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ON IMPROPER INTEGRALS OF PRODUCTS OF LOGARITHMIC, POWER AND BESSEL FUNCTIONS

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In this paper we have evaluated the integral $\int_0^\infty x^{\nu+1} J_\nu(ax) K_0(bx) \ln(x/2b) dx$. A new integral representation of the Euler constant is shown. Some special cases of the result are discussed and an open problem is posed.

INTRODUCTION

Integrals of products of Bessel functions and powers have been a matter of curiosity for a long time [**1**, **2**, **7**]. Askey, Koornwinder and Rahman [**1**] have evaluated integrals involving powers and product of Bessel functions. A little attempt has been made to evaluate integrals involving powers, logarithmic and Bessel functions [**4**, **5**]. These integrals are of extreme importance in many branches of mathematical physics, elasticity, potential theory and applied probability [**6**, **7**]. In this paper we have evaluated the integral

$$\int_0^\infty x^{\nu+1} J_\nu(ax) K_0(bx) \ln(x/2b) dx. \tag{1}$$

We have obtained a new integral representation of the Euler constant

$$\gamma = -\int_0^\infty J_0\left(\sqrt{t}\right) K_0\left(\sqrt{t}\right) \ln\left(\sqrt{t}\right) dt, \tag{2}$$

as a special case of our result. Some other special cases are also discussed and an open problem is posed.

THEOREM. *For $a > 0$, $b > 0$, and $\nu > -1$,*

$$\int_0^\infty x^{\nu+1} J_\nu(ax) K_0(bx) \ln(x/2b) dx = (2a)^\nu \frac{\Gamma(\nu+1)}{(a^2+b^2)^{\nu+1}} [\psi(\nu+1) - \ln(a^2+b^2)].$$

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PROOF: It is known [**5**, p.340] that

$$\int_0^\infty x^{\alpha-1}\exp\left(-sx-bx^{-1}\right)dx = 2(b/s)^{\alpha/2}K_\alpha\left(2\sqrt{bs}\right), \quad a>0, s>0. \tag{3}$$

A formal differentiation with respect to α yields

$$\int_0^\infty x^{\alpha-1}(\ln x)\exp\left(-sx-bx^{-1}\right)dx = 2(b/s)^{\alpha/2}\left\{\frac{1}{2}\ln(b/s)K_\alpha\left(2\sqrt{bs}\right)+\frac{\partial}{\partial\alpha}K_\alpha\left(2\sqrt{bs}\right)\right\}. \tag{4}$$

The process of differentiation in (4) is justified [**3**, p.430].

Substituting $\alpha = 0$ in (4) and using the fact that $(\partial/\partial\alpha)[K_\alpha(z)]_{\alpha=0} = 0$ [**5**, p.358] we get

$$\int_0^\infty \frac{\ln x}{x}\exp\left(-sx-bx^{-1}\right)dx = \ln(b/s)K_0\left(2\sqrt{bs}\right), \quad b>0,\ s>0. \tag{5}$$

We can write equation (5) in the operational form as follows

$$L\left\{\frac{\ln x}{x}\exp(-b/x);s\right\} = \ln(b/s)K_0\left(2\sqrt{bs}\right) \tag{6}$$

where L is the Laplace transform operator [**4**, p.129].

However, [**4**, p.132]

$$L\{x^{\nu-1}f(1/x);s\} = s^{-\nu/2}\int_0^\infty x^{\nu/2}J_\nu\left(2\sqrt{xs}\right)F(x)dx \tag{7}$$

where $F(s)$ is the Laplace transform of $f(x)$. It follows from (6) and (7) that

$$\int_0^\infty x^{\nu/2}J_\nu\left(2\sqrt{sx}\right)K_0\left(2\sqrt{bx}\right)\ln(x/b)dx = s^{\nu/2}\int_0^\infty x^\nu(\ln x)\exp(-(b+s)x)dx \tag{8}$$

However [**5**, p.576]

$$\int_0^\infty x^{\nu-1}(\ln x)e^{-\mu x}dx = \frac{\Gamma(\nu)}{\mu^\nu}[\psi(\nu)-\ln\mu], \quad \mu>0, \nu>0 \tag{9}$$

