

# The character table of a group of shape $(2 \times 2 \cdot G):2$

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## ABSTRACT

We use the technique of Fischer matrices to write a program to produce the character table of a group of shape  $(2 \times 2 \cdot G):2$  from the character tables of  $G$ ,  $G:2$ ,  $2 \cdot G$  and  $2 \cdot G:2$ .

### 1. Introduction

Let  $G$  be a finite group with an automorphism of order two and a double cover. We may form a group of shape  $(2 \times 2 \cdot G):2$ , which we write as

$$(\langle x \rangle \times \langle z \rangle \cdot G) : \langle \sigma \rangle,$$

so that  $\sigma$  acts to swap  $x$  with  $xz$  and  $\langle x, z, \sigma \rangle \cong D_8$ . (Here and throughout we use ATLAS [2] notation for products of groups.) This group has three interesting subgroups of index two:

$$G^+ = \langle z \rangle \cdot G : \langle \sigma \rangle, \quad G^- = \langle z \rangle \cdot G \cdot \langle x \sigma \rangle, \quad G^0 = \langle x \rangle \times \langle z \rangle \cdot G.$$

Here  $G^+$  and  $G^-$  are representatives of the two isomorphism classes of groups of shape  $2 \cdot G:2$  (they are isoclinic, as explained in [2]). To produce the character table of  $(2 \times 2 \cdot G):2$ , all we must do is determine the class fusion from  $G^0$  and the character values on the outer conjugacy classes. The ATLAS [2] map of the character table of  $(2 \times 2 \cdot G):2$  is

$G$	$G : \langle \sigma \rangle$	$r$
$\langle x \rangle \times G$	$\langle x \rangle \times G : \langle \sigma \rangle$	$r$
$\langle z \rangle \cdot G$	$\langle z \rangle \cdot G : \langle \sigma \rangle$	$s$
$\langle xz \rangle \cdot G$	$\langle xz \rangle \cdot G : \langle \sigma \rangle$	$s$
$r$	$t$	

Here  $G$  has  $r$  conjugacy classes,  $2 \cdot G$  has  $s$  characters that are faithful on  $z$  and  $G:2$  has  $t$  conjugacy classes of outer elements.

Let  $\chi$  be a character in the  $\langle z \rangle \cdot G$  square; then  $\chi(z) = \chi(xz) = -\chi(1)$  and  $\chi(x) = \chi(1)$ . However,  $\chi^\sigma(x) = \chi(xz) = -\chi(1)$  and  $\chi^\sigma(xz) = \chi(x) = \chi(1)$ . Thus,  $\sigma$  must fuse characters in the  $\langle z \rangle \cdot G$  square with characters in the  $\langle xz \rangle \cdot G$  square. Therefore, the  $\langle z \rangle \cdot G : \langle \sigma \rangle$  and  $\langle xz \rangle \cdot G : \langle \sigma \rangle$  squares contain only zeros.

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Each outer class  $[g]$  of  $G:2$  lifts to two classes in  $2^2 \cdot G:2$ , namely  $[g, zg]$  and  $[xg, xzg]$ . To see this, observe that  $g^x = xgx = xxzg = zg$ , which also shows that if the order of  $g$  is indivisible by four then the order of  $xg$  is twice the order of  $g$ . A character  $\chi \in \text{Irr } G:2$  takes value  $\chi(xg) = -\chi(g)$ , which, it would seem, completes the table.

We wish to improve the above analysis to sufficient precision to allow the character table to actually be produced. There are about twelve different ways in which conjugacy classes of  $G:2$  can lift to  $2^2 \cdot G:2$ , and these are explained in the case analysis beginning in § 4. (This number is approximate because there are some subtleties involving the element orders which one may use to produce further cases.) We choose to work with the example  $G = \text{Fi}_{22}$ , because the character table of  $2^2 \cdot \text{Fi}_{22}:2$  is not stored in GAP [4], and most of the conjugacy class types occur (the three that do not can be seen in  $A_5$  or  $L_2(17)$ ). Also,  $2^2 \cdot \text{Fi}_{22}$  is a maximal subgroup of  $\text{Fi}_{24}$ , so we can compute the class fusion and then restrict characters in order to check our calculations.

