for any $w \in \Theta_\epsilon$, the inequality

$$||w - P(\delta)w|| \leq \delta^{-1} ||A_\epsilon w - \alpha w||$$

is valid (7, p. 34). In particular, $w$ can be replaced by any $x \in \Phi_{\alpha \epsilon}$. Then (3.9) combined with the lemma (3.2) yields

$$||x - P(\delta)z|| \leq c\phi(\epsilon)\delta^{-1} ||x||.$$

This implies (7, p. 35) that there are at least $m$ eigenvalues $\beta_i$ ($i = 1, 2, \ldots, m$) of $A_\epsilon$ in the interval $[\alpha - c\phi(\epsilon), \alpha]$. Since $\lambda, \mu_i$ are reciprocals of $\alpha, \beta_i$, there are at least $m$ eigenvalues $\mu_i$ of the perturbed problem (2.2) in the interval $[\lambda, \lambda + c_1 \phi(\epsilon)]$, where $c_1$ is another positive constant.

To prove there are exactly $m$, let $\lambda^i$ denote the $i$th distinct eigenvalue, of multiplicity $m_i$, $\lambda^1 < \lambda^2 < \ldots$. Since the eigenvalues do not accumulate and since $\phi(\epsilon) \to 0$ as $\epsilon \to 0$, a positive interval $(0, \epsilon_1]$ can be selected so that $c_1 \phi(\epsilon)$ is less than the minimum of all the differences $\lambda^{i+1} - \lambda^i$ for $\epsilon$ on this
interval; and accordingly at least \( m_i \) eigenvalues \( \mu \) are included in the sub-interval \( [\lambda^i, \lambda^{i+1} + c_i \phi(\epsilon)] \) of \( [\lambda^i, \lambda^{i+1}] \), \( i = 1, 2, \ldots \). Since \( \lambda^i \leq \mu_i \) for each \( i \) is known as a general consequence of \( M_r \subset M_i \) an easy induction argument shows that exactly \( m_i \) eigenvalues \( \mu \) are included in \( [\lambda^i, \lambda^{i+1} + c_i \phi(\epsilon)] \).

In order to prove the second part of the theorem, we select the number \( \delta \) in (3.9) as the distance between \( \alpha \) and the nearest eigenvalue \( \alpha^i \neq \alpha \). Then \( \delta \) is \( m \)-dimensional by the first part of the theorem, and \( P(\delta)x = 0 \) implies \( x = 0 \) by (3.10). Hence there exist \( m \) uniquely determined linearly independent eigenfunctions \( z_i \in \mathbb{A}_t \) which \( P(\delta) \) maps into the orthonormal eigenfunctions \( y_i \), and by (3.10), \( ||z_i - y_i|| = O(\phi), i = 1, 2, \ldots, m \). We use the Schwartz inequality to obtain

\[
|\langle z_i, z_j \rangle - (y_i, y_j) | \leq ||y_i|| ||z_j - y_j|| + ||z_i|| ||z_i - y_i||.
\]

Hence

\[
(3.11) \quad (z_i, z_j) = \delta_{ij} + O(\phi), \quad i, j = 1, 2, \ldots, m.
\]

Since the \( z_i \) are linearly independent, an orthonormal sequence \( \{x_i\} \) can be obtained by the Schmidt process as linear combinations of the \( z_i \). Because of (3.11) it turns out that \( ||x_i - z_i|| = O(\phi) \), and hence \( ||x_i - y_i|| = O(\phi) \).

Omitting the subscript \( i \), we select a typical \( x \) in the set \( \{x_i\} \) and the corresponding \( y \) in \( \{y_i\} \). It follows from the uniform estimate

\[
|y(p) - \lambda A_x x(p)| = |A_x(\mu y - \lambda x)(p)| \leq ||A_x|| (\mu ||y - x|| + |\mu - \lambda|| ||x||)
\]

and the first part of the theorem that there exists a constant \( c \) such that

\[
(3.12) \quad |y(p) - \lambda A_x x(p)| \leq c\phi(\epsilon) \quad (p \in M_\epsilon; 0 < \epsilon \leq \epsilon_1).
\]

The function \( g \) defined by

\[
g(p) = \lambda A_x x(p) - x(p) + f(p),
\]

where \( f \) is given by (3.3), is a solution of the Dirichlet problem \( Lg = 0 \), \( g = \frac{\partial g}{\partial n} = 0 \) on \( B \cup \gamma \), and hence \( g \) is identically zero. The following estimate is then a direct consequence of (3.12):

\[
y(p) = x(p) - f(p) + O(\phi), \quad 0 < \epsilon \leq \epsilon_1.
\]

Since \( f(p) = O(\phi) \) uniformly in \( M_\epsilon \) by the lemma, (3.7) is established. The proof of (3.8) is similar and will be omitted.

