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EXISTENCE OF POSITIVE GLOBAL SOLUTIONS OF MIXED SUBLINEAR-SUPERLINEAR PROBLEMS

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1. Introduction. Consider the elliptic quasilinear problem:

(1) $\quad l_0(u) = f(x, u, \nabla u)$

in $\mathbf{R}^n$, $n \geqq 3$, where

$$l_0(u) = -\sum D_i(a_{ij}D_ju).$$

We are interested in establishing sufficient conditions on f for the existence of positive solutions $u(x)$ with specified behaviour at ∞. Of special interest to us are criteria which guarantee that $u(x)$ decays at least as fast as $|x|^{-\alpha}$ for some $\alpha \geqq 0$, given below, in the case $f(x, u, \nabla u)$ contains terms of type

$$p(x)u^\gamma + q(x)u^\delta \quad \text{with } 0 < \gamma < 1 < \delta.$$

That is: f is of mixed sublinear-superlinear type. Our main result is Theorem 3 below which explicitly states sufficient conditions for the existence of such solutions.

The variational prototype of this problem is the equation:

(1*) $\quad -\Delta u = p(x)u^\gamma$

where $p:\mathbf{R}^n \rightarrow \mathbf{R}$ is continuous. The existence of positive solutions for (1*) has been established by many authors under various conditions. For example, the critical case ($\gamma = (n + 2)/(n - 2)$) is studied by Ni, [**24**], and Ding and Ni, [**6**]; the singular case ($\gamma < 0$) by Kusano and Swanson, [**20**]; the sublinear case ($0 < \gamma < 1$) by Fukagai, [**8**], and Kusano and Swanson, [**21**]; the superlinear case ($1 < \gamma < (n + 2)/(n - 2)$) by Fukagai *et al*, [**9**]; Gidas and Spruck, [**11**]; and Joseph and Lundgren, [**15**]. See also the mini-survey paper, [**27**], by Swanson. Even though bifurcation methods, [**6**], and variational methods, [**25**], [**26**], are also used, radial arguments (i.e., ordinary differential equations) and sub- and super-solutions play an important role in these investigations and the statements of the theorems obtained. More specifically, Ni, [**24**], proved the existence

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of infinitely many bounded positive solutions for $f = p(x)u^\gamma$ ($\gamma \geqq 1$) if $|p(x)| \leqq c/|x|^l$, $l > 2$. If

$$|p(x)| \leqq \varphi(|x|) \quad \text{and} \quad \int_0^\infty r\varphi(r)dr < \infty,$$

similar results have been established by Kawano [**16**] and Kusano and Oharu [**18**] for

$$f = p(x)u^\gamma(\log(1 + u))^\delta$$

with γ and δ satisfying one of the following: (i) $\gamma > 1$ and $\delta \geqq 0$; (ii) $\gamma = 1$ and $\delta \neq 0$; (iii) $\gamma < 1$, $\gamma \neq 0$ and δ arbitrary. Fukagai [**8**] showed the existence of entire positive solution with specific behavior at ∞ for $f = p(x)u^\gamma$, $0 < \gamma < 1$, $p(x)$ as above. Kusano and Trench [**22**] proved the existence of decaying positive solution for the following mixed sublinear-superlinear equation:

$$-\Delta u + \varphi(|x|)u^\lambda + \psi(|x|)u^\mu = 0$$

in $\mathbf{R}^n$ with $0 < \lambda < 1 < \mu$, while the case of nonradial φ and ψ was given as an open problem in their paper. There are considerably fewer results along these lines in the literature dealing with the quasilinear case. We mention, however, that Furusho, [**10**], recently obtained criteria for the existence of positive solutions for mixed problems under integrability conditions for radial majorants of the coefficients. The case where φ, ψ may not admit such radial majorants was left open, and is the case on which we focus in this paper. Earlier, in [**21**], Kusano and Swanson proved the existence of decaying positive entire solution for f depending on u and ∇u sublinearly. Usami [**28**] established the existence of bounded positive solutions which are bounded away from zero, and Kusano and Oharu [**19**] further gave the existence of infinitely many such solutions, both for f depending on u and ∇u either sublinearly or superlinearly. Again, radial ideas and strong sub- and supersolution methods were extensively used.

