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Symmetries of Kirchberg Algebras

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Abstract. Let G_0 and G_1 be countable abelian groups. Let γ_i be an automorphism of G_i of order two. Then there exists a unital Kirchberg algebra A satisfying the Universal Coefficient Theorem and with $[1_A] = 0$ in $K_0(A)$, and an automorphism $\alpha \in \mathrm{Aut}(A)$ of order two, such that $K_0(A) \cong G_0$, such that $K_1(A) \cong G_1$, and such that $\alpha_* \colon K_i(A) \to K_i(A)$ is γ_i. As a consequence, we prove that every $\mathbb{Z}_2$-graded countable module over the representation ring $R(\mathbb{Z}_2)$ of $\mathbb{Z}_2$ is isomorphic to the equivariant K-theory $K^{\mathbb{Z}_2}(A)$ for some action of $\mathbb{Z}_2$ on a unital Kirchberg algebra A.

Along the way, we prove that every not necessarily finitely generated $\mathbb{Z}[\mathbb{Z}_2]$-module which is free as a $\mathbb{Z}$-module has a direct sum decomposition with only three kinds of summands, namely $\mathbb{Z}[\mathbb{Z}_2]$ itself and $\mathbb{Z}$ on which the nontrivial element of $\mathbb{Z}_2$ acts either trivially or by multiplication by -1.

1 Introduction

Following Definition 4.3.1 of Part 1 of [22], we use the term Kirchberg algebra for a purely infinite simple separable nuclear C^*-algebra. In this paper, we prove that any order two automorphism of the K-theory of a unital Kirchberg algebra A satisfying the Universal Coefficient Theorem, and with $[1_A] = 0$ in $K_0(A)$, lifts to an order two automorphism of A.

Recall the classification theorem for unital Kirchberg algebras satisfying the Universal Coefficient Theorem ([14], Theorem 4.2.4 of [19]): if A and B are such algebras, and if $\theta \colon K_*(A) \to K_*(B)$ is a graded isomorphism such that $\theta([1_A]) = [1_B]$, then there is an isomorphism $\varphi \colon A \to B$ such that $\varphi_* = \theta$. The statement also holds with "isomorphism" replaced by "homomorphism" everywhere ([14]; in [19] see Theorem 4.1.3 and the proofs of Corollary 4.4.2 and Theorem 4.2.4). Moreover, every pair of countable abelian groups occurs as the K-theory of a unital Kirchberg algebra satisfying the Universal Coefficient Theorem, and the K_0-class of the identity can be arbitrary (Section 4.4 in Part 1 of [22] or Theorem 5.2 of [9]). Thus, the classification functor is surjective on isomorphism classes in this case.

George Elliott has asked to what extent this functor "splits", in the same sense in which a surjection between abelian groups might (or might not) split. That is: How close can one come to constructing a functor F, from the values of the invariant, to Kirchberg algebras satisfying the Universal Coefficient Theorem, such that $K_*\big(F(G_*)\big) \cong G_*$, etc.? One can't do this exactly. (See Example 1.1 below.) We may then ask for some weaker sort of splitting, or we can try to eliminate the problem by reducing the number of morphisms in the categories. The question we consider here,

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of lifting finite order automorphisms of the K-theory to automorphisms of the algebra of the same order, is a special case of what one gets by restricting the morphisms to be isomorphisms.

Our method is to find, for given countable abelian groups G_i and order two automorphisms γ_i, some separable nuclear C^*-algebra A_0 whose K-theory is G_* and some order two automorphism α_0 of A_0 which lifts the γ_i. In fact, A_0 will be type I. Then we apply the construction of [15] to get a Kirchberg algebra with the same properties. To construct A_0, we regard G_* as a module over the group ring $\mathbb{Z}[\mathbb{Z}_2]$. (Throughout this paper, $\mathbb{Z}_p$ denotes $\mathbb{Z}/p\mathbb{Z}$, not the p-adic integers.) Generalizing an old result for the finitely generated case (see Lemma 1 of [11]), we prove that every $\mathbb{Z}[\mathbb{Z}_2]$-module which is free as a $\mathbb{Z}$-module has a direct sum decomposition with only three kinds of summands, namely $\mathbb{Z}[\mathbb{Z}_2]$ itself and $\mathbb{Z}$ on which the nontrivial element of $\mathbb{Z}_2$ acts either trivially or by multiplication by -1. (Butler and Kovács have also obtained similar results, by slightly different methods [5]; a more general result is to appear in [4].) We produce A_0 by combining this structure theorem with part of Schochet's geometric realization technique from [24].

We want to explicitly point out that the K-theory of a C^*-algebra A with an action of $\mathbb{Z}_2$, regarded as a $\mathbb{Z}[\mathbb{Z}_2]$-module, is not the same as the $\mathbb{Z}_2$-equivariant K-theory of A, regarded as a module over the representation ring $R(\mathbb{Z}_2) \cong \mathbb{Z}[\mathbb{Z}_2]$. It is, however, the same as the $\mathbb{Z}_2$-equivariant K-theory of $C^*(\mathbb{Z}_2, A)$, using the dual action. (Compare with Lemma 4.4.) Thus, as a corollary of our construction we obtain a realization theorem, Theorem 4.7, for $R(\mathbb{Z}_2)$-modules as the equivariant K-theory of actions of $\mathbb{Z}_2$ on Kirchberg algebras. There is also a version for type I C^*-algebras.

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This paper is organized as follows. In the rest of this section, we demonstrate the failure of the classification functor to split. In Section 2 we prove a structure theorem for $\mathbb{Z}[\mathbb{Z}_2]$-modules which are free as $\mathbb{Z}$-modules. In Section 3 we prove the weak form of realization, using some type I C^*-algebra with the right K-theory. In Section 4 we prove the main result, and also the realization theorems for $\mathbb{Z}_2$-equivariant K-theory. Section 5 contains an explicit formula for the order two automorphism in the simplest nontrivial case of our theorem, namely the nontrivial automorphism of $K_*\big(M_3(\mathcal{O}_4)\big)$.

Here is the example which shows that the most obvious sort of splitting of the classification functor is not possible.

Example 1.1 There is no functor F, from the category of $\mathbb{Z}_2$-graded abelian groups G_* with distinguished element g in degree zero and graded homomorphisms preserving the distinguished elements, to the category of unital Kirchberg algebras satisfying the Universal Coefficient Theorem of [23] and unital homomorphisms, such that

$K_*\big(F(G_*, g)\big) \cong G_*$ with $[1_{F(G_*,g)}] \mapsto g$ for all G_* and g, and such that the diagram

$$\begin{array}{ccc} K_*\big(F(G_*, g)\big) & \xrightarrow{F(\theta)_*} & K_*\big(F(H_*, h)\big) \\ \big\downarrow & & \big\downarrow \\ G_* & \xrightarrow{\theta} & H_* \end{array}$$

commutes whenever $\theta\colon G_* \to H_*$ is a graded homomorphism such that $\theta(g) = h$.

Suppose we had such a functor. Take

$$G_0 = 0, \quad G_1 = 0, \quad H_0 = \mathbb{Z}, \quad \text{and} \quad H_1 = 0.$$

Take $g = 0$ and $h = 0$. Let $\theta\colon G_* \to H_*$ and $\rho\colon H_* \to G_*$ be the unique homomorphisms between these groups. Then $\rho \circ \theta = \mathrm{id}_{G_*}$. Therefore, using functoriality,

$$F(\rho) \circ F(\theta) = F(\rho \circ \theta) = F(\mathrm{id}_{G_*}) = \mathrm{id}_{F(G_*)} .$$

In particular, $F(\rho)$ is surjective. However, $F(G_*, g)$ (which is isomorphic to the Cuntz algebra $\mathcal{O}_2$ [6]) and $F(H_*, h)$ (which is isomorphic to a suitable corner of $\mathcal{O}_\infty$) are nonisomorphic simple C^*-algebras, so there are no surjective homomorphisms from $F(H_*, h)$ to $F(G_*, g)$.

Remark 1.2 The problem here has nothing to do with units, since the same argument works even if one considers stable Kirchberg algebras satisfying the Universal Coefficient Theorem. Moreover, one can even restrict, in either case, to the case $K_1 = 0$.

2 Modules over the Group Ring of $\mathbb{Z}_2$

In this section, we prove a structure theorem for modules over the group ring of $\mathbb{Z}_2$ which are free as $\mathbb{Z}$-modules. We have learned that Butler and Kovác proved the main result of this section a little earlier than we did, using slightly different methods [5]. They use a sophisticated argument to reduce to the countably generated case first, which is avoided in our analysis. There is also a subsequent generalization to modules over the group ring of $\mathbb{Z}_p$ for an arbitrary prime p [4]. We have decided to retain our proof for completeness of exposition, and because it gives the result we need with a minimum of machinery.

