On the Direction-Cosines of the Axes of the Conicoid

\[ f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1. \]

Some time ago I received from Dr. Muirhead the following theorem:—“If \( l_r, m_r, n_r \) \((r = 1, 2, \ldots 3)\) are the direction-cosines of the axes of the conicoid

\[ f(x, y, z) = 1, \quad f l_1 l_2 l_3 + g m_1 m_2 m_3 + h n_1 n_2 n_3 = 0. \]

In this note a proof and extension of the theorem are given.

The equations for the direction-cosines are

\[
\frac{al + hm + gn}{l} = \frac{bl + fm + cn}{m} = \frac{cl + gm + fn}{n} \quad \ldots \ldots (1)
\]

Therefore

\[
a + \frac{hm_2 + gn_2}{l_2} = \frac{hl_2}{m_2} + b + \frac{fn_2}{m_2}
\]

and

\[
a + \frac{hm_3 + gn_3}{l_3} = \frac{hl_3}{m_3} + b + \frac{fn_3}{m_3}
\]

Subtracting, and remembering that \( m_2 n_3 - m_3 n_2 = l_1 \), etc., we obtain

\[
g m_1 - h n_1 = \frac{hl_1 - f l_1}{m_2 m_3} = (\text{similarly}) \frac{fl_1 - g m_1}{n_2 n_3}. \]

Multiplying numerators and denominators by \( fl_1 \), \( gm_1 \), and \( h n_1 \), respectively, and adding, we get

\[ fl_1 l_2 l_3 + g m_1 m_2 m_3 + h n_1 n_2 n_3 = 0. \] ................. (2)

If \( D = ab c + 2fg h - af^2 - bg^2 - ch^2 \) and \( A = bc - f^2 \), etc., we find that each ratio in (1)

\[
\frac{D l}{Al + H m + G n} = \frac{D m}{H l + B m + F n} = \frac{D n}{Gl + F m + C n}
\]

(Geometrically, these follow from the fact that a cone and its reciprocal are coaxal, and (1) gives the direction-cosines of the axes of the cone \( f(x, y, z) = 0 \).

Therefore as above, we prove

\[ fl_1 l_2 l_3 + G m_1 m_2 m_3 + H n_1 n_2 n_3 = 0. \] ................. (3)

From (2) and (3)

\[
\frac{l_1 l_2 l_3}{G H - h G} = \frac{m_1 m_2 m_3}{h F - f H} = \frac{n_1 n_2 n_3}{f G - g F}.
\]

Now if the axes are \( OA, OB, OC \), the cone through the coordinate axes and \( OA, OB, OC \) is easily seen to be

\[
l_1 l_2 l_3 + m_1 m_2 m_3 + n_1 n_2 n_3 = 0,
\]

\[ (229) \]

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and the cone which touches the planes $BOC$, $COA$, $AOB$ and the coordinate planes is

$$\sqrt{l_1 l_2 l_3} x + \sqrt{m_1 m_2 m_3} y + \sqrt{n_1 n_2 n_3} z = 0.$$  

Substituting for $l_1 l_2 l_3$, $m_1 m_2 m_3$, and $n_1 n_2 n_3$, we obtain the equations of these cones.

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A Method of obtaining Examples on the Multiplication of Determinants.

In the ordinary text-books on Algebra there is a lack of suitable examples on Multiplication of Determinants. Most of the examples that are given are particular cases of the theorem

$$D \triangle = D^n,$$

in which

$$D = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \quad \triangle = \begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & B_2 & \cdots & B_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

where $A_1$, $A_2$, ..., $B_1$, ..., are the co-factors of $a_1$, $a_2$, ..., $b_1$, ..., in $D$.

If the determinant $D$ is chosen at random, in most cases the second determinant $\triangle$ will be too complicated. It is easy, however, to choose $D$ so that factors can be taken out of $\triangle$; and thus a sufficiently simple second determinant is obtained.

For example, let

$$D = \begin{vmatrix} b & a & a \\ a & b & a \\ a & a & b \end{vmatrix} = (2a+b)(a-b)^2.$$  

Then

$$\triangle = \begin{vmatrix} b^2-a^2 & a^2-ab & a^2-ab \\ a^2-ab & b^2-a^2 & a^2-ab \\ a^2-ab & a^2-ab & b^2-a^2 \end{vmatrix}.$$  

Let the factor $b-a$ be taken out of each row of $\triangle$. Then, multiplying the determinant so obtained by $D$, we have

$$D \begin{vmatrix} b & a & a \\ a & b & a \\ a & a & b \end{vmatrix} = \begin{vmatrix} b^2 + ba - 2a^2 & 0 & 0 \\ 0 & b^2 + ba - 2a^2 & 0 \\ 0 & 0 & b^2 + ba - 2a^2 \end{vmatrix} = (b-a)^2 (b+2a)^3.$$  

(230)