Our method originates from the procedures employed in [**1**], and consists of a combination of *a-priori* estimates and of sub-supersolution arguments. Since we do not use radial arguments nor variational techniques, it is possible for us to deal with some problems which are not amenable to such procedures. Another feature of our method is that relevant constants can be estimated. This is the key step in answering the open question mentioned at the end of the paper by Kusano and Trench, [**22**], for the cases not covered by [**10**].

Our procedure is as follows: Our methods begin with Theorem 2 which shows, under suitable conditions, the existence of a weak supersolution to (1). Theorem 2 can be viewed as a nonlinear perturbation result about positive constant solutions of a linear elliptic equation. As we illustrate below, see e.g. Example 4, in some cases the existence of positive solutions to (1) follows immediately from Theorem 2. However, an immediate consequence of the maximum principle shows that

$$-\Delta u = f(x, u, \nabla u) \leqq 0$$

cannot have positive solutions which decay to zero at infinity. Since the existence conditions in Theorem 2 only require f to be small in some norm, they are not sufficient for the existence of a decaying solution by the above observation. Consequently, simple spectral estimates are introduced in the proof of Theorem 3 to construct a nonnegative nontrivial weak subsolution of small L^∞ norm. This spectral procedure is motivated by the nature of the previous results to which we wish to refer, and only requires a local structure condition on f at a point of $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$. The final part of the proof of Theorem 3 follows from the recall of a global weak sub- and supersolution method. This procedure, with a long history, has been extended to unbounded domains by several authors. See, e.g., the articles by Hess, **[13]**, Boccardo *et al.*, **[3]**, Donato and Giachetti, **[7]**, Fukagai, **[8]**, Noussair and Swanson, **[25]**, and the references therein. We wish to consider classical solutions but, as is apparent below, it is convenient for us to allow weak sub- and supersolutions which belong only to local Sobolev Spaces. Theorem 5.3 of **[7]** gives a result very close to the one we need, and for convenience, we explicitly state the modified result which we require in this paper. We conclude the paper by giving several examples which explicitly show the connection of our results to some earlier work, and by obtaining estimates for the critical constants which appear in our existence criteria.

The key steps in our arguments involve *a-priori* L^∞_{loc} estimates for u and $|\nabla u|$. Such estimates are a well-known important part of existence results for a bounded domain Ω, see e.g. **[4]** and **[5]**. The proof of Theorem 2, however, follows the more classical method of estimates based on the linear part of (1) rather than estimates which also involve the nonlinear structure of (1) as in **[4]**, **[5]**. Our estimates are thus not as sharp, but they do imply that the constants involved can be estimated with a reasonable amount of effort. In any case, if $\Omega = \mathbf{R}^n$ then the absence of a finite boundary makes calculations easier, since only interior estimates need be used. Our procedure, however, requires that $-\sum D_i(a_{ij}D_ju)$ satisfy an explicit Hardy inequality. Alternatively put, $-\sum D_i(a_{ij}D_ju)$ needs to be subcritical or satisfy the λ-property or admit a Green's function (see e.g. **[2]** for clarification of these ideas). In particular, these restrictions mean that we can consider $-\Delta$ only for $n \geqq 3$. The restrictions on (a_{ij}) are more complicated since the structure of linear critical operators is not well known. In any case, this is only a difficulty in $\mathbf{R}^n$ and not in any proper subdomain of $\mathbf{R}^n$. We again refer the interested reader to **[2]** for proofs and discussion of these ideas. In practice, we obtain the needed explicit Hardy inequality either directly or from Sobolev's inequalities. Heuristically, this can be viewed as a bound on Green's function. In cases where Green's function can be found, such estimates may be obtained more easily directly.