For any unital ring R and any discrete group Γ, we let $R[\Gamma]$ denote the ordinary algebraic group ring of Γ with coefficients in R. Although we are ultimately interested in the case $R = \mathbb{Z}$, we will need the cases $R = \mathbb{F}_2$, the field with two elements, and $R = \mathbb{Q}$. Also, any $\mathbb{Z}[\Gamma]$-module is automatically a $\mathbb{Z}$-module (abelian group) by restriction of scalars.

The structure theorem is already known for the finitely generated case (see the first lemma), and the main point of this section is to remove the finite generation hypothesis.

Lemma 2.1 *Let N be a finitely generated $\mathbb{Z}[\mathbb{Z}_2]$-module which is free as a $\mathbb{Z}$-module. Then N is a finite direct sum $\bigoplus_{i\in I} N_i$ in which each N_i is isomorphic to one of the following three $\mathbb{Z}[\mathbb{Z}_2]$-modules:*

- *$T_1 = \mathbb{Z}$ with the nontrivial element of $\mathbb{Z}_2$ acting trivially.*
- *$T_2 = \mathbb{Z}$ with the nontrivial element of $\mathbb{Z}_2$ acting by multiplication by -1.*
- *$T_3 = \mathbb{Z}[\mathbb{Z}_2]$.*

The multiplicities of T_1, T_2 and T_3 in such a direct sum decomposition are independent of the decomposition.

Proof This is an immediate consequence of the canonical form for invertible elements $a \in M_n(\mathbb{Z})$ with $a^2 = 1$, given in Lemma 1 of [11], and the discussion following the proof of that lemma. For the matrix L of [11], the computation

$$L = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

shows that it is similar over $\mathbb{Z}$ to the action of the nontrivial element of $\mathbb{Z}_2$ on $\mathbb{Z}[\mathbb{Z}_2]$ in its usual basis. ■

Theorem 74.3 of [8] gives a structure theorem for finitely generated $\mathbb{Z}[\mathbb{Z}_p]$-modules that are free as $\mathbb{Z}$-modules, for arbitrary primes p. As described there, the direct sum decomposition is in general not unique.

Lemma 2.2 *Let M be a $\mathbb{Z}[\mathbb{Z}_2]$-module. Suppose M is free as a $\mathbb{Z}$-module and $M/2M$ is free as an $\mathbb{F}_2[\mathbb{Z}_2]$-module. Then M is free as a $\mathbb{Z}[\mathbb{Z}_2]$-module.*

Proof Theorem 6.1 of [2], with $G = \mathbb{Z}_2$ and $H = \{0\}$, shows that M is projective as a $\mathbb{Z}[\mathbb{Z}_2]$-module.

If M is finitely generated, apply Lemma 2.1. Since T_1 and T_2 are not projective as $\mathbb{Z}[\mathbb{Z}_2]$-modules, they can't appear in the direct sum. Therefore M is free.

So suppose M is not finitely generated. A direct calculation shows that $\mathbb{Z}[\mathbb{Z}_2]$ has no nontrivial idempotents. (This is true for any finite group in place of $\mathbb{Z}_2$, by Corollary 8.1 of [26].) Therefore Corollary 4.5 of [1] applies, and shows that M is free. ■

Notation 2.3 For the rest of this section, we regard $\mathbb{Z}[\mathbb{Z}_2]$ as a subring of $\mathbb{Q}[\mathbb{Z}_2]$. If M is a $\mathbb{Z}[\mathbb{Z}_2]$-module, then the corresponding $\mathbb{Q}[\mathbb{Z}_2]$-module is $\mathbb{Q} \otimes_{\mathbb{Z}} M$, and if M is free as a $\mathbb{Z}$-module then we may clearly regard M as a submodule of $\mathbb{Q} \otimes_{\mathbb{Z}} M$. We let s be the nontrivial element of $\mathbb{Z}_2$, and use the same notation for the corresponding element of $\mathbb{Z}[\mathbb{Z}_2]$. Further set

$$e = \tfrac{1}{2}(1+s) \in \mathbb{Q}[\mathbb{Z}_2] \quad \text{and} \quad f = \tfrac{1}{2}(1-s) \in \mathbb{Q}[\mathbb{Z}_2],$$

which are idempotents with $e + f = 1$.

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Lemma 2.4 *Let M be a $\mathbb{Z}[\mathbb{Z}_2]$-module which is free as a $\mathbb{Z}$-module. Let $m \in M$, and assume $(1+s)m \in 2M$. Then, following Notation 2.3,*

$$m \in (M \cap eM) + (M \cap fM) \subset \mathbb{Q} \otimes_{\mathbb{Z}} M.$$

Proof We have $(1-s)m = (1+s)m - 2m \in 2M$. Write $(1+s)m = 2m_0$ and $(1-s)m = 2m_1$ with $m_0, m_1 \in M$. In $\mathbb{Q} \otimes_{\mathbb{Z}} M$ we have $m_0 = \frac{1}{2}(1+s) = em$ and $m_1 = fm$. Then $m_0 \in M \cap eM$ and $m_1 \in M \cap fM$. Moreover, $2m = 2(m_0 + m_1)$, so $m = m_0 + m_1$. ∎

Lemma 2.5 *Let M be a $\mathbb{Z}[\mathbb{Z}_2]$-module which is free as a $\mathbb{Z}$-module. Following Notation 2.3, suppose that $M \cap eM = 2eM$ and $M \cap fM = 2fM$. Then M is free as a $\mathbb{Z}[\mathbb{Z}_2]$-module.*

Proof By Lemma 2.2, it suffices to show that $M/2M$ is free as an $\mathbb{F}_2[\mathbb{Z}_2]$-module.

Set $N = (2eM + 2fM)/M$. Consider $1+s$ as a multiplication map on $M/2M$. We claim that

$$\operatorname{Ker}(1+s) = (1+s)(M/2M) = N.$$

We consider $\operatorname{Ker}(1+s)$ first. Suppose $m \in M$ and $m + 2M \in \operatorname{Ker}(1+s)$. Then $(1+s)m \in 2M$. Using Lemma 2.4 at the first step and the hypothesis at the second, we get

$$m \in (M \cap eM) + (M \cap fM) = 2eM + 2fM,$$

so $m + 2M \in N$. For the reverse, suppose $m \in 2eM + 2fM$. Write

$$m = (1+s)m_0 + (1-s)m_1$$

with $m_0, m_1 \in M$. Then

$$(1+s)m = (1+s)^2 m_0 + (1+s)(1-s)m_1 = 2(1+s)m_0 \in 2M,$$

so $m + 2M \in \operatorname{Ker}(1+s)$.

Now we consider the range of $1+s$. Since $(1+s)^2 = 0$ in $\mathbb{F}_2[\mathbb{Z}_2]$, we get $(1+s)(M/2M) \subset \operatorname{Ker}(1+s) = N$. For the reverse, let $m \in 2eM + 2fM$, and again write $m = (1+s)m_0 + (1-s)m_1$ with $m_0, m_1 \in M$. Then

$$\begin{aligned} m + 2M = (1+s)(m_0 + m_1) - 2sm_1 + 2M &= (1+s)(m_0 + m_1) + 2M \\ &\in (1+s)(M/2M). \end{aligned}$$

This completes the proof of the claim.

Now N is a $\mathbb{F}_2$-vector subspace of $M/2M$, so there exists a $\mathbb{F}_2$-vector subspace $V \subset M/2M$ such that $V \oplus N = M/2M$. We claim that $V \cap sV = \{0\}$ and $V + sV = M/2M$. To prove the first, suppose $v, w \in V$ and $v = sw$. Then $(1+s)v = (1+s)sw = (1+s)w$, so that $(1+s)(v-w) = 0$. Therefore $v - w \in \operatorname{Ker}(1+s) \cap V = N \cap V$, whence $v = w$. Now $(1+s)v = v + w = 2v = 0$, so $v \in (1+s)(M/2M) \cap V = N \cap V$, whence $v = 0$. For the second, let $d \in M/2M$. Using the first claim, we have

$N = (1+s)(M/2M) = (1+s)(V)$. So we can write $d = x + (1+s)w$ with $w, x \in V$. Then $d = v + sw$ with $v = x + w$ and w both in V. This proves the claim.

Given the last claim, it is immediate that $M/2M \cong \mathbb{F}_2[\mathbb{Z}_2] \otimes_{\mathbb{F}_2} V$, which is a free $\mathbb{F}_2[\mathbb{Z}_2]$-module. ■

We only need the next two lemmas for the prime 2. Since they hold for arbitrary primes, with the same proof, we may as well give them in that generality.