As a consequence of Theorem 1, we shall establish the following asymptotic variational formulae for eigenvalues:

\[
(3.13) \quad \mu_\epsilon - \lambda_\epsilon = \int_\gamma \left[ \bar{\phi}_i(p) \frac{\partial \phi_i(p)}{\partial n} - \Delta \phi_i(p) \frac{\partial \phi_i(p)}{\partial n} \right] ds + O(\phi^2)
\]

\( (i = 1, 2, \ldots; 0 < \epsilon \leq \epsilon_1) \).
where \( f_i \) is the solution of the Dirichlet problem \( Lf = 0, f = \partial f / \partial n = 0 \) on \( B \), \( f = x_i \) on \( \gamma \), \( \partial f / \partial n = \partial x_i / \partial n \) on \( \gamma \).

The subscripts \( i \) will be deleted in the proof of (3.13). Since \( \Delta x = \lambda x \) and \( \Delta y = \mu y \), the Rayleigh–Green formula gives

\[
(3.14) \quad (\lambda - \mu) \int_{M} x(p) \bar{y}(p) \, dp = \int_{B \cup \gamma} \left( \bar{y} \frac{\partial \Delta x}{\partial n} + \Delta \bar{y} \frac{\partial x}{\partial n} - x \frac{\partial \Delta \bar{y}}{\partial n} - \Delta x \frac{\partial \bar{y}}{\partial n} \right) \, ds.
\]

By Theorem 1, \( y(p) = x(p) + O(\phi) \) on \( M \). Since \( y = \partial y / \partial n = 0 \) on \( B \cup \gamma \), \( x = \partial x / \partial n = 0 \) on \( B \), and \( ||x|| = 1 \),

\[
(\lambda - \mu)[1 + O(\phi)] = - \int_{\gamma} \left( \Delta \bar{y} \frac{\partial x}{\partial n} - x \frac{\partial \Delta \bar{y}}{\partial n} \right) \, ds.
\]

Similarly,

\[
-\mu \int_{M} f(p) \bar{y}(p) \, dp = - \int_{\gamma} \left( \Delta \bar{y} \frac{\partial x}{\partial n} - x \frac{\partial \Delta \bar{y}}{\partial n} \right) \, ds.
\]

Since \( \mu = \lambda + O(\phi) \) and \( y(p) = x(p) + O(\phi) \), the following asymptotic variational formula is obtained:

\[
\mu - \lambda = \lambda \int_{M} \bar{x}(p)f(p) \, dp + O(\phi^2).
\]

An identity similar to (3.14) then yields the result (3.13).

4. Asymptotic estimates for problem 2. The analogue of Theorem 1 will now be obtained directly from the corresponding known result (8) for the fixed-edge membrane problem for the operator \( -\Delta \).

Let \( \mathcal{B} \) be the set of all complex-valued functions on \( \bar{M} \) that are of class \( C^2[\bar{M}] \), continuous on \( \bar{M} \), and zero on \( B \). The basic eigenvalue problem corresponding to this boundary condition is (fixed-edge membrane problem)

\[
(4.1) \quad -\Delta x = \nu x, \quad x \in \mathcal{B}.
\]

Let \( \mathcal{B}_e \) be the set of all complex-valued functions on \( \bar{M}_e \) that are of class \( C^2[\bar{M}_e] \), continuous on \( \bar{M}_e \), and zero on \( B \cup \gamma \). The perturbed problem corresponding to these boundary conditions is

\[
(4.2) \quad -\Delta y = \sigma y, \quad y \in \mathcal{B}_e.
\]

The basic and perturbed problems for \( L = \Delta \Delta \) corresponding to boundary conditions 2 are

\[
(4.3) \quad Lx = \lambda x, \quad x \in \mathcal{D}^2,
\]

\[
(4.4) \quad Ly = \mu y, \quad y \in \mathcal{D}_e^2.
\]