We compute the character table using Fischer matrices, as in this case we can eliminate all uncertainty about the entries of the matrices. Thus, assembling the character table is easy. We shall need the character table of  $G:2$  and the projective character table of  $G$ . This is no more information that we needed above, for the projective character table of  $G$  is easily deduced from the character table of  $2 \cdot G$ .

### 2. Fischer matrices

The technique of Fischer matrices [3] seems generally to have been used to calculate character tables of maximal subgroups of sporadic simple groups and their automorphism groups, and recently has enjoyed something of a revival; see, for example, [1] and similar papers.

The Fischer matrices method relies on the fact that every irreducible character can be obtained by induction from the inertia groups. Specifically, let  $\bar{G} = N.G$  be a group and let  $\theta_1, \theta_2, \dots, \theta_t$  be representatives for the orbits of  $G$  on  $\text{Irr } N$  (by convention  $\theta_1 = 1$ ). Let  $\bar{H}_i$  be the inertia group of  $\theta_i$  in  $\bar{G}$ . If  $\psi_i$  is a (possibly projective) extension of  $\theta_i$  to  $\bar{H}_i$ , then  $\psi_i \eta \uparrow \bar{G}$  is irreducible, where  $\eta$  is inflated from  $H_i = \bar{H}_i/N$ . In fact,

$$\text{Irr } \bar{G} = \bigcup_{i=1}^t \{(\psi_i \eta) \uparrow \bar{G} \mid \eta \in \text{Irr } \bar{H}_i \text{ and } N \subseteq \ker \eta\}.$$

This is shown, for example, in [5] and [6].

We now let  $[g]$  be a conjugacy class of  $G$ . The set  $H_i \cap [g]$  splits into a number of  $H_i$  conjugacy classes; let representatives be  $y_k \in H_i$  for  $1 \leq k \leq r$ . Let these lift to classes with representatives  $y_{l_k}$  in  $\bar{H}_i$ . When  $\hat{\eta} \in \text{Irr } H_i$  lifts to  $\eta \in \text{Irr } \bar{H}_i$ , this gives  $\eta(y_{l_k}) = \hat{\eta}(y_k)$  for all  $l$ .

LEMMA 1. *With notation as above,*

$$(\psi_i \eta) \uparrow \bar{G}(x_j) = \sum_{k=1}^r \hat{\eta}(y_k) \sum_{\{l \mid y_{l_k} \sim_{\bar{H}_i} x_j\}} \frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|} \psi_i(y_{l_k}).$$

*Proof.* The induction formula for chosen  $i$  and  $j$  gives

$$(\psi_i \eta) \uparrow \bar{G}(x_j) = \frac{1}{|\bar{H}_i|} \sum_{\{\bar{h} \in \bar{G} \mid x_j^{\bar{h}} \in \bar{H}_i\}} \psi_i \eta(x_j^{\bar{h}}).$$

Now the classes  $[y_{l_k}]_{\bar{H}_i}$  partition  $\bar{H}_i$ , so the statement  $x_j^{\bar{h}} \in \bar{H}_i$  is equivalent to  $x_j^{\bar{h}} \sim_{\bar{H}_i} y_{l_k}$  for those  $y_{l_k}$  that are  $\bar{G}$ -conjugate to  $x_j$ , that is,

$$(\psi_i \eta) \uparrow \bar{G}(x_j) = \frac{1}{|\bar{H}_i|} \sum_{k=1}^r \sum_{\{l|y_{l_k} \sim_{\bar{G}} x_j\}} \sum_{\{\bar{h} \in \bar{G} | x_j^{\bar{h}} \sim_{\bar{H}_i} y_{l_k}\}} \psi_i \eta(x_j^{\bar{h}}).$$