Lemma 2.6 *Let M be a free $\mathbb{Z}$-module, and let p be a prime number. Let V be an $\mathbb{F}_p$-vector subspace of M/pM. Then there exists a direct summand L of M such that the image of L under the map $M \to M/pM$ is exactly V.*

Proof Let $\pi\colon M \to M/pM$ be the quotient map. Choose an $\mathbb{F}_p$-vector subspace W of M/pM such that $M/pM = V \oplus W$. Let $\sigma\colon M/pM \to W$ be the projection onto W obtained from this direct sum decomposition. Using an $\mathbb{F}_p$-basis for W, find a free $\mathbb{Z}$-module P_0 with an isomorphism $P_0/pP_0 \to W$. Let $\kappa_0\colon P_0 \to W$ be the composite of this isomorphism with the quotient map $P_0 \to P_0/pP_0$. Since M is a projective $\mathbb{Z}$-module and κ_0 is surjective, there exists a $\mathbb{Z}$-module homomorphism $\varphi_0\colon M \to P_0$ making the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\varphi_0} & P_0 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\kappa_0} \\ M/pM & \xrightarrow{\sigma} & W \end{array}$$

The image $\varphi_0(M)$ is a submodule of a free $\mathbb{Z}$-module, hence free (Theorem 14.5 of [10]). Let $(b_i)_{i\in I_0}$ be a $\mathbb{Z}$-basis for $\varphi_0(M)$. Since $\kappa_0 \circ \varphi_0 = \sigma \circ \pi$ is surjective, $\{\kappa_0(b_i)\colon i \in I_0\}$ spans W. Choose a subset $I \subset I_0$ such that $\bigl(\kappa_0(b_i)\bigr)_{i\in I}$ is an $\mathbb{F}_p$-basis for W. Set $P = \operatorname{span}_{\mathbb{Z}}(\{b_i : i \in I\}) \subset P_0$, which is a free $\mathbb{Z}$-module with basis $(b_i)_{i\in I}$. For each $j \in I_0 \setminus I$, choose $\alpha_{i,j} \in \mathbb{F}_p$ for $i \in I$, all but finitely many of which are zero, such that $\kappa_0(b_j) = \sum_{i\in I} \alpha_{i,j}\kappa_0(b_i)$. For $j \in I_0 \setminus I$ and $i \in I$, choose $\beta_{i,j} \in \mathbb{Z}$ whose image in $\mathbb{F}_p$ is $\alpha_{i,j}$, and such that if $\alpha_{i,j} = 0$ then $\beta_{i,j} = 0$. For each $j \in I_0 \setminus I$, all but finitely many of the $\beta_{i,j}$ are zero. Therefore there is a well defined surjective $\mathbb{Z}$-module homomorphism $\mu\colon \varphi_0(M) \to P$ such that $p(b_j) = b_j$ for $j \in I$ and $\mu(b_j) = \sum_{i\in I} \beta_{i,j} b_i$ for $j \in I_0 \setminus I$. By construction, we have $\kappa_0 \circ \mu(b_i) = \kappa_0(b_i)$ for all $i \in I$. Therefore $(\kappa_0|_P) \circ \mu = \kappa_0$. Setting $\varphi = \mu \circ \varphi_0$ and $\kappa = \kappa_0|_P$, we thus have a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & P \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\kappa} \\ M/pM & \xrightarrow{\sigma} & W \end{array}$$

The $\mathbb{Z}$-module P is still free, and in addition φ is surjective. Furthermore, $\operatorname{Ker}(\kappa) = pP$, since a linear combination of $(b_i)_{i\in I}$ has image in W equal to zero if and only if all coefficients are divisible by p.

Set $L = \mathrm{Ker}(\varphi)$. Since P is projective, there is a $\mathbb{Z}$-module homomorphism $\eta \colon P \to M$ such that $\varphi \circ \eta = \mathrm{id}_P$. Then $M = L \oplus \eta(P)$. We claim that $\pi(L) = V$. First, if $m \in L$ then $\sigma\big(\pi(m)\big) = \kappa\big(\varphi(m)\big) = 0$, so $\pi(m) \in \mathrm{Ker}(\sigma) = V$. Now suppose $v \in V$. Choose $m_0 \in M$ such that $\pi(m_0) = v$. Then $\kappa\big(\varphi(m_0)\big) = 0$, whence $\varphi(m_0) \in pP$. Write $\varphi(m_0) = px$ with $x \in P$. Then $m = m_0 - p\eta(x) \in \mathrm{Ker}(\varphi)$ by a diagram chase, and $\pi(m) = \pi(m_0) = v$. This proves the claim, and the lemma. ■

Lemma 2.7 *Let M be a free $\mathbb{Z}$-module and let p be a prime number. Let N be a $\mathbb{Z}$-submodule such that $pM \subset N \subset M$. Then there exist $\mathbb{Z}$-submodules $N_0, N_1 \subset M$ such that $M = N_0 \oplus N_1$ and $N = pN_0 \oplus N_1$.*

Proof Let $\pi \colon M \to M/pM$ be the quotient map. Using Lemma 2.6, write $M = N_0 \oplus N_1$ for $\mathbb{Z}$-submodules $N_0, N_1 \subset M$, with $\pi(N_1) = \pi(N)$. The result will now follow if we prove that $N = pN_0 + N_1$.

We prove $pN_0 + N_1 \subset N$. Clearly $pN_0 \subset N$. So let $m \in N_1$. Choose $n \in N$ such that $\pi(n) = \pi(m)$. Then $\pi(m - n) = 0$, so $m - n \in pM \subset N$. Thus $m \in N$. We have $N_1 \subset N$.

Now we prove the reverse inclusion. Let $n \in N$. Write $n = r + m_1$ with $r \in N_0$ and $m_1 \in N_1$. Then $\pi(r) = \pi(n) - \pi(m_1)$. Because $N_1 \subset N$, we get $\pi(r) \in \pi(N_0) \cap \pi(N)$. By considering a $\mathbb{Z}$-basis for M made up of bases for N_0 and N_1, we see that $\pi(N_0) \cap \pi(N) = \{0\}$. Therefore $\pi(r) = 0$. So we can write $r = pm_0$ with $m_0 \in M$. Since M is torsion free and N_0 is a summand, we get $m_0 \in N_0$. Therefore $n = pm_0 + m_1 \in pN_0 + N_1$. ■

The uniqueness statement in the following theorem will not be used in the rest of the paper, but we include it because it seems of independent interest.

Theorem 2.8 *Let M be a $\mathbb{Z}[\mathbb{Z}_2]$-module which is free as a $\mathbb{Z}$-module. Then M is a direct sum $\bigoplus_{i \in I} M_i$ in which each M_i is isomorphic to one of the three modules T_1, T_2, or T_3 of Lemma 2.1. The multiplicities of T_1, T_2 and T_3 in such a direct sum decomposition are independent of the decomposition.*

Proof We use Notation 2.3.

We first prove uniqueness of the multiplicities. Suppose the $\mathbb{Z}[\mathbb{Z}_2]$-module M is isomorphic to the direct sum of n_1 copies of T_1, n_2 copies of T_2, and n_3 copies of T_3. Then n_1 is the $\mathbb{F}_2$-vector space dimension of $(M \cap eM)/2eM \subset eM/2eM$, and similarly $n_2 = \dim_{\mathbb{F}_2}\big((M \cap fM)/2fM\big)$. Finally, one checks that $n_3 = \dim_{\mathbb{F}_2}\big(eM/(M \cap eM)\big)$, which is the same as $\dim_{\mathbb{F}_2}\big(fM/(M \cap fM)\big)$.