In conclusion, we remark that some of these results were recently presented at the Canadian Mathematics Society 1987 Winter Meeting in Vancouver.

2. Preliminaries. We briefly recall for convenience some of the definitions and notations of [**1**]. For any given function $0 < t \in C^\infty(\mathbf{R}^n)$ we denote by $L_t^q(D)$ the associated weighted L^q space in the domain D with norm

$$||\varphi||^q_{L_t^q(D)} = \int_D t|\varphi|^q.$$

For any $x \in \mathbf{R}^n$ we define

$$B_i(x) = \{y| \quad |y - x| < i\} \quad \text{and}$$

$$N(\varphi, q, i, D) = \sup_{x \in D} [\, ||\varphi||_{L^q(B_i(x))}].$$

By $\vec{1}$ we denote the vector $(1, \ldots, 1)$ and vector inequalities are understood componentwise. A function $v \in H^{1,2}_{\text{loc}}(\mathbf{R}^n)$ is a weak supersolution of (1), if

$$\int_{\mathbf{R}^n} \sum a_{ij} D_i v D_j \varphi \geqq \int_{\mathbf{R}^n} f(x, v, \nabla v)\varphi$$

for any $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\varphi \geqq 0$. A weak subsolution is defined accordingly. For convenience, we always assume that

$$a_{ij} = a_{ji} \in C^{1+\alpha}_{\text{loc}}(\mathbf{R}^n) \quad \text{with } \xi_0 I \leqq (a_{ij}) \leqq \xi_1 I$$

for some positive constants ξ_0, ξ_1. While the differentiability assumption can be relaxed to e.g. $a_{ij} \in L^\infty$ for some results, it is sometimes needed in our methods as we indicate below.

Finally, assume $f(x, u, \vec{\xi})$ satisfies the Nagumo condition: for $(x, u, \vec{\xi}) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ we have:

$$(2) \qquad |f(x, u, \vec{\xi})\,| \leqq b(\,|u|\,)[h(x) + k_1|\vec{\xi}|^2]$$

with b: $\mathbf{R}^+ \to \mathbf{R}^+$ nondecreasing,

$$b \in L^\infty_{\text{loc}}(\mathbf{R}^+), \quad k_1 \in \mathbf{R}^+, \quad h \in L^\infty_{\text{loc}}(\mathbf{R}^n).$$

We state:

THEOREM 1. *Let f satisfy* (2) *and be locally Hölder continuous with exponent $\mu \in (0, 1)$. Suppose $w, v \in H^{1,\infty}_{\text{loc}}(\mathbf{R}^n)$ form a weak sub-supersolution pair with $w \leqq v$ and $w \leqq 0 \leqq v$ near ∞. Then* (1) *has a solution $u \in C^2(\mathbf{R}^n)$ with $w \leqq u \leqq v$.*

This is contained in Theorem 5.3 of [7] except for the regularity of u which can easily be established by bootstrap arguments. We remark that assumption (2) is only needed when sub-supersolution arguments are used, but for simplicity is assumed everywhere. Similarly, f need not be Hölder continuous if we wish to deal with generalized solutions.

Let λ be a smooth positive function with

$$0 < \lambda^{-1} \in L^{n/2}(\mathbf{R}^n)$$

and set $t = (1 + |x|^2)$. The specific choices of λ will depend on the problem considered, as we indicate below. Let q denote a positive number, $n < q$, fixed in the sequel. Consider the space $S \subset L^q_{\text{loc}}(\mathbf{R}^n)$ equipped with norm

$$||s||_S = N(s, q, 2, \mathbf{R}^n)$$

and observe that $\{S, || \ ||_S\}$ is a Banach Space. We form $\mathscr{L}_1 = L^2_t \cap S$ with norm