But,  $\psi_i \eta$  is a class function of  $\bar{H}_i$  and so its value is constant over the range of the rightmost sum above. Hence,

$$\begin{aligned} (\psi_i \eta) \uparrow \bar{G}(x_j) &= \frac{1}{|\bar{H}_i|} \sum_{k=1}^r \sum_{\{l|y_{l_k} \sim_{\bar{G}} x_j\}} |\{\bar{h} \in \bar{G} | x_j^{\bar{h}} \sim_{\bar{H}_i} y_{l_k}\}| \psi_i \eta(y_{l_k}) \\ &= \frac{1}{|\bar{H}_i|} \sum_{k=1}^r \sum_{\{l|y_{l_k} \sim_{\bar{G}} x_j\}} |[y_{l_k}]_{\bar{H}_i}| |\{\bar{h} \in \bar{G} | x_j^{\bar{h}} = y_{l_k}\}| \psi_i \eta(y_{l_k}) \\ &= \frac{1}{|\bar{H}_i|} \sum_{k=1}^r \sum_{\{l|y_{l_k} \sim_{\bar{G}} x_j\}} |[y_{l_k}]_{\bar{H}_i}| |\{\bar{h} \in \bar{G} | x_j^{\bar{h}} = x_j\}| \psi_i \eta(y_{l_k}) \\ &= \sum_{k=1}^r \sum_{\{l|y_{l_k} \sim_{\bar{G}} x_j\}} \frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|} \psi_i \eta(y_{l_k}) \\ &= \sum_{k=1}^r \hat{\eta}(y_{l_k}) \sum_{\{l|y_{l_k} \sim_{\bar{G}} x_j\}} \frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|} \psi_i(y_{l_k}). \end{aligned} \quad \square$$

Note that of course  $r$  depends on  $g$ , and [3] uses  $x_j$  for our  $g$  and  $R(x_j)$  for our  $r$ . By writing

$$a_{kj}^{(i)} = \sum_{\{l|y_{l_k} \sim_{\bar{G}} x_j\}} \frac{|C_{\bar{G}}(x_j)|}{|C_{\bar{H}_i}(y_{l_k})|} \psi_i(y_{l_k}) \tag{2.1}$$

(cf. [3, equation 1.10.5]), Lemma 1 becomes

$$\psi_i \eta \uparrow \bar{G}(x_j) = \sum_{k=1}^r a_{kj}^{(i)} \hat{\eta}(y_k).$$

This can be interpreted as multiplication of the matrix  $M_i(g) = (a_{kj}^{(i)})$  by a portion of the character table of  $H_i$ . The Fischer matrix for the class  $[g]_G$  is

$$M(g) = \begin{pmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{pmatrix}$$

where, if  $H_i \cap [g] = \emptyset$ , then  $M_i(g)$  is not defined and is omitted from  $M(g)$ .

The various properties of character tables can be used to deduce constraints on the entries of the Fischer matrices; for example, the orthogonality relations give a weighted orthogonality for the Fischer matrices. The Fischer matrices method relies on using these properties to deduce the entries of the matrices. See, for example, [3], [7] or [8] and subsequent works.

### 3. The character table of $2^2 \cdot \text{Fi}_{22}:2$

The non-split extension of  $N = 2^2$  by  $G = \text{Fi}_{22}:2$  provides a suitable example of a Fischer matrices calculation. We use information from the character tables of the groups  $2 \cdot \text{Fi}_{22}:2$  and  $2 \times 2 \cdot \text{Fi}_{22}:2$  to check our calculations.