Now we prove existence. We have

$$2eM \subset M \cap eM \subset eM \quad \text{and} \quad 2fM \subset M \cap fM \subset fM.$$

Apply Lemma 2.7 twice, to get $\mathbb{Z}$-submodules $R_0, R_1 \subset eM$ and $S_0, S_1 \subset fM$ such that

$$eM = R_0 \oplus R_1 \quad \text{and} \quad M \cap eM = 2R_0 \oplus R_1$$

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and

$$fM = S_0 \oplus S_1 \quad \text{and} \quad M \cap fM = 2S_0 \oplus S_1.$$

Since $eM \cap fM = \{0\}$ and $e + f = 1$, we can write

$$(M \cap eM) \oplus (M \cap fM) \subset M \subset eM \oplus fM.$$

Substituting from above, we have the inclusions of $\mathbb{Z}$-module direct sums

$$2R_0 \oplus R_1 \oplus 2S_0 \oplus S_1 \subset M \subset R_0 \oplus R_1 \oplus S_0 \oplus S_1.$$

Since $R_1, S_1 \subset M$, we have the $\mathbb{Z}$-module direct sum decomposition

$$M = R_1 \oplus S_1 \oplus N \quad \text{with} \quad N = M \cap (R_0 \oplus S_0).$$

Now $R_1 \subset eM$ and $sm = m$ for all $m \in eM$. So R_1 is actually a $\mathbb{Z}[\mathbb{Z}_2]$-module. Since it is a $\mathbb{Z}$-submodule of M, it is a free $\mathbb{Z}$-module (Theorem 14.5 of [10]), and therefore as a $\mathbb{Z}[\mathbb{Z}_2]$-module it is a direct sum of $\mathbb{Z}[\mathbb{Z}_2]$-modules isomorphic to T_1. Similarly, from $S_1 \subset fM$ and $sm = -m$ for all $m \in fM$, we see that S_1 is a $\mathbb{Z}[\mathbb{Z}_2]$-module which is a direct sum of $\mathbb{Z}[\mathbb{Z}_2]$-modules isomorphic to T_2. We will complete the proof by showing that N is a $\mathbb{Z}[\mathbb{Z}_2]$-module, so that the direct sum decomposition above is really a $\mathbb{Z}[\mathbb{Z}_2]$-module direct sum decomposition, and that it is free, so that it is a direct sum of $\mathbb{Z}[\mathbb{Z}_2]$-modules isomorphic to T_3.

To prove that N is a $\mathbb{Z}[\mathbb{Z}_2]$-module, we simply observe that R_0 and S_0 are $\mathbb{Z}[\mathbb{Z}_2]$-modules, for the same reason that R_1 and S_1 are. For freeness, we know as above that N is a free $\mathbb{Z}$-module because it is a $\mathbb{Z}$-submodule of M. By Lemma 2.5, it now suffices to prove that $N \cap eN = 2eN$ and $N \cap fN = 2fN$. That $2eN \subset N \cap eN$ and $2fN \subset N \cap fN$ are clear, so we prove the reverse inclusions.

Let $m \in N \cap eN$. Use $m \in N$ to write $m = a + b$ with $a \in R_0$ and $b \in S_0$. Use $m \in eN$ to write $m = e(x + y)$ with $x \in R_0$ and $y \in S_0$. Note that $ex = x$ and $ey = 0$, and combine the two expressions for m to get $x - a = b$. Since $x - a \in eM$ and $b \in fM$, we get $b = 0$. Therefore $m = a \in R_0$. From $m \in M \cap eM = 2R_0 \oplus R_1$ we get $m \in 2R_0$. Now $R_0 \subset eM$ implies $eR_0 = R_0$, so $R_0 \subset N$ implies $R_0 \subset eN$. Therefore $m \in 2eN$, as desired.

Now suppose $m \in N \cap fN$. Write $m = a + b = f(x + y)$ with $a, x \in R_0$ and $b, y \in S_0$. From $fx = 0$ and $fy = y$, we get $y - b = a \in eM \cap fM = \{0\}$. So $m = b \in S_0$. From $m \in M \cap fM = 2S_0 \oplus S_1$ we now get $m \in 2S_0$. Since $S_0 \subset N \cap fM \subset fN$, it follows that $m \in 2fN$, as desired. This completes the proof. ■

3 Geometric Resolution

In this section, we prove that for every order two automorphism γ of a countable $\mathbb{Z}_2$-graded abelian group G_*, there is some separable C^*-algebra A with an order two automorphism $\alpha \in \operatorname{Aut}(A)$, such that there is a graded isomorphism $\mu \colon G_* \to K_*(A)$ which identifies γ with α_*. We borrow the method of geometric realization of

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resolutions from Schochet's proof of the Künneth formula for C^*-algebras [24]. As it turns out, the C^*-algebra A we produce will be of type I and nonunital.

Although we do not state this version formally, the same proofs show that if G_* is not countable, we can still produce an automorphism of a nonseparable type I C^*-algebra satisfying the remaining conditions.

For any C^*-algebra A, we let A^+ denote its usual unitization, in which we add a new unit even if A is already unital.

Lemma 3.1 *Let $C = \bigoplus_{n\in\mathbb{N}}[C_0(\mathbb{R}) \oplus C_0(\mathbb{R})]$, and let $\gamma \in \mathrm{Aut}(C)$ be the order two automorphism given by*

$$\gamma\big((f_1, g_1), (f_2, g_2), \dots\big) = \big((g_1, f_1), (g_2, f_2), \dots\big).$$

Let M be a subgroup of $K_1(C)$ which is invariant under γ_*. Then there exists a separable Hilbert space H with an action of $\mathbb{Z}_2$, a separable commutative C^*-algebra B with an automorphism β of order two, and a $\mathbb{Z}_2$-equivariant homomorphism $\varphi\colon B \to K(H)\otimes C$, where $\mathbb{Z}_2$ acts on $K(H)$ by conjugation, such that $K_0(B) = 0$, such that $\varphi_*\colon K_1(B) \to K_1(C)$ is injective, and such that the image of φ_* is exactly M.

Proof The K-theory $K_*(C)$ is a countable abelian group with an automorphism γ_* of order two, and hence in an obvious way a $\mathbb{Z}[\mathbb{Z}_2]$-module. As $\mathbb{Z}[\mathbb{Z}_2]$-modules, $K_1(C)$ is a countable direct sum of copies of $\mathbb{Z}[\mathbb{Z}_2]$ and $K_0(C) = 0$. Moreover, the hypothesis on M is just that it is a $\mathbb{Z}[\mathbb{Z}_2]$-submodule. It is a free $\mathbb{Z}$-module because subgroups of free abelian groups are free (Theorem 14.5 of [10]), and it is clearly countably generated, so by Theorem 2.8 there is an isomorphism $M \cong \bigoplus_{i\in I} M_i$ with I finite or countable, and with each M_i isomorphic to one of the three modules T_j of Lemma 2.1. We regard each M_i as a $\mathbb{Z}[\mathbb{Z}_2]$-submodule of $K_1(C)$.

For each $i \in I$, we construct a separable Hilbert space H_i with an action of $\mathbb{Z}_2$, a separable commutative C^*-algebra B_i with an automorphism β_i of order two, and a $\mathbb{Z}_2$-equivariant homomorphism $\varphi_i\colon B_i \to K(H_i)\otimes C$, such that $K_0(B_i) = 0$, such that $(\varphi_i)_*\colon K_1(B_i) \to K_1(C)$ is injective, and such that the image of $(\varphi_i)_*$ is exactly M_i. There are three cases, but we begin by introducing notation common to all three. Define $u\colon \mathbb{R} \to S^1$ by

$$u(t) = \exp\Big(\pi i\big(1 + t(1+t^2)^{-1/2}\big)\Big).$$

Then u is a unitary in the unitization $C_0(\mathbb{R})^+$ such that $[u]$ generates $K_1\big(C_0(\mathbb{R})\big)$, and $u - 1$ generates $C_0(\mathbb{R})$ as a C^*-algebra. In particular, if E is any C^*-algebra and $v \in E^+$ is any unitary such that $v - 1 \in E$, then there is a unique homomorphism $\psi\colon C_0(\mathbb{R}) \to E$ such that $\psi(u-1) = v - 1$. Further let s be the nontrivial element of $\mathbb{Z}_2$, and use the same notation for the corresponding element of $\mathbb{Z}[\mathbb{Z}_2]$. We will also write elements of $K_1(C)$ as sequences

$$(k_1 + l_1 s, k_2 + l_2 s, \dots) \in \bigoplus_{n\in\mathbb{N}} \mathbb{Z}[\mathbb{Z}_2],$$

where, with all nontrivial entries being in the n-th position, $(0, \ldots, 0, 1, 0, \ldots)$ corresponds to the K_1-class of the unitary

$$1 + \big((0,0), \ldots, (0,0), (u-1, 0), (0,0), \ldots\big) \in C^+$$

and $(0, \ldots, 0, s, 0, \ldots)$ corresponds to the K_1-class of the unitary

$$1 + \big((0,0), \ldots, (0,0), (0, u-1), (0,0), \ldots\big) \in C^+.$$

The first case is $M_i \cong T_1$. Let m be a generator of M_i as a $\mathbb{Z}$-module. Then $sm = m$. Therefore m has the form

$$m = (k_1 + k_1 s, k_2 + k_2 s, \ldots),$$

with all but finitely many of the k_j equal to zero. Take $H_i = \mathbb{C}$ with the trivial action of $\mathbb{Z}_2$, take $B_i = C_0(\mathbb{R})$, take $\beta_i = \mathrm{id}_{B_i}$, and take φ_i to be the homomorphism determined by

$$\varphi_i(u-1) = \big((u^{k_1} - 1, u^{k_1} - 1), (u^{k_2} - 1, u^{k_2} - 1), \ldots\big).$$

The required properties are immediate.

