Bull. Aust. Math. Soc. **101** (2020), 233–237
doi:10.1017/S0004972719000820

GENERALISATION OF A RESULT ON DISTINCT PARTITIONS WITH BOUNDED PART DIFFERENCES

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(Received 22 May 2019; accepted 14 June 2019; first published online 24 July 2019)

Abstract

We generalise a result of Chern ['A curious identity and its applications to partitions with bounded part differences', *New Zealand J. Math.* **47** (2017), 23–26] on distinct partitions with bounded difference between largest and smallest parts. The generalisation is proved both analytically and bijectively.

2010 *Mathematics subject classification*: primary 11P84; secondary 05A17.

Keywords and phrases: partition, distinct partition, bounded part difference.

1. Introduction

Throughout the paper, we adopt the following q-series notation:

$$(a;q)_0 = 1,$$

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad n \in \mathbb{N}.$$

A *partition* λ of a positive integer n is a nonincreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ (see [1]). The terms λ_i are called the *parts* of λ, and the number of parts of λ is called the *length* of λ.

Recently, partitions with fixed or bounded difference between the largest and smallest parts have received much attention. In 2015, Andrews *et al.* [2] initiated the study of partitions where the difference between largest and smallest parts is a fixed positive integer t, and proved the following surprising result.

Theorem 1.1 [2]. *Let $\mathcal{P}(t,n)$ be the set of partitions of n with fixed difference t between largest and smallest parts. For $t > 1$,*

$$\sum_{n=1}^{\infty} |\mathcal{P}(t,n)|q^n = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q;q)_t} + \frac{q^{t-1}}{(1-q^{t-1})(q;q)_t}.$$

This work was supported by the National Natural Science Foundation of China (Nos. 11401080 and 11871246).

This work motivated studies of partitions with the difference between largest and smallest parts being at most t (see [2–9]). The results on distinct partitions and odd partitions, which were found by Chern [5], interest us the most.

THEOREM 1.2 [5]. *Let $\mathcal{P}_d(t, n)$ be the set of partitions of n into distinct parts in which the difference between the largest and smallest parts is at most t. For $t \geq 1$,*

$$\sum_{n=1}^{\infty} |\mathcal{P}_d(t, n)| q^n = \frac{1}{1 - q^{t+1}}((-q; q)_{t+1} - 1).$$

THEOREM 1.3 [5]. *Let $\mathcal{P}_o(t, n)$ be the set of partitions of n into odd parts in which the difference between the largest and smallest parts is at most t. For $t \geq 1$,*

$$\sum_{n=1}^{\infty} |\mathcal{P}_o(t, n)| q^n = \frac{1}{1 - q^{2t}}\left(\frac{1}{(q; q^2)_t} - 1\right).$$

More recently, Lin [8] generalised the result on odd partitions to k-regular partitions (partitions with no part divisible by k).

THEOREM 1.4 [8]. *Let $\mathcal{R}_k(t, n)$ be the set of k-regular partitions of n with the difference between the largest and smallest parts at most kt. For $t \geq 1$ and $k \geq 2$,*

$$\sum_{n=1}^{\infty} |\mathcal{R}_k(t, n)| q^n = \frac{1}{1 - q^{kt}}\left(\frac{(q^k; q^k)_t}{(q; q)_{kt}} - 1\right).$$

Our aim is to generalise Theorem 1.2. The rest of the paper is organised as follows. In Section 2, we present the main result accompanied by an analytic proof. A bijective proof is given in Section 3.

2. The main result

Given integers $k \geq 2$ and $t \geq 0$, let $\mathcal{P}_k(t, n)$ be the set of partitions of n in which the difference between the largest and smallest parts is at most $t + 1$, each part occurs at most $k - 1$ times and the largest and smallest parts occur at most $k - 1$ times together if the difference between them is exactly $t + 1$. Denote by $\mathcal{P}_k(t, m, n)$ the set of partitions in $\mathcal{P}_k(t, n)$ with m parts.