$$||u||_{\mathscr{L}_1} = ||u||_{L^2_t(\mathbf{R}^n)} + ||u||_S$$

and $\mathscr{L}_2 = L^2_\lambda \cap S$ with norm

$$||u||_{\mathscr{L}_2} = e||u||_{L^2_\lambda(\mathbf{R}^n)} + ||u||_S$$

where e is a positive constant, explicitly chosen below. Observe that since $t^{-1} \notin L^{n/2}(\mathbf{R}^n)$ it is possible to find a function $u \in C^\infty \cap L^\infty(\mathbf{R}^n)$ such that $u \in L^2_t(\mathbf{R}^n)$ and yet $u \notin L^2_\lambda(\mathbf{R}^n)$ for any λ such that $0 < \lambda^{-1} \in L^{n/2}(\mathbf{R}^n)$. We thank A. Meir for showing us an elegant proof of a more general version of this result (which is to appear elsewhere). Conversely, it is possible to select λ and $u \in C^\infty \cap L^\infty(\mathbf{R}^n)$ such that $u \in L^2_\lambda(\mathbf{R}^n)$ and yet $u \notin L^2_t(\mathbf{R}^n)$. These remarks show that $\mathscr{L}_1$ and $\mathscr{L}_2$ are different spaces. Consider the tensor product:

$$P = \{ (u_1, u_2) \mid \ u_1 \in \mathscr{L}_1, u_2 \in \mathscr{L}_2\}$$

and define on P the equivalence relation $\sim$ given by: $(u_1, u_2) \sim (u_3, u_4)$ if and only if $u_1 + u_2 = u_3 + u_4$ a.e. Let $\mathscr{H}$ be the quotient space $\mathscr{H} = P/\sim$ and define on $\mathscr{H}$ the norm $|| \ ||_{\mathscr{H}}$ given by:

$$|| [(u_1, u_2)] ||_{\mathscr{H}} = \inf\{ ||v_1||_{\mathscr{L}_1} + ||v_2||_{\mathscr{L}_2} | \quad (v_1, v_2) \sim (u_1, u_2) \}.$$

We now define a map $J:\mathscr{H} \to L^2_{\text{loc}}(\mathbf{R}^n)$ by

$$J([(u_1, u_2)]) = u_1 + u_2$$

and observe that J is well defined and one to one by construction. Clearly J is linear and furthermore the range of J has the following order property: Let $f \in \text{Range}(J)$ and $|g| \leqq |f|$ then $g \in \text{Range}(J)$. Indeed, if $f = J([(f_1, f_2)])$ we need only observe

$$g = J([(g_1, g_2)])$$

with

$$g_1 = g|f_1|/(|f_1| + |f_2|), \quad g_2 = g|f_2|/(|f_1| + |f_2|)$$

where $|f_1| + |f_2| \neq 0$ while $g_1 = g_2 = 0$ if $|f_1| + |f_2| = 0$. We now define on Range(J) a norm $M(\ )$ given by:

$$M(f) = \|J^{-1}(f)\|_{\mathscr{H}}.$$

We note that if $|g| \leqq |f|$ then $M(g) \leqq M(f)$ while if $f = f_1 + f_2$ with $f_1 \in \mathscr{L}_1$, $f_2 \in \mathscr{L}_2$ then

$$M(f) \leqq \|f_1\|_{\mathscr{L}_1} + \|f_2\|_{\mathscr{L}_2}$$

by definition.

Finally, we observe that if the problem (1) is semilinear, then the procedures we will introduce may be simplified and the constants changed. In particular, we only require $q > n/2$ if no gradient estimates are desired.