Next, suppose $M_i \cong T_2$. Let m be a generator of M_i as a $\mathbb{Z}$-module. Then $sm = -m$. Therefore m has the form

$$m = (k_1 - k_1 s, k_2 - k_2 s, \ldots),$$

with all but finitely many of the k_j equal to zero. Take $H_i = \mathbb{C}$ with the trivial action of $\mathbb{Z}_2$, and take $B_i = C_0(\mathbb{R})$. Take $\beta_i \colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ to be the homomorphism $\beta_i(f)(t) = f(-t)$, which is the unique homomorphism such that $\beta_i(u-1) = u^* - 1$. Take φ_i to be the homomorphism determined by

$$\varphi_i(u-1) = \big((u^{k_1} - 1, u^{-k_1} - 1), (u^{k_2} - 1, u^{-k_2} - 1), \ldots\big).$$

Equivariance follows from

$$\varphi_i(u^* - 1) = \big((u^{-k_1} - 1, u^{k_1} - 1), (u^{-k_2} - 1, u^{k_2} - 1), \ldots\big),$$

and the rest of the required properties are immediate.

Finally, suppose $M_i \cong T_3$. Let $m \in K_1(C)$ be the image of $1 \in T_3$ under the isomorphism $T_3 \to M_i$. Write $m = (k_1 + l_1 s, k_2 + l_2 s, \ldots)$. Then the image of s is

$$sm = (l_1 + k_1 s, l_2 + k_2 s, \ldots),$$

and these two elements are linearly independent over $\mathbb{Z}$. Take $H_i = \mathbb{C}^2$, with $s(\xi_1, \xi_2) = (\xi_2, \xi_1)$. Take $B_i = C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$, with $\beta_i(f_1, f_2) = (f_2, f_1)$. To define

$$\varphi_i \colon C_0(\mathbb{R}) \oplus C_0(\mathbb{R}) \to L(\mathbb{C}^2) \otimes C,$$

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let $p_1, p_2 \in L(\mathbb{C}^2)$ be the projections on the first and second coordinates, and let $\psi_1, \psi_2 \colon C_0(\mathbb{R}) \to C$ be the unique homomorphisms such that

$$\psi_1(u-1) = \big((u^{k_1}-1, u^{l_1}-1), (u^{k_2}-1, u^{l_2}-1), \dots\big)$$

and

$$\psi_2(u-1) = \big((u^{l_1}-1, u^{k_1}-1), (u^{l_2}-1, u^{k_2}-1), \dots\big).$$

Then define

$$\varphi_i(f_1, f_2) = p_1 \otimes \psi_1(f_1) + p_2 \otimes \psi_2(f_2).$$

This is a homomorphism because p_1 and p_2 are orthogonal. Equivariance follows from the fact that the action of s exchanges p_1 and p_2, and the formulas for m and sm. That $(\psi_i)_*$ is injective follows from the fact that m and sm are linearly independent over $\mathbb{Z}$, and the remaining required properties are clear.

Now let H be the Hilbert direct sum $H = \bigoplus_{i\in I} H_i$, and let B be the C^*-algebra direct sum $B = \bigoplus_{i\in I} B_i$. Give both the action of $\mathbb{Z}_2$ coming from the actions on the summands. Define $\varphi \colon B \to K(H) \otimes C$ by taking $\varphi\big((b_i)_{i\in I}\big)$ to be the block diagonal element $\bigoplus_{i\in I} \varphi_i(b_i)$. This element is in fact in $K(H) \otimes C$ because for every $\varepsilon > 0$, we have $\|b_i\| < \varepsilon$ for all but finitely many i. Then φ is the required homomorphism. ■

Lemma 3.2 *Let G be a countable abelian group, and let $\nu \colon G \to G$ be an automorphism of G of order two. Then there exist a separable type I C^*-algebra A such that $K_1(A) = 0$, an automorphism $\alpha \in \mathrm{Aut}(A)$ of order two, and an isomorphism $\mu \colon G \to K_0(A)$, such that $\mu \circ \nu = \alpha_* \circ \mu$.*

Proof The group G is a $\mathbb{Z}[\mathbb{Z}_2]$-module in an obvious way, and it is countable by assumption. Set $N = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}[\mathbb{Z}_2]$, and choose a surjective $\mathbb{Z}[\mathbb{Z}_2]$-module homomorphism $\tau \colon N \to G$. Set $M = \mathrm{Ker}(\tau)$. Let C be as in Lemma 3.1, and identify N with $K_1(C)$. Apply Lemma 3.1 with this M, obtaining a $\mathbb{Z}_2$-equivariant homomorphism of separable type I C^*-algebras $\varphi \colon B \to K(H) \otimes C$ such that (by abuse of notation) $K_1(B) = M$, $K_1\big(K(H) \otimes C\big) = N$, and φ_* is the inclusion. To simplify the notation, set $D = K(H) \otimes C$.

Let A be the mapping cone

$$A = \{(f, b) \in C([0,1], D) \oplus B : f(0) = 0 \text{ and } f(1) = \varphi(b)\}.$$

Since φ is equivariant, this algebra has an obvious action of $\mathbb{Z}_2$. (The action is trivial on $[0,1]$.) Moreover, with $SD = C_0\big((0,1), D\big)$ being the usual suspension of D, there is an equivariant short exact sequence

$$0 \longrightarrow SD \longrightarrow A \longrightarrow B \longrightarrow 0.$$

By naturality the corresponding six term exact sequence in K-theory is actually a sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules. Since $K_0(B) = 0$ and $K_0(D) = 0$, this sequence reduces to

$$0 \longrightarrow K_1(A) \longrightarrow K_1(B) \longrightarrow K_0(SD) \longrightarrow K_0(A) \longrightarrow 0.$$

Using the Bott periodicity isomorphism $K_0(SD) \cong K_1(D)$, and appropriately identifying the maps following Theorem 3.5 of [25], this sequence can be naturally identified with

$$0 \longrightarrow K_1(A) \longrightarrow K_1(B) \xrightarrow{\varphi_*} K_1(D) \longrightarrow K_0(A) \longrightarrow 0.$$

By our construction and by naturality of the sequence, we therefore have the sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$0 \longrightarrow K_1(A) \longrightarrow M \xrightarrow{\tau} N \longrightarrow K_0(A) \longrightarrow 0,$$

from which it follows that $K_1(A) = 0$ and that $K_0(A) \cong N/M \cong G$ as $\mathbb{Z}[\mathbb{Z}_2]$-modules. ■

Proposition 3.3 *Let G_0 and G_1 be countable abelian groups. Let γ_i be an automorphism of G_i of order two. Then there exist a separable type I C^*-algebra A, an automorphism $\alpha \in \mathrm{Aut}(A)$ of order two, and a graded isomorphism $\mu\colon G_* \to K_*(A)$, such that the diagram*

$$\begin{array}{ccc} G_* & \xrightarrow{\gamma_*} & G_* \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ K_*(A) & \xrightarrow{\alpha_*} & K_*(A) \end{array}$$

commutes.

Proof Apply Lemma 3.2 to G_0 and γ_0, obtaining a C^*-algebra A_0 with an order two automorphism α_0. Apply Lemma 3.2 to G_1 and γ_1, obtaining a C^*-algebra A_1 with an order two automorphism α_1. Take $A = A_0 \oplus SA_1$ and $\alpha = \alpha_0 \oplus S\alpha_1$. This C^*-algebra and automorphism satisfy the conclusions by Bott periodicity. ■

4 From Type I to Purely Infinite Simple

For countable abelian groups G_0 and G_1 and order two automorphisms γ_i of G_i, we can now produce a separable type I C^*-algebra A with K-theory $G_0 \oplus G_1$, and an order two automorphism which induces γ_i on K-theory. We want to use the construction of [15] to produce a unital Kirchberg algebra. However, A is not unital, and this construction will then not produce a unital C^*-algebra. Unitizing A changes the K-theory. To remedy this problem, we introduce in the next proposition a kind of unitization functor which does not change the K-theory. Our functor preserves nuclearity and the Universal Coefficient Theorem, but not type I (or simplicity or finiteness). Although we will not use this fact, it also seems interesting to observe that the functor sends exact C^*-algebras to exact C^*-algebras.