Therefore, it follows from (8) and (9) that

$$\int_0^\infty x^{\nu/2}J_\nu\left(2\sqrt{sx}\right)K_0\left(2\sqrt{bx}\right)\ln(x/b)dx = s^{\nu/2}\frac{\Gamma(\nu+1)}{(b+s)^{\nu+1}}[\psi(\nu+1)-\ln(b+s)]. \quad (b>0, s>0, \nu>-1). \tag{10}$$

Now replacing s by a^2, b by b^2 and then by using the transformation $2\sqrt{x} = t$, we get the proof of the theorem. ∎

COROLLARY 1.

$$\int_0^\infty J_0(2\sqrt{ax})K_0\left(2\sqrt{bx}\right)\ln(x/b)dx = \frac{-1}{(a+b)}(\gamma + \ln(a+b)), \quad a > 0, b > 0 \tag{11}$$

where γ is the Euler constant [5].

PROOF: This follows from (10) when we take $\nu = 0$ and $s = a$. ∎

COROLLARY 2.

$$\gamma = -\int_0^\infty J_0\left(\sqrt{t}\right)K_0\left(\sqrt{t}\right)\ln\left(\sqrt{t}\right)dt. \tag{12}$$

PROOF: This follows from (11) when we take $a = b = 1/2$ and use the transformation $2x = t$. ∎

COROLLARY 3.

(13)

$$\int_0^\infty \sin\left(2\sqrt{ax}\right)K_0\left(2\sqrt{bx}\right)\ln(x/b)dx = \frac{\pi\sqrt{a}}{(a+b)^{3/2}}\left[\psi(3/2) - \ln(a+b)\right], \quad a > 0, b > 0$$

PROOF: This follows from (10) when we take $\nu = 1/2$ and use the fact that [5, p.966]

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z.$$

In particular, when $a = b = 1/2$ in (13) we get

$$\int_0^\infty \sin\left(\sqrt{2x}\right)K_0\left(\sqrt{2x}\right)\ln\left(\sqrt{2x}\right)dx = \frac{\pi}{2\sqrt{2}}[2 - 2\ln 2 - \gamma] = 0.6816 - 1.1107\gamma$$

or

$$\int_0^\infty x\sin(x)K_0(x)\ln(x)dx = \frac{\pi}{4\sqrt{2}}[2 - 2\ln 2 - \gamma] = 0.3408 - 0.5554\gamma.$$

∎

COROLLARY 4.

$$\int_0^\infty \cos\left(2\sqrt{ax}\right)K_0\left(2\sqrt{bx}\right)\ln(x/b)\frac{dx}{\sqrt{x}} = \frac{\pi}{\sqrt{a+b}}\left[\psi(1/2) - \ln(a+b)\right]. \tag{14}$$

PROOF: This follows from (10) when we taken $\nu = -1/2$ and use the fact [5, p.966]

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z.$$

In particular when $a = b = 1/2$ in (14) we get

$$\int_0^\infty \cos\left(\sqrt{2x}\right) K_0\left(\sqrt{2x}\right) \ln\left(\sqrt{2x}\right) \frac{dx}{\sqrt{x}} = -\frac{\pi}{2}(\gamma + 2\ln 2)$$

or

$$\int_0^\infty \cos(t) K_0(t) \ln(t) dt = -\frac{\pi}{2\sqrt{2}}(\gamma + 2\ln 2).$$

□

COROLLARY 5. See [5, p.577].

$$\int_0^\infty \frac{\ln x}{x} \exp(-\mu(x/c + c/x)) dx = 2\ln c K_0(2\mu), \quad c > 0, \ \mu > 0.$$

PROOF: This follows from (5) when we take $s = \mu/c$ and $b = \mu c$. □

For some suitable values of the constants ρ, a, b and c evaluate the integral. The following problem remains open

$$\int_0^\infty x^\rho (\ln(ax))^n J_\alpha(bx) K_\beta(cx) dx, \quad n \geqslant 2.$$

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