For every \( m \)-fold degenerate eigenvalue \( \lambda \) of (4.3), there is an \( m \)-fold degenerate
eigenvalue \( \nu \) of (4.1) such that \( \lambda = \nu^2 \), and the eigenfunctions \( x_i \) associated with \( \nu \) are identical with the eigenfunctions associated with \( \lambda \). Likewise \( \mu_i = \sigma_i^2 \) is the \( i \)th eigenvalue of (4.4) corresponding to the \( i \)th eigenvalue \( \sigma_i \) of (4.2), and the associated eigenfunctions are identical. According to (8),

\[
(4.5) \quad \sigma_i - \nu = -[|x_i(p_0)|^2 + O(\psi)] \int_\gamma \nabla h \cdot \mathbf{n} \, ds = O(\psi),
\]

where

\[
\psi(\epsilon) = [\log(1/\epsilon)]^{-1} \quad (0 < \epsilon < 1; \quad i = 1, 2, \ldots, m),
\]

and \( h \) is the solution of the Dirichlet problem \( \Delta h = 0 \) in \( \Omega_\epsilon \), \( h = 0 \) on \( \partial \), and \( h = 1 \) on \( \gamma \). Furthermore, the following uniform estimates are valid:

\[
(4.6) \quad \gamma_i(p) = x_i(p) - x_i(p_0)h(p) + O(\psi), \quad 0 < \epsilon \leq \epsilon_0, \quad i = 1, 2, \ldots.
\]

It follows from (4.5) that \( \mu_i - \lambda = (\sigma_i - \nu)(\sigma_i + \nu) = O(\psi) \), and hence

\[
(4.7) \quad \mu_i - \lambda = (\sigma_i - \nu)[2\nu + (\sigma_i - \nu)]
\]

\[= [-2\lambda |x_i(p_0)|^2 + O(\psi)] \int_\gamma \nabla h \cdot \mathbf{n} \, ds, \quad 0 < \epsilon \leq \epsilon_0, \quad i = 1, 2, \ldots, m.\]

**Theorem 2.** Corresponding to each eigenvalue \( \lambda \) of (4.3), of multiplicity \( m \), there exist positive numbers \( \epsilon_1 \) and \( \epsilon_2 \) such that exactly \( m \) eigenvalues \( \lambda_i \) of (4.4) lie in the interval \([\lambda, \lambda + \epsilon_1 \psi(\epsilon)]\) provided \( 0 < \epsilon \leq \epsilon_1 \). The asymptotic variational formulae (4.7) hold for the eigenvalues and the uniform estimates (4.6) hold for corresponding orthonormal eigenfunctions.

### 5. Asymptotic estimates for problem 3.

Let \( \mathfrak{W}' \) be the set of complex-valued functions \( u \) on \( \Omega \) that are of class \( C^2(\Omega) \) and satisfy the boundary condition \( \partial u / \partial n = 0 \) on \( \partial \). Let \( \mathfrak{W}' \) be the set of complex-valued functions on \( \Omega \) that are of class \( C^2(\Omega_\epsilon) \) and satisfy the same boundary condition on \( \partial \cup \gamma \). The basic and perturbed problems for \( -\Delta \) (free-edge membrane problems) are

\[
(5.1) \quad -\Delta x = \nu x, \quad x \in \mathfrak{W}'; \quad (5.2) \quad -\Delta y = \sigma y, \quad y \in \mathfrak{W}.'
\]

Let \( x \) be an arbitrary eigenfunction associated with an \( m \)-fold degenerate eigenvalue \( \nu \) of (5.1). Let \( \Phi \) be the linear integral operator whose kernel \( \Phi(p, q) \) is the Neumann function (Green’s function of the second kind) for the operator \( J = -\Delta + 1 \) in \( \Omega_\epsilon \). Consider the function \( f = \Phi x - \rho x \) in \( \Omega_\epsilon \),where \( \rho = 1/(\nu + 1) \). Then \( Jf = 0 \) in \( \Omega_\epsilon \), \( \partial f / \partial n = 0 \) on \( \partial \), and \( \partial f / \partial n = -\rho \partial x / \partial n \) on \( \gamma \). It follows from the standard representation formula for solutions of the differential equation \( Jf = 0 \) (3, p. 160) that

\[
(5.3) \quad f(p) = \rho \int_\gamma \Phi(p, q) \frac{\partial x}{\partial n_q} \, ds_q.
\]
The inequality
\[ \int_{\gamma} N_t(p, q) \, ds \leq c \epsilon N(p, p_0) \]
is valid, where \( c \) is independent of \( \epsilon \), and \( N(p, p_0) \) is the Neumann function for \( J \) in \( M \). Hence (5.3) yields
\[ |f(p)| \leq c \epsilon [\max_{\gamma} |\partial x/\partial n|] N(p, p_0). \]