Proposition 4.1 *There is a functor F from the category of all C^*-algebras and C^*-algebra homomorphisms to the subcategory of all unital C^*-algebras and unital C^*-algebra homomorphisms, and a natural transformation η from the identity functor to F, such that:*

(1) *For every C^*-algebra A, the homomorphism $\eta_A\colon A \to F(A)$ is a KK-equivalence, and in particular is an isomorphism on K-theory.*
(2) *If A is separable, or nuclear, or exact, or satisfies the Universal Coefficient Theorem of [23], then $F(A)$ also has the same property.*

Proof We use the Cuntz algebras [6], [7]. Choose and fix a nonzero projection $e \in \mathcal{O}_\infty$ such that $[e] = 0$ in $K_0(\mathcal{O}_\infty)$, a projection $q \leq e$ such that $[q] = [1_{\mathcal{O}_\infty}]$ in $K_0(\mathcal{O}_\infty)$, and a unital subalgebra $D \subset e\mathcal{O}_\infty e$ such that $D \cong \mathcal{O}_2$. For any C^*-algebra A, let A^+ be the unitization (as in Section 3), let $\pi_A\colon A^+ \to \mathbb{C}$ be the canonical map, and define

$$F(A) = \{b \in e\mathcal{O}_\infty e \otimes A^+ : (\mathrm{id}_{e\mathcal{O}_\infty e} \otimes \pi_A)(b) \in D\}.$$

Let $\iota_A\colon A \to A^+$ be the inclusion, and define $\eta_A\colon A \to F(A)$ by $\eta_A(a) = q \otimes \iota_A(a)$. This element is in $F(A)$ because $(\mathrm{id}_{e\mathcal{O}_\infty e} \otimes \pi_A)\big(q \otimes \iota_A(a)\big) = 0$.

Since the map $\lambda \mapsto \lambda q$ from $\mathbb{C}$ to $e\mathcal{O}_\infty e$ is a KK-equivalence, so is the map $a \mapsto q \otimes a$ from A to $e\mathcal{O}_\infty e \otimes A$. (See Example 19.1.2(c) of [3].) Moreover, there is a split short exact sequence

$$0 \longrightarrow e\mathcal{O}_\infty e \otimes A \longrightarrow F(A) \longrightarrow \mathcal{O}_2 \longrightarrow 0.$$

Since $\mathcal{O}_2$ is KK-equivalent to the zero C^*-algebra, it follows that, for every separable C^*-algebra B, the inclusion $\varphi\colon e\mathcal{O}_\infty e \otimes A \to F(A)$ induces isomorphisms

$$\varphi_B^*\colon KK\big(F(A), B\big) \to KK(e\mathcal{O}_\infty e \otimes A, B)$$

and

$$\varphi_*^B\colon KK(B, e\mathcal{O}_\infty e \otimes A) \to KK\big(B, F(A)\big).$$

It now follows from the Yoneda Lemma that φ is a KK-equivalence (see Section III.2 of [16]), but we give the easy direct argument here. In terms of the Kasparov product, these isomorphisms have the formulas $\varphi_B^*(\gamma) = [\varphi] \cdot \gamma$ and $\varphi_*^B(\gamma) = \gamma \cdot [\varphi]$. (See Proposition 18.7.2 of [3].) Choose

$$\alpha \in KK\big(F(A), e\mathcal{O}_\infty e \otimes A\big) \quad \text{and} \quad \beta \in KK\big(e\mathcal{O}_\infty e \otimes A, F(A)\big)$$

such that

$$\varphi_{e\mathcal{O}_\infty e \otimes A}^*(\alpha) = [\mathrm{id}_{e\mathcal{O}_\infty e \otimes A}] \quad \text{and} \quad \varphi_*^{F(A)}(\beta) = [\mathrm{id}_{F(A)}].$$

Then

$$[\varphi] \cdot \alpha = [\mathrm{id}_{e\mathcal{O}_\infty e \otimes A}] \quad \text{and} \quad \beta \cdot [\varphi] = [\mathrm{id}_{F(A)}],$$

from which it follows that φ is a KK-equivalence. So η_A is a KK-equivalence. This proves the property (1).

It is immediate from the KK-equivalence of (1) that $F(A)$ satisfies the Universal Coefficient Theorem if A does. That F preserves separability and nuclearity is immediate from the split exact sequence above, since $e\mathcal{O}_\infty e$ and $\mathcal{O}_2$ are separable and nuclear. The same argument works for exactness: the minimal tensor product of exact C^*-algebras is exact by Proposition 7.1(iii) of [13], and extensions with completely positive splittings preserve exactness by Proposition 7.1(vi) of [13]. ∎

Next, we need an equivariant version of the construction of [15].

Proposition 4.2 *Let $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of a countable discrete group Γ on a separable unital C^*-algebra A. Then there exists a separable unital purely infinite simple C^*-algebra B, an action $\beta\colon \Gamma \to \mathrm{Aut}(B)$, and an equivariant unital homomorphism $\varphi\colon A \to B$, such that φ is a KK-equivalence. Moreover, if A is nuclear then so is B, and if A satisfies the Universal Coefficient Theorem then so does B.*

Proof Choose an injective unital representation $\pi_0\colon A \to L(H_0)$ of A on a separable Hilbert space H_0 such that $\pi_0(A) \cap K(H_0) = \{0\}$. Let $H = l^2(\Gamma, H_0)$, and let $u\colon \Gamma \to U(H)$ and $\pi\colon A \to L(H)$ be the components of the regular covariant representation of (Γ, A) associated to π_0, as in 7.7.1 in [17]. Then $u_\gamma \pi(a) u_\gamma^* = \pi\big(\alpha_\gamma(a)\big)$ for all $a \in A$ and $\gamma \in \Gamma$. (See [17].) Moreover, $\pi(A) \cap K(H) = \{0\}$.

Now we follow the proof of Proposition 2.1 of [15]. Let $E = H \otimes_{\mathbb{C}} A$ be the Hilbert A-bimodule defined there, and let $\varphi\colon A \to L(E)$ be as there. Each $\gamma \in \Gamma$ induces a isometric $\mathbb{C}$-linear map $v_\gamma\colon E \to E$ given by $v_\gamma(\xi \otimes a) = u_\gamma \xi \otimes \alpha_\gamma(a)$ for $\xi \in H$ and $a \in A$. These maps have the following properties:

(1) $v_\gamma(\xi a) = v_\gamma(\xi)\alpha_\gamma(a)$ for $\gamma \in \Gamma$, $\xi \in E$, and $a \in A$.
(2) $\langle v_\gamma \xi, v_\gamma \eta\rangle = \alpha_\gamma(\langle \xi, \eta\rangle)$ for $\gamma \in \Gamma$ and $\xi, \eta \in E$.
(3) $\|v_\gamma \xi\| = \|\xi\|$ for $\gamma \in \Gamma$ and $\xi \in E$.
(4) $v_{\gamma^{-1}} = v_\gamma^{-1}$ for $\gamma \in \Gamma$.
(5) $v_\gamma \varphi(a) v_\gamma^{-1} = \varphi\big(\alpha_\gamma(a)\big)$ for $\gamma \in \Gamma$ and $a \in A$.

Caution: v_γ is not a right A-module homomorphism. Nevertheless, these properties imply that $b \mapsto v_\gamma b v_\gamma^{-1}$ is a $*$-automorphism of the C^*-algebra $L(E)$ of all adjointable right A-module morphisms of E; moreover, φ is equivariant.

Next, let $\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^{\otimes n}$ be the Fock space of E, as after Definition 1.3 of [15] or as at the beginning of Section 1 of [20]. We let $\gamma \in \Gamma$ act on $E^{\otimes n}$ via $v_\gamma^{\otimes n}$ (via α_γ on $E^{\otimes 0} = A$), and on $\mathcal{E}_+$ via the direct sum w_γ of these actions. This action is still isometric, and we again have the analogs of Properties (1)–(5) above. The role of φ is now played by the homomorphism $\varphi_+\colon A \to L(\mathcal{E}_+)$ of the discussion after Definition 1.3 of [15]. Following the construction there, the Toeplitz algebra $\mathcal{T}_E$ is by definition the C^*-subalgebra of $L(\mathcal{E}_+)$ generated by the operators T_ξ defined there, for $\xi \in E$. One checks that $T_{v_\gamma \xi} = w_\gamma T_\xi w_\gamma^{-1}$ for $\gamma \in \Gamma$ and $\xi \in E$. Therefore Γ acts on $\mathcal{T}_E$ via $*$-automorphisms. Since $\pi(A) \cap K(H) = \{0\}$, Lemma 2.1 of [15] and Corollary 3.14 and Theorem 3.13 of [20] show that the canonical map from $\mathcal{T}_E$ to the Cuntz-Pimsner algebra $\mathcal{O}_E$ is an isomorphism. Therefore we have an action of Γ on $\mathcal{O}_E$, and $\varphi_+\colon A \to \mathcal{O}_E$ is equivariant.