3. Results. We henceforth assume that f satisfies the conditions of Theorem 1.

We introduce a positive function z such that:

$$\text{(i) } z \in C^1 \cap H^{2,2}_{\text{loc}}(\mathbf{R}^n); \quad \text{(ii) } l_0(z) \geqq 0; \quad \text{(iii) } z, \nabla z \in L^\infty(\mathbf{R}^n);$$

$$\text{(iv) } z \to 0 \text{ as } r \to \infty; \quad \text{(v) } \left(l_0\varphi - 2\sum \beta_j \frac{\partial\varphi}{\partial x_j}, \varphi\right) \geqq \delta(-\Delta\varphi, \varphi)$$

for a fixed $\delta > 0$ and any $\varphi \in C_0^\infty$, where

$$\beta_j = \sum a_{ij}\frac{\partial}{\partial x_i}(\ln z)$$

is assumed in L^∞.

We now set the constant e in the definition of $\|\ \|_{\mathscr{L}_2}$ to be

$$e = [(n-2)/2]T^{1/2}\|\lambda^{-1}\|^{1/2}_{L^{n/2}(\mathbf{R}^n)}$$

where T is the optimum embedding constant:

$$H^{1,2} \to L^{2n/(n-2)}.$$

THEOREM 2. *Let*

$$F(x, a, b) = \sup_{\substack{0 < \xi \leqq a \\ -b\vec{1} \leqq \vec{\tau} \leqq b\vec{1}}} \frac{|f(x, \xi z, \xi \nabla z + z\vec{\tau})|}{z} \tag{3}$$

satisfy $M(F(x, a, b)) < \infty$ *for any positive constants* a, b. *Then there exists a positive constant* E_1, *independent of* f, *such that if for some positive constants* a, b, σ *with* $\sigma < 1$ *we have*

$$\left.\begin{aligned} &E_1 M(F(x, a, b)) - b \leqq 0 \\ &E_1 M(F(x, a, b)) - \frac{(1-\sigma)}{2}a \leqq 0 \end{aligned}\right\} \tag{4}$$

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then (1) *has a positive weak supersolution* $v \in H^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^n)$ *such that* $v \sim z$ *at* ∞. *Furthermore if* $|\beta_j| < c/(1 + |x|)$ *and* $F = J([F_1, F_2])$ *with*:

$$\lim_{|x|\to\infty} ||F_1||_{L_q(B_2(x))} = \lim_{|x|\to\infty} ||F_2||_{L_q(B_2(x))} = 0$$

then $v/z \to c$, *for some positive constant* c, *as* $|x| \to \infty$.

Proof. Observe that setting $u = \hat{u}z$ in (1) reduces (1) to

$$l_0(\hat{u}) - 2 \sum \beta_j D_j \hat{u} + \frac{l_0(z)}{z}\hat{u} = \frac{f(x, \hat{u}z, \hat{u}\nabla z + z \nabla \hat{u})}{z}.$$

Since $l_0 z \geqq 0$, it suffices to show the existence of a solution $\hat{u}$ (bounded above and below by positive constants) of

$$(5) \qquad \begin{aligned} l_1 \hat{u} \triangleq l_0(\hat{u}) - 2 \sum \beta_j D_j \hat{u} &= \frac{f(x, \hat{u}z, \hat{u}\nabla z + z \nabla u)}{z} \\ &= \tau(x, \hat{u}, \nabla \hat{u}). \end{aligned}$$

Assume $\{t_m\}$, $\{\varphi_m\}$ denote a sequence of positive numbers and $C_0^\infty(\mathbf{R}^n)$ functions respectively such that

$$t_m \uparrow +\infty, t_1 > 3, 0 \leqq \varphi_m \leqq 1, \varphi_m \in C_0^\infty(|x| < t_m - 2), \varphi_m \equiv 1$$

in $(|x| \leqq t_m - 3)$. For any chosen m, set

$$\mathscr{B} = C^0(|x| \leqq t_m) \cap C^1(|x| \leqq t_m - 2)$$

and norm $\mathscr{B}$ with

$$||u||_{\mathscr{B}} = \max\left(||u||_{C^0(|x|\leqq t_m)}; \frac{(1-\sigma)}{2}\frac{a}{b}||\nabla u||_{C^0(|x|\leqq t_m - 2)}\right),$$