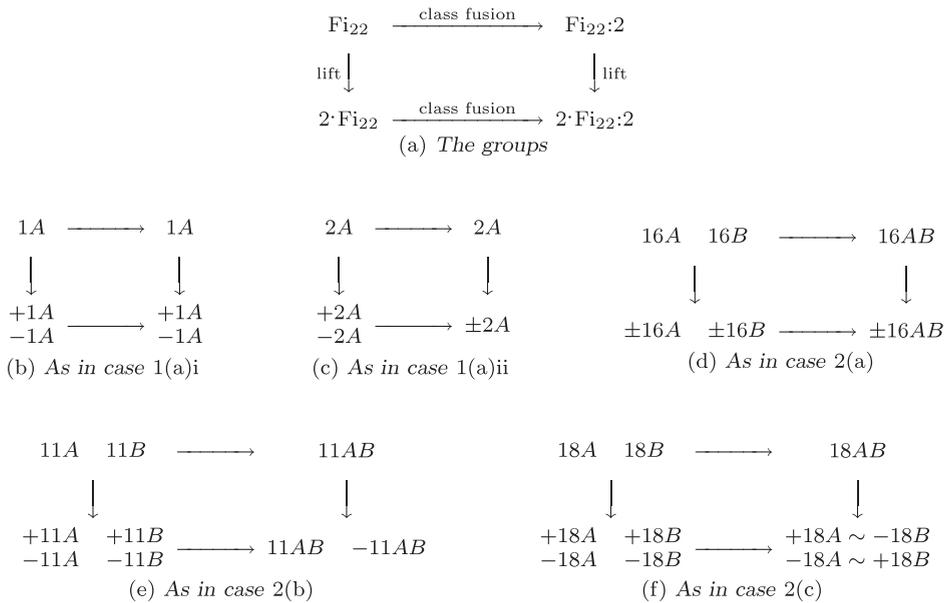


FIGURE 1. The relationship between conjugacy classes of  $\text{Fi}_{22}$  and  $2 \cdot \text{Fi}_{22}:2$ . Horizontal arrows are for class fusion under the automorphism, and vertical arrows show lifting to the double cover. Diagram (a) shows the groups involved. The other diagrams give examples of some of the possibilities. Class names are for  $\text{Fi}_{22}$ .

We write our group  $\bar{G} = 2^2 \cdot \text{Fi}_{22}:2 \cong (2 \times 2 \cdot \text{Fi}_{22}):2$  as

$$(\langle x \rangle \times \langle z \rangle \cdot \text{Fi}_{22}) : \langle \sigma \rangle,$$

so that  $\sigma$  acts to swap  $x$  with  $xz$  and  $\langle x, z, \sigma \rangle \cong D_8$ . The group  $\bar{G}$  has three orbits on elements of the normal  $2^2$  group and the stabilizers are the following.

- (1)  $\bar{G}$  fixing the identity. The trivial character of  $2^2$  extends to  $\bar{G}$ .
- (2)  $2 \times 2 \cdot \text{Fi}_{22}$  fixing  $x$  and  $xz$ . Neither of the other two characters of the  $2^2$  that represent  $z$  faithfully can extend to the inertia group for the following reason. Choose  $g \in 2^2 \cdot \text{Fi}_{22}$  that is conjugate to  $zg$  [9] and let  $\chi$  be such an extension. Then  $\chi(g) = \chi(zg)$ , which is non-zero as  $\chi$  is linear. But,  $z$  is represented as  $-1$ , which forces  $\chi(g) = -\chi(zg)$ .
- (3)  $\bar{G}$  fixing  $z$ . The character that takes values  $-1$  on  $x$  and  $xz$  extends to the inertia group.

The required inertia factors are therefore  $\text{Fi}_{22}:2$ ,  $\text{Fi}_{22}$  and  $\text{Fi}_{22}:2$ , respectively, and we must use the projective character table of  $\text{Fi}_{22}$ .

The program to construct the character table (`ct22g2.gap`) is included as an electronic appendix to this paper. It defines a function, `CharacterTableTwoSquaredGsplitTwo`, for constructing the character table of such a group.

The calculations are described in the following sections.

#### 4. Conjugacy classes and Fischer matrices

We can easily find the number of conjugacy classes of  $2^2 \cdot \text{Fi}_{22}:2$  lying above a class  $[g]$  of  $\text{Fi}_{22}:2$  by counting class fusions from our inertia groups and using the fact that Fischer matrices are square. With a little more work, we can also compute the class fusion from  $2 \times 2 \cdot \text{Fi}_{22}$ , which we use later to check our calculations.

We consider the diagram in Figure 1(a) and, for  $g \in 2 \cdot \text{Fi}_{22}:2$ , write  $\hat{g}$  for the image of  $g$  under the natural homomorphism to  $\text{Fi}_{22}:2$ . We follow the method explained in detail in [8],

writing  $k$  for the number of orbits of  $N = 2^2$  acting by conjugation on the coset  $Ng$ , and  $f_j$  for the numbers of these fused by the action of  $C_G(g)$ . The following cases arise.