THEOREM 2.1. *For $|q| < 1$ and $|zq| < 1$,*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\mathcal{P}_k(t, m, n)| z^m q^n = \frac{1}{1 - q^{t+1}}\left(\frac{(z^k q^k; q^k)_{t+1}}{(zq; q)_{t+1}} - 1\right).$$

PROOF. The standard methods for producing partition generating functions reveal directly that

https://doi.org/10.1017/S0004972719000820 Published online by Cambridge University Press

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|\mathcal{P}_k(t,m,n)|z^m q^n &= \sum_{n=1}^{\infty}\sum_{\substack{1\le i\le k-1\\ 0\le j<k-i}} z^i q^{in}\cdot z^j q^{j(n+t+1)}\cdot\frac{(z^kq^{k(n+1)};q^k)_t}{(zq^{n+1};q)_t}\\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{k-1}\sum_{i+j=1}^{k-1} z^{i+j}q^{(i+j)n}\cdot q^{j(t+1)}\cdot\frac{(z^kq^{k(n+1)};q^k)_t}{(zq^{n+1};q)_t}\\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{k-1}\sum_{j=0}^{i-1} z^i q^{in}\cdot q^{j(t+1)}\cdot\frac{(z^kq^{k(n+1)};q^k)_t}{(zq^{n+1};q)_t}\\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{k-1} z^i q^{in}\cdot\frac{1-q^{i(t+1)}}{1-q^{t+1}}\cdot\frac{(z^kq^{k(n+1)};q^k)_t}{(zq^{n+1};q)_t}\\
&= \frac{1}{1-q^{t+1}}\sum_{n=1}^{\infty}\sum_{i=0}^{k-1}(z^iq^{in}-z^iq^{i(n+t+1)})\cdot\frac{(z^kq^{k(n+1)};q^k)_t}{(zq^{n+1};q)_t}\\
&= \frac{1}{1-q^{t+1}}\sum_{n=1}^{\infty}\Big(\frac{1-z^kq^{kn}}{1-zq^n}-\frac{1-z^kq^{k(n+t+1)}}{1-zq^{n+t+1}}\Big)\frac{(z^kq^{k(n+1)};q^k)_t}{(zq^{n+1};q)_t}\\
&= \frac{1}{1-q^{t+1}}\sum_{n=1}^{\infty}\Big(\frac{(z^kq^{kn};q^k)_{t+1}}{(zq^n;q)_{t+1}}-\frac{(z^kq^{k(n+1)};q^k)_{t+1}}{(zq^{n+1};q)_{t+1}}\Big)\\
&= \frac{1}{1-q^{t+1}}\Big(\frac{(z^kq^k;q^k)_{t+1}}{(zq;q)_{t+1}}-1\Big),\end{aligned}$$

which completes the proof. □

Remark 2.2. We claim that $\mathcal{P}_2(t,n)$ is the set of partitions of n into distinct parts with bounded difference t between largest and smallest parts. Since the largest and smallest parts appear at least twice together, the difference between them cannot be $t+1$. In addition, each part occurs at most once. Therefore, the claim holds. Taking $z=1$ and $k=2$ in Theorem 2.1,

$$\sum_{n=1}^{\infty}|\mathcal{P}_2(t,n)|q^n = \frac{1}{1-q^{t+1}}\Big(\frac{(q^2;q^2)_{t+1}}{(q;q)_{t+1}}-1\Big) = \frac{1}{1-q^{t+1}}((-q;q)_{t+1}-1),$$

which is Theorem 1.2.

3. A bijective proof

In this section, we present a bijective proof of Theorem 2.1.

Denote by $\mathcal{B}_k(t,m,n)$ the set of bipartitions $(\mu;\nu)$ of n where the nonempty partition μ has exactly m parts, each being at most $t+1$ and occurring fewer than k times, and ν can only have $t+1$ as a part. It is clear that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|\mathcal{B}_k(t,m,n)|z^m q^n = \frac{1}{1-q^{t+1}}\Big(\frac{(z^kq^k;q^k)_{t+1}}{(zq;q)_{t+1}}-1\Big).$$

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Theorem 3.1. *There is a bijection φ between $\mathcal{P}_k(t,m,n)$ and $\mathcal{B}_k(t,m,n)$.*