Theorem 2.8 of [15] shows that $\mathcal{O}_E$ is purely infinite and simple. Separability is clear. The proof of Theorem 3.1 of [15] shows that if A is nuclear, then so is $\mathcal{O}_E$. Corollary 4.5 of [20] shows that φ_+ is a KK-equivalence, from which it is immediate that if A satisfies the Universal Coefficient Theorem then so does $\mathcal{O}_E$. ■

Theorem 4.3 *Let G_0 and G_1 be countable abelian groups. Let γ_i be an automorphism of G_i of order two. Then there exist a unital Kirchberg algebra A satisfying the Universal*

Coefficient Theorem and with $[1_A] = 0$ *in* $K_0(A)$, *an automorphism* $\alpha \in \operatorname{Aut}(A)$ *of order two, and a graded isomorphism* $\mu\colon G_* \to K_*(A)$, *such that the diagram*

$$\begin{array}{ccc} G_* & \xrightarrow{\gamma_*} & G_* \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ K_*(A) & \xrightarrow{\alpha_*} & K_*(A) \end{array}$$

commutes.

Proof Use Proposition 3.3 to find a separable type I C^*-algebra A_0, an automorphism $\alpha_0 \in \operatorname{Aut}(A_0)$ of order two, and a graded isomorphism $\mu_0\colon G_* \to K_*(A_0)$, such that $\mu_0 \circ \gamma = (\alpha_0)_* \circ \mu_0$.

Let F be the functor of Proposition 4.1, and let $\eta\colon A_0 \to F(A_0)$ be the natural transformation from there. By (2) of Proposition 4.1, the algebra $F(A_0)$ is separable, unital, nuclear, and satisfies the Universal Coefficient Theorem; also, η is a KK-equivalence. Since F is a functor, $F(\alpha_0)$ is an order two automorphism of $F(A_0)$, and naturality implies that $\eta \circ \alpha_0 = F(\alpha_0) \circ \eta$.

Now use Proposition 4.2 to find a unital Kirchberg algebra A satisfying the Universal Coefficient Theorem, an automorphism $\alpha \in \operatorname{Aut}(A)$ of order two, and a $\mathbb{Z}_2$-equivariant homomorphism $\varphi\colon F(A_0) \to A$ which is a KK-equivalence. Then $\varphi \circ \eta\colon A_0 \to A$ is $\mathbb{Z}_2$-equivariant and is a KK-equivalence. The theorem is therefore proved by taking $\mu = \varphi_* \circ \eta_* \circ \mu_0$. ∎

It should be easy to arrange to have $[1_A]$ correspond to any element in G of the form $\eta + \gamma_0(\eta)$, but we don't know how to get arbitrary γ_0-invariant elements of G_0.

We now turn to the problem of realizing $R(\mathbb{Z}_2)$-modules as equivariant K-theory for actions on C^*-algebras. The main point is contained in the next lemma, which we state in greater generality.

Before stating it, recall (Section 2.2 of [18]) that if a compact group G acts on a C^*-algebra A, then the equivariant K-theory $K_*^G(A)$ is, in a canonical way, a module over the representation ring $R(G)$ of G. Further recall that when G is abelian, the representation ring $R(G)$ is just $\mathbb{Z}[\widehat{G}]$; in particular, if G is a countable abelian group, then $R(\widehat{G})$ is canonically isomorphic to $\mathbb{Z}[G]$.

Lemma 4.4 *Let* G *be a countable abelian group, and let* $\alpha\colon G \to \operatorname{Aut}(A)$ *be an action of* G *on a* C^**-algebra* A. *Regard* $K_*(A)$ *as a* $\mathbb{Z}[G]$*-module via the action of* G *on* A. *Regard the equivariant* K*-theory* $K_*^{\widehat{G}}\big(C^*(G,A,\alpha)\big)$ *of the dual action* $\widehat{\alpha}\colon \widehat{G} \to \operatorname{Aut}\big(C^*(G,A,\alpha)\big)$ *as a* $\mathbb{Z}[G]$*-module as discussed above. Then* $K_*^{\widehat{G}}\big(C^*(G,A,\alpha)\big) \cong K_*(A)$ *as* $\mathbb{Z}[G]$*-modules.*

Proof Let $B = C^*\big(\widehat{G}, C^*(G,A,\alpha), \widehat{\alpha}\big)$ be the second crossed product, with second dual action $\beta\colon G \to \operatorname{Aut}(B)$. Regard $K_*(B)$ as a $\mathbb{Z}[G]$-module via this action of G. Proposition 2.7.10 of [18], applied to the action $\widehat{\alpha}$ of $\widehat{G}$ on $C^*(G,A,\alpha)$, shows that $K_0^{\widehat{G}}\big(C^*(G,A,\alpha)\big) \cong K_0(B)$ as $\mathbb{Z}[G]$-modules. Applying this result to the suspension

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SA, we obtain the same isomorphism for K_1 as well. By Takai duality [27], there is an isomorphism $B \cong A \otimes K\big(l^2(G)\big)$ which intertwines the action β with the tensor product of α and an inner action on $K\big(l^2(G)\big)$. It is therefore immediate that $K_*(B) \cong K_*(A)$ as $\mathbb{Z}[G]$-modules. ■

We immediately get a realization theorem using type I C^*-algebras.

Theorem 4.5 *Let G_* be a countable $\mathbb{Z}_2$-graded $R(\mathbb{Z}_2)$-module. Then there exists a separable type I C^*-algebra A, and an action α of $\mathbb{Z}_2$ on A, such that $K_*^{\mathbb{Z}_2}(A) \cong G_*$ as $R(\mathbb{Z}_2)$-modules.*

Proof Identify $R(\mathbb{Z}_2)$ with $\mathbb{Z}[\widehat{\mathbb{Z}}_2]$ as discussed before Lemma 4.4. Since $\widehat{\mathbb{Z}}_2 \cong \mathbb{Z}_2$, we can use Proposition 3.3 to find a separable type I C^*-algebra B and an action $\beta\colon \widehat{\mathbb{Z}}_2 \to \operatorname{Aut}(B)$ such that $K_*(B) \cong G_*$ as $R(\mathbb{Z}_2)$-modules. Let $A = C^*(\widehat{\mathbb{Z}}_2, B, \beta)$, equipped with the dual action $\alpha = \widehat{\beta}$ of $\mathbb{Z}_2$. Lemma 4.4 implies that $K_*^{\mathbb{Z}_2}(A) \cong G_*$ as $R(\mathbb{Z}_2)$-modules. Clearly A is separable, and Theorem 4.1 of [21] implies that A is type I. ■

For the realization theorem using Kirchberg algebras, we need another lemma.

Lemma 4.6 *Let*

$$A_0 \xrightarrow{\rho_1} A_1 \xrightarrow{\rho_2} A_2 \xrightarrow{\rho_3} \cdots$$

be a direct system of C^*-algebras in which each A_n is a finite direct sum of Kirchberg algebras satisfying the Universal Coefficient Theorem. Suppose that for each n the partial map determined by ρ_n from any summand of A_{n-1} to any other summand of A_n is nonzero. Then the direct limit of this system is again a Kirchberg algebra satisfying the Universal Coefficient Theorem.

Proof Let $A = \varinjlim A_n$. Clearly A is separable and nuclear. The condition that all the partial maps be nonzero, together with the simplicity of all the summands, guarantees that the algebraic direct limit of the A_n is simple. A standard argument now shows that A is simple. It is easy to check, using standard direct limit methods, that A has real rank zero and that every nonzero projection in A is properly infinite. (For the second statement, use the fact that every projection in A is equivalent to a projection in the algebraic direct limit.) Therefore A is purely infinite simple. Finally, A satisfies the Universal Coefficient Theorem because the class of such algebras is closed under direct limits. This is essentially Proposition 2.3(b) of [23], in view of Theorem 4.1 of [23] and its converse (the converse being trivial). ■

Theorem 4.7 *Let G_* be a countable $\mathbb{Z}_2$-graded $R(\mathbb{Z}_2)$-module. Then there exists a unital Kirchberg algebra A, and an action α of $\mathbb{Z}_2$ on A, such that $K_*^{\mathbb{Z}_2}(A) \cong G_*$ as $R(\mathbb{Z}_2)$-modules, and such that $C^*(\mathbb{Z}_2, A, \alpha)$ is a Kirchberg algebra.*

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Proof Identify $R(\mathbb{Z}_2)$ with $\mathbb{Z}[\widehat{\mathbb{Z}}_2]$ as discussed before Lemma 4.4. Let $\tau \in \widehat{\mathbb{Z}}_2$ be the nontrivial element.