(1) Let  $[\hat{g}]$  be a conjugacy class of  $\text{Fi}_{22}$  that does not fuse with another class under the action of  $\sigma$ .

- (i) If  $g$  is not conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}$ , then we have the following.
  - (a) If  $g$  is not conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}:2$  (for example,  $\hat{g} \in 1A$ , as in Figure 1(b)), then we obtain three conjugacy classes with representatives  $g$ ,  $xg$  and  $zg$ , respectively. (The second column corresponds to a class of twice the size of the other two, so  $xg$  and  $xzg$  must fuse to this class.) Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

with  $k = 4$  and  $f_1 = 1, f_2 = 2, f_3 = 1$ . The class  $[\hat{g}]$  lifts to the first of these new classes and these elements have order  $|g|$ . If  $g$  has odd order, then the other two classes contain elements of order  $2|g|$ , otherwise they contain elements of order  $|g|$ . (It is possible that  $|g| = 2|\hat{g}|$ . This case does not occur for  $\text{Fi}_{22}$ ; the smallest ATLAS group where it does occur is  $L_2(17)$ .)

(b) If  $g$  is conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}:2$  (for example,  $\hat{g} \in 2A$ , as in Figure 1(c)), then we obtain three conjugacy class representatives  $xg, g$  and  $xzg$ . Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & -1 \end{pmatrix}$$

with  $k = 2$  and  $f_1 = 1, f_2 = 2, f_3 = 1$ . The class  $[\hat{g}]$  must lift to the second of these new classes, which is twice the size of the other two. The order of the elements in both classes is the same as the order of  $g$ .

- (ii) (a) Suppose that  $g$  is conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}$  and  $g$  has the same order as  $\hat{g}$  (for example,  $\hat{g} \in 2C$ ). Then  $g$  and  $zg$  are also conjugate in  $2 \cdot \text{Fi}_{22}:2$  and  $xg$  is conjugate to  $xzg$  in  $\bar{G}$ . We thus obtain two conjugacy classes and our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with  $k = 2$  and  $f_1 = 1, f_2 = 1$ . The class  $[\hat{g}]$  lifts to the first of these new classes. Both new classes contain elements with the same order as  $\hat{g}$ .

(b) Suppose that  $g$  is conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}$  and  $g$  has twice the order of  $\hat{g}$ . This case does not occur for  $\text{Fi}_{22}$ . An example is class  $2A$  of  $A_5$ . The only difference from case 1(b)i is the element orders, which are doubled.

(2) Let  $[\hat{g}]$  be a conjugacy class of  $\text{Fi}_{22}$  that is fused with a class  $[\hat{h}]$  under the action of  $\sigma$ .

- (i) If  $g$  is conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}$  (for example,  $g \in 16A$ , as in Figure 1(d)), then we obtain two conjugacy classes with representatives  $g \sim zg \sim h \sim zh$  and  $xg \sim xzg \sim xh \sim xzh$ . The Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with  $k = 2$  and  $f_1 = 1, f_2 = 1$ , and the class  $[\hat{g}]$  lifts to the first of these new classes.

- (ii) Suppose that  $g$  is not conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}$  and  $[g]$  fuses with  $[h]$  in  $2 \cdot \text{Fi}_{22}:2$  (for example,  $\hat{g} \in 11A$  and  $\hat{h} \in 11B$ , as in Figure 1(e)). It follows that  $g \sim h, xg \sim xh, xzg \sim xzh$  and  $zg \sim zh$  and these are representatives for the four new conjugacy classes

in that order. Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

with  $k = 4$  and  $f_1 = f_2 = f_3 = f_4 = 1$ .