Proof. Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \cup_{n\geq 1}\mathcal{P}_k(t,m,n)$, we define an operation ρ on μ by

$$\rho(\mu) = \begin{cases} (\mu_2, \mu_3, \ldots, \mu_m, \mu_1 - (t+1)) & \text{if } \mu_1 > t+1, \\ \mu & \text{otherwise.} \end{cases}$$

We claim that $\rho(\mu) \in \cup_{n\geq 1}\mathcal{P}_k(t,m,n)$. It is sufficient to consider the case where $\mu_1 > t+1$. Since $\mu \in \cup_{n\geq 1}\mathcal{P}_k(t,m,n)$, we have $\mu_1 - \mu_m \leq t+1$, which means that $\mu_m \geq \mu_1 - (t+1) > 0$ and $\mu_2 - (\mu_1 - (t+1)) \leq t+1$. Therefore, $\rho(\mu)$ is a partition with bounded difference $t+1$ between the largest and smallest parts and has the same length as μ.

Let $ml(\mu)$ and $ms(\mu)$ be the numbers of appearances of the largest and smallest parts of μ, respectively. If $\mu_1 = \mu_2$, then the difference between the largest and smallest parts of $\rho(\mu)$ is exactly $t+1$. Now

$$ml(\rho(\mu)) + ms(\rho(\mu)) = \begin{cases} ml(\mu) + ms(\mu) & \text{if } \mu_1 - \mu_m = t+1, \\ ml(\mu) & \text{otherwise.} \end{cases}$$

So, $ml(\rho(\mu)) + ms(\rho(\mu)) < k$ in this case. If $\mu_1 > \mu_2$, then the difference between the largest and smallest parts of $\rho(\mu)$ is at most t. Thus,

$$ms(\rho(\mu)) = \begin{cases} ms(\mu) + 1 & \text{if } \mu_1 - \mu_m = t+1, \\ 1 & \text{otherwise.} \end{cases}$$

This implies that $ms(\rho(\mu)) < k$. Since the number of appearances of other parts remains unchanged under ρ, we can conclude that $\rho(\mu) \in \cup_{n\geq 1}\mathcal{P}_k(t,m,n)$.

We now establish the bijection φ from $\mathcal{P}_k(t,m,n)$ to $\mathcal{B}_k(t,m,n)$. For $\lambda \in \mathcal{P}_k(t,m,n)$, let r be the smallest nonnegative integer i such that the largest part of $\rho^i(\lambda)$ is not greater than $t+1$, where $\rho^0(\lambda) = \lambda$ and $\rho^i(\lambda) = \rho(\rho^{i-1}(\lambda))$ for $i \geq 1$. Then the partition λ is mapped to the bipartition

$$\varphi(\lambda) = (\rho^r(\lambda); \underbrace{(t+1, t+1, \ldots, t+1)}_{r}).$$

It is easy to see that $\varphi(\lambda) \in \mathcal{B}_k(t,m,n)$.

We now show that φ is invertible. Given a bipartition $(\mu; \nu) \in \mathcal{B}_k(t,m,n)$, if ν is the empty partition, then $\varphi^{-1}((\mu;\nu)) = \mu$. Otherwise, we remove a part $t+1$ from ν, add it to the smallest part of μ and shift the last part to the first position. Repeat this transformation until the partition related to ν is completely emptied. □

Example 3.2. Let $\lambda = (9,9,7,7,6,6,6,3) \in \mathcal{P}_4(5,8,53)$; then

$$\rho(\lambda) = (9,7,7,6,6,6,3,3),$$
$$\rho^2(\lambda) = (7,7,6,6,6,3,3,3),$$
$$\rho^3(\lambda) = (7,6,6,6,3,3,3,1),$$
$$\rho^4(\lambda) = (6,6,6,3,3,3,1,1).$$

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The largest part of $\rho^4(\lambda)$ is now not greater than $t + 1 = 6$. Thus,

$$\varphi(\lambda) = ((6, 6, 6, 3, 3, 3, 1, 1); (6, 6, 6, 6)) \in \mathcal{B}_4(5, 8, 53).$$

Acknowledgement

We thank the anonymous referee for reading the manuscript carefully and providing helpful comments.

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