First suppose that the action of τ on the $R(\mathbb{Z}_2)$-module G_* is nontrivial. Since $\widehat{\mathbb{Z}}_2 \cong \mathbb{Z}_2$, we can use Theorem 4.3 to find a unital Kirchberg algebra B and an action $\beta\colon \widehat{\mathbb{Z}}_2 \to \mathrm{Aut}(B)$ such that $K_*(B) \cong G_*$ as $R(\mathbb{Z}_2)$-modules. Let $A = C^*(\widehat{\mathbb{Z}}_2, B, \beta)$, equipped with the dual action $\alpha = \widehat{\beta}$ of $\mathbb{Z}_2$. Lemma 4.4 implies that $K_*^{\mathbb{Z}_2}(A) \cong G_*$ as $R(\mathbb{Z}_2)$-modules. Moreover, β_τ is outer (because it is nontrivial on K-theory), so Corollary 4.4 of [12] implies that A is a Kirchberg algebra. Furthermore, $C^*(\mathbb{Z}_2, A, \alpha) \cong M_2 \otimes B$ is also a Kirchberg algebra.

Now suppose that the action of τ on the $R(\mathbb{Z}_2)$-module G_* is trivial. Choose a unital Kirchberg algebra A_0 satisfying the Universal Coefficient Theorem such that $K_*(A_0) \cong G_*$ as abelian groups. Let $\alpha^{(0)}\colon \mathbb{Z}_2 \to \mathrm{Aut}(A_0 \oplus A_0)$ be the action such that $\alpha_\tau^{(0)}(a, b) = (b, a)$. We identify the crossed product $C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)})$. Let $u \in C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)})$ be the canonical unitary of order two. Then there is an isomorphism $\sigma\colon C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)}) \to M_2(A_0)$, determined by

$$\sigma(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

for $a, b \in A_0$, and

$$\sigma(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The dual action is inner, so $K_*^{\mathbb{Z}_2}(A_0 \oplus A_0) \cong G_*$ as $R(\mathbb{Z}_2)$-modules.

The algebra $A_0 \oplus A_0$ is not a Kirchberg algebra. To remedy this, choose a nonzero projection $q \in A_0$ such that $[q] = 0$ in $K_0(A_0)$. Set $p = 1 - q$, so that $[p] = [1_{A_0}]$ in $K_0(A_0)$. Choose ([14]; or Theorem 4.1.3 and the proofs of Corollary 4.4.2 and Theorem 4.2.4 of [19]) unital homomorphisms $\varphi\colon A_0 \to pA_0p$ and $\psi\colon A_0 \to qA_0q$ such that

$$A_0 \xrightarrow{\varphi} pA_0p \longrightarrow A_0$$

is the identity on K-theory and $\psi_* = 0$. Define a unital equivariant homomorphism $\rho\colon A_0 \oplus A_0 \to A_0 \oplus A_0$ by $\varphi(a, b) = \big(\varphi(a) + \psi(b), \varphi(b) + \psi(a)\big)$ for $a, b \in A_0$. Let A be the direct limit of the system

$$A_0 \oplus A_0 \xrightarrow{\rho} A_0 \oplus A_0 \xrightarrow{\rho} A_0 \oplus A_0 \xrightarrow{\rho} \cdots.$$

Then A is a unital Kirchberg algebra by Lemma 4.6. Let $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ be the direct limit action.

We next show that $K_*^{\mathbb{Z}_2}(A) \cong G_*$ as $R(\mathbb{Z}_2)$-modules. We do this by proving that $\rho_*\colon K_*^{\mathbb{Z}_2}(A_0 \oplus A_0) \to K_*^{\mathbb{Z}_2}(A_0 \oplus A_0)$ is an isomorphism and using Proposition 2.5.4 of [18] on the direct system and its suspension. We claim that the corresponding homomorphism

$$\overline{\rho}\colon C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)}) \to C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)})$$

is, under the identification of the crossed product above, given by

$$\overline{\rho}(x) = (\mathrm{id}_{M_2} \otimes \varphi)(x) + u(\mathrm{id}_{M_2} \otimes \psi)(x)u^*.$$

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It is enough to check this on the image of $A_0 \oplus A_0$, namely the diagonal matrices, and on u; this is easy. It is immediate from the choice of φ and ψ that this map is the identity on K-theory.

It remains to show that $C^*(\mathbb{Z}_2, A, \alpha)$ is a Kirchberg algebra. This algebra is the direct limit of the system of crossed products

$$C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)}) \xrightarrow{\overline{\rho}} C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)}) \xrightarrow{\overline{\rho}} C^*(\mathbb{Z}_2, A_0 \oplus A_0, \alpha^{(0)}) \xrightarrow{\overline{\rho}} \cdots$$

which, by the computation at the beginning of this case, can be rewritten as

$$M_2(A_0) \longrightarrow M_2(A_0) \longrightarrow M_2(A_0) \longrightarrow \cdots .$$

The direct limit of this system is a Kirchberg algebra by Lemma 4.6. ■

Remark 4.8 When the action of the nontrivial element of $\widehat{\mathbb{Z}}_2$ is nontrivial, we do not know whether the C^*-algebra in Theorem 4.7 satisfies the Universal Coefficient Theorem.

5 An Explicit Example

In this section, we write down a formula for an automorphism in the smallest nontrivial case of Theorem 4.3. This is the case $K_0(A) \cong \mathbb{Z}_3$, $K_1(A) = 0$, and γ_0 is the unique nontrivial automorphism of $\mathbb{Z}_3$. Since γ_0 fixes only the identity element of $\mathbb{Z}_3$, the automorphism will exist only when $[1_A] = 0$ in $K_0(A)$. Let $\mathcal{O}_4$ be the Cuntz algebra [6], with K-theory as computed in [7]. Then the unital Kirchberg algebra A with this K-theory and satisfying the Universal Coefficient Theorem is $M_3 \otimes \mathcal{O}_4$.

Example 5.1 We give an explicit formula, in terms of the standard generators, for an order two automorphism φ of $A = M_3 \otimes \mathcal{O}_4$ which is nontrivial on $K_0(A)$.

Let s_1, s_2, s_3, and s_4 be the standard generating isometries of $\mathcal{O}_4$, and define $p_m = s_m s_m^*$. Further let $e_{j,k}$, for $1 \leq j, k \leq 3$, be the standard matrix units of M_3, satisfying $e_{j,k} e_{k,l} = e_{j,l}$, etc. Then φ is determined by the formulas:

$$\begin{aligned}
e_{1,1} \otimes 1 &\mapsto f_{1,1} = (e_{2,2} + e_{3,3}) \otimes 1 \\
e_{2,2} \otimes 1 &\mapsto f_{2,2} = e_{1,1} \otimes (p_1 + p_2) \\
e_{3,3} \otimes 1 &\mapsto f_{3,3} = e_{1,1} \otimes (p_3 + p_4) \\
e_{1,2} \otimes 1 &\mapsto f_{1,2} = e_{2,1} \otimes s_1^* + e_{3,1} \otimes s_2^* \\
e_{1,3} \otimes 1 &\mapsto f_{1,3} = e_{2,1} \otimes s_3^* + e_{3,1} \otimes s_4^* \\
e_{1,1} \otimes s_1 &\mapsto v_1 = e_{2,2} \otimes s_1 + e_{2,3} \otimes s_2 \\
e_{1,1} \otimes s_2 &\mapsto v_2 = e_{2,2} \otimes s_3 + e_{2,3} \otimes s_4 \\
e_{1,1} \otimes s_3 &\mapsto v_3 = e_{3,2} \otimes s_1 + e_{3,3} \otimes s_2 \\
e_{1,1} \otimes s_4 &\mapsto v_4 = e_{3,2} \otimes s_3 + e_{3,3} \otimes s_4 .
\end{aligned}$$

One must check two things: that φ is a homomorphism, and that $\varphi^2 = \mathrm{id}_A$. For the first, one checks the following relations:

- $f_{1,1}$, $f_{2,2}$, and $f_{3,3}$ are orthogonal projections which sum to 1.
- $f_{1,j} f_{1,j}^* = f_{1,1}$ and $f_{1,j}^* f_{1,j} = f_{j,j}$ for $j = 2, 3$.
- $v_m^* v_m = f_{1,1}$ for $1 \leq m \leq 4$ and $\sum_{m=1}^4 v_m v_m^* = f_{1,1}$.

The details of the computation are somewhat long, and are omitted.

To prove that $\varphi^2 = \mathrm{id}_A$, one checks that $\varphi\big(\varphi(a)\big) = a$ for each of the generators used above. Again, we omit the details.

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