(iii) Suppose that  $g$  is not conjugate to  $zg$  in  $2 \cdot \text{Fi}_{22}$  and  $[g]$  fuses with  $[zh]$  in  $2 \cdot \text{Fi}_{22}:2$  (for example,  $\hat{g} \in 18A$  and  $\hat{h} \in 18B$ , as in Figure 1(f)). The four new conjugacy classes are those of  $xg \sim xzh$ ,  $g \sim zh$ ,  $zg \sim h$  and  $xzg \sim xh$  in that order. The Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

with  $k = 4$  and  $f_1 = f_2 = f_3 = f_4 = 1$ .

(3) Suppose now that  $\hat{g} \in \text{Fi}_{22}:2 \setminus \text{Fi}_{22}$ . Then  $g^x = xgx = xxzg = zg$ , so  $g$  is always conjugate to  $zg$  in  $G$ .

(i) Suppose that  $\hat{g}$  has order indivisible by four and lifts to one class in  $2 \cdot \text{Fi}_{22}:2$  of elements of the same order, for example,  $\hat{g} \in 2D$ . Now  $(xg)(xg) = zg^2$ , which must have even order, so  $\hat{g}$  lifts to two classes in  $\bar{G}$ , the second (with representatives  $xg$  and  $xzg$ ) consisting of elements of twice the order of those in the first, which have the same order as  $\hat{g} \in \text{Fi}_{22}:2$ .

(ii) Suppose that  $\hat{g}$  has order indivisible by four and lifts to one class in  $2 \cdot \text{Fi}_{22}:2$  of elements of twice the order, for example,  $\hat{g} \in 2F$ . This means that  $(g)(g) = zg^2$ , so  $\hat{g}$  lifts to two classes in  $\bar{G}$ , the first (with representatives  $g$  and  $zg$ ) consisting of elements of twice the order of those in the second, which have the same order as  $\hat{g} \in \text{Fi}_{22}:2$ .

(iii) If  $\hat{g}$  has order indivisible by four and lifts to two classes in  $2 \cdot \text{Fi}_{22}:2$  of elements of the same order, for example,  $\hat{g} \in 6M$ , then  $(zg)(zg) = g^2$  and  $(xg)(xg) = zg^2$ . Therefore,  $\hat{g}$  lifts to two classes in  $\bar{G}$ , the second (with representatives  $xg$  and  $xzg$ ) consisting of elements of twice the order of those in the first, which have the same order as  $\hat{g} \in \text{Fi}_{22}:2$ .

(iv) Suppose that  $\hat{g}$  has order divisible by four and lifts to two classes of elements of the same order as  $\hat{g}$ , for example,  $\hat{g} \in 8F$ . Then  $\hat{g}$  lifts to two conjugacy classes in  $2 \cdot \text{Fi}_{22}:2$ , both consisting of elements of the same order as  $\hat{g} \in \text{Fi}_{22}:2$ .

(v) Suppose that  $\hat{g}$  has order divisible by four and lifts to two classes of elements of the same order as  $\hat{g}$ . This case does not occur in  $\text{Fi}_{22}$ ; an example is class  $4A$  of  $A_5$ . In this case  $\hat{g}$  lifts to two conjugacy classes in  $2 \cdot \text{Fi}_{22}:2$ , both consisting of elements of twice the order as  $\hat{g} \in \text{Fi}_{22}:2$ .

In all of these cases the Fischer matrix is  $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  with  $k = 2$  and  $f_1 = f_2 = 1$ .

### 5. From class functions to characters

Our Fischer matrices are only defined up to multiplication of rows by  $-1$ , and in the  $4 \times 4$  matrices any permutation of the bottom three rows is possible. However, we have chosen class fusion from  $2 \times 2 \cdot \text{Fi}_{22}$ , which forces us to use the matrices given above.

Using the Fischer matrices from above, and from GAP the character table of  $\text{Fi}_{22}:2$  and the projective character table of  $\text{Fi}_{22}$ , we assemble a table of class functions  $\chi_i$  for  $\bar{G}$  that obey row and column orthogonality. The element orders follow from the calculations above, as does class fusion from  $2 \times 2 \cdot \text{Fi}_{22}$ . Because of our choice of ordering of the conjugacy classes, we can also write down the projection map to  $2 \cdot \text{Fi}_{22}:2$ .