Since \( N(p, p_0) \in L^2[M] \), the inequality
\[ (5.4) \quad ||f|| = ||N_t x - px|| \leq c_1 \epsilon ||x|| \]
is obtained immediately as an analogue of (3.2).

**Theorem 3.** Corresponding to each positive eigenvalue \( \nu \) of (5.1), of multiplicity \( m \), there exist positive numbers \( \sigma_i \) and \( c \) such that exactly \( m \) eigenvalues \( \sigma_i \) of (5.2) are in the interval \([\nu, \nu + c \epsilon]\) provided \( 0 < \epsilon \leq \epsilon_1 \). There are orthonormal eigenfunctions \( x_i \) associated with \( \nu \) and corresponding orthonormal \( y_i \) associated with \( \sigma_i \) such that
\[ (5.5) \quad y_i(p) = x_i(p) - h_i(p) + O(\epsilon), \quad p \in M, \quad 0 < \epsilon \leq \epsilon_1, \quad i = 1, 2, \ldots, m, \]
where \( h_i \) is the solution of the boundary-value problem
\[ (5.6) \quad Jh = 0 \text{ in } M, \quad \partial h/\partial n = 0 \text{ on } B, \quad \partial h/\partial n = \partial x_i/\partial n \text{ on } \gamma. \]

Furthermore, the following asymptotic variation formulae are valid:
\[ (5.7) \quad \sigma_i = \nu - [\bar{x}_i(p_0) + O(\epsilon)] \int_\gamma \nabla x_i \cdot n \, ds. \]

**Proof.** The first statement follows from (5.4) similarly to Theorem 1. One obtains also \( ||x_i - y_i|| = O(\epsilon) \) by the method of Theorem 1.

To prove (5.5), observe first that there is a number \( c \) independent of \( \epsilon \) such that
\[ (5.8) \quad |(y_i - (\nu + 1)N_t x_i)(p)| = |N_t[(\sigma_i + 1)y_i - (\nu + 1)x_i](p)| \leq ||N_t(p, q)|| \[(\sigma_i - \nu) ||y_i|| + (\nu + 1) ||x_i - y_i||\] \leq c \epsilon. \]

Since the function \( f = (\nu + 1)N_t x_i - x_i + h_i \) is the unique solution of the Neumann problem (3, p. 153) \( Jf = 0 \) in \( M \), \( \partial f/\partial n = 0 \) on \( B \cup \gamma \), \( f \) is identically zero. The uniform estimates (5.5) then follow from (5.8).

To prove (5.7), we first obtain
\[ \sigma_i - \nu = -\int_\gamma \bar{y}_i \nabla x_i \cdot n \, ds[1 + O(\epsilon)] \]
in a standard way from Green's identity (8, p. 18). Use of (5.5) then leads to (5.7).

The basic and perturbed problems for \( L \) corresponding to boundary conditions 3 are given by (2.1), (2.2) (with \( j = 3 \)). By the methods of section 4 we obtain the following analogue of Theorem 2.
THEOREM 4. Corresponding to each eigenvalue $\lambda$ of (2.1) (with $j = 3$), of multiplicity $m$, there exist positive numbers $\epsilon_1$ and $c_1$ such that exactly $m$ eigenvalues $\mu_i$ of (2.2) lie in the interval $[\lambda, \lambda + c_1 \epsilon]$ provided that $0 < \epsilon < \epsilon_1$. The uniform estimates (5.5) for eigenfunctions and the asymptotic formulae

$$
\mu_i = \lambda + [\frac{2}{\lambda} \xi_i(p_0) + O(\phi)] \int_\gamma \nabla x_1 \cdot n \, ds
$$

for eigenvalues are valid, $i = 1, 2, \ldots, m$.

REFERENCES


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