We use these maps to restrict each of our class functions to a class function  $\psi$  of the group  $\text{Fi}_{22}:2$ . We then check that  $\langle \psi, \chi \rangle \in \mathbb{N} \cup \{0\}$  for all  $\chi \in \text{Irr}(\text{Fi}_{22}:2)$ . (Whenever it was not, the

reason was always because we had not used the correct Fischer matrix for our chosen ordering of the conjugacy classes.)

## 6. Power maps

To compute the power maps, we observe that classes lying above  $[\hat{g}]$  must power up to classes lying above  $[\hat{g}^p]$  for all  $p$  and, for odd  $p$ , elements in  $[ng]$  must  $p$ -power to elements in  $[ng^p]$  for all  $n \in \langle x, z \rangle$ . For  $p = 2$  and  $\hat{g} \in \text{Fi}_{22}$ , elements in  $[ng]$  square to elements in  $[g]$  and, for outer elements, the 2-power map is clear from item 4 of the case analysis.

We can also use GAP to compute possible power maps from the character table. Unusually, it produces unique  $p$ -power maps for our table and these agree with ours. Furthermore, these agree with the power maps of  $2 \times 2 \cdot \text{Fi}_{22}$ .

To further test our class functions and power maps, we check that all symmetric and anti-symmetric parts of all irreducibles have non-negative inner products with all irreducibles.

Finally, GAP produces four possible class fusions to  $\text{Fi}_{24}$ . There are two independent choices.

- There are two classes of elements of order 26 lying above class 13A of  $\text{Fi}_{22}:2$ . These could fuse either way round to the algebraically conjugate classes 26B and 26C of  $\text{Fi}_{24}$ .
- Our labelling of the involutions  $x$  and  $xz$  was arbitrary; if we swap them then our choice of class representatives in cases 1(a)ii, 2(b) and 2(c) is affected. For example, in case 2(b), the class representatives would become (in order)  $g \sim h$ ,  $xzg \sim xzh$ ,  $zg \sim zh$  and  $zg \sim zh$ . The labelling of  $x$  and  $xz$  corresponds to the other choice for class fusion. (The classes of elements of order four lying above 11A are not affected, since they fuse to the same class in  $\text{Fi}_{24}$ .)

## 7. A GAP function for the general case

The function

`CharacterTableTwoSquaredGsplitTwo(t_g, t_g2, t_2g, t_2g2, proj1, proj2)`

assembles the character table. The arguments are:

- (1) `t_g`, the character table of  $G$ ;
- (2) `t_g2`, the character table of  $G:2$ ;
- (3) `t_2g`, the character table of  $2 \cdot G$ ;
- (4) `t_2g2`, the character table of  $2 \cdot G:2$ ;
- (5) `proj1`, the index in `ProjectivesInfo(t_g)` for the record with name  $2 \cdot G$ . This is usually 1;
- (6) `proj2`, the index in `ProjectivesInfo(t_g2)` for the record with name  $2 \cdot G:2$ . This is usually 1.

The function takes only a few seconds, so such character tables can be computed on demand.

## 8. Conclusions

Finally, a word of warning. We have always assumed that in  $2 \cdot G:2$  the normal subgroup  $2 \cdot G$  has a complement. However, this need not always be the case: for example,  $A_6$  has three involutory automorphisms,  $\sigma$ ,  $\tau$  and  $\rho = \sigma\tau$  say, with

$$A_6 \cdot \langle \sigma \rangle \cong S_6 \quad A_6 \cdot \langle \tau \rangle \cong \text{PGL}_2(9) \quad A_6 \cdot \langle \rho \rangle \cong M_{10}.$$

Our program can produce character tables for  $2^2 \cdot S_6$  and  $2^2 \cdot \text{PGL}_2(9)$ , but not for  $2^2 \cdot M_{10}$ . This is because there is no group  $2 \cdot A_6 \cdot 2_3$ , just a group  $(4 \circ 2 \cdot A_6) \cdot 2_3$  which is isoclinic to  $(2 \times 2 \cdot A_6) \cdot 2_3$ .

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