where a, b, σ satisfy (4). Clearly $\{\mathscr{B}, || \ ||_{\mathscr{B}}\}$ forms a Banach space and $\mathscr{B} \hookrightarrow \mathrm{Range}(J)$ (by defining $u \equiv 0$ outside $(|x| \leqq t_m)$) with continuous embedding. Furthermore, we observe that if l_1^{-1} denotes the Dirichlet inverse in $(|x| < t_m)$ then

$$l_1^{-1}: \mathrm{Range}\ J \to \mathscr{B}$$

(see **[1]**, **[12]**, **[23]**) and there exists a constant E_1 independent of m such that for $g \in \mathrm{Range}(J)$:

$$||l_1^{-1}(g)||_{C^0(|x|\leqq t_m)} \leqq E_1 M(g)$$
$$||\nabla(l_1^{-1}(g))||_{C^0(|x|\leqq t_m - 2)} \leqq E_1 M(g).$$

Let K denote the ball in $\mathscr{B}$ with center at $a(1 + \sigma)/2$ and radius $a(1 - \sigma)/2$, and define T on K by:

$$T(u) = \frac{a(1+\sigma)}{2} + l_1^{-1}(\tau(x, u, \varphi_m \nabla u)).$$

Note that since $u \in K$ then

$$|\tau(x, u, \varphi_m \nabla u)| \leqq F(x, a, b),$$

whence $\tau(x, u, \varphi_m \nabla u) \in \text{Range}(J)$ and:

$$M(\tau(x, u, \varphi_m \nabla u)) \leqq M(F(x, a, b)).$$

Our estimates thus imply that $T:K \rightarrow K$ by (4) and, furthermore, T is a compact continuous map by the coerciveness of l_1. By the Schauder Fixed Point Theorem, we conclude the existence of $u_m \in K$ such that $T(u_m) = u_m$. Equivalently,

$$l_1\left(u_m - \frac{a(1 + \sigma)}{2}\right) = \tau(x, u_m, \varphi_m \nabla u_m)$$

with $a\sigma \leqq u_m \leqq a$, $|\varphi_m \nabla u_m| \leqq b$. Since $\varphi_m \equiv 1$ in $|x| < t_m - 2$, a diagonal argument, whose details may be found in **[1]** and elsewhere, shows the existence of a function $\hat{u}$ with the desired properties. We need only notice that $\{u_m\}$ uniformly locally bounded in C^1 implies that $\{u_m\}$ is in $C^{1+\alpha}(B_R)$ for some $\alpha > 0$ ([**23**, p. 203]). There only remains to show that $\hat{u} = v/z \rightarrow c$ as $|x| \rightarrow \infty$ for some constant c. This follows from the observation that

$$\tau(x, u_m, \varphi_m \nabla u_m) \in \text{Range } J$$

by construction, whence $\tau = g_1 + g_2$ with $g_i \in \mathscr{L}_i$ $i = 1, 2$ uniformly bounded in m. Choose $\alpha > 0$ and a function $h \in C^1$ such that $h(x) = |x|^{-\alpha}$ for $|x| > 2$, $|h(x)| < D(\alpha)$ for $|x| < 2$ and $|D_j h/h| < C(\alpha)$, $C(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0$, $h(0) = 1$. Select $x_0 \in \mathbf{R}^n$ and define

$$h_0(x) = h(x - x_0), \quad \beta_j^* = D_j h_0/h_0.$$

We note that $\omega = \bar{u}_m h_0$ with $\bar{u}_m = \hat{u}_m - a(1 + \sigma)/2$ satisfies for some $\tau = g_1 + g_2 \in \text{Range}(J)$

$$\begin{aligned}\hat{l}\omega &= -\sum D_i(a_{ij}(D_j\omega - \beta_j^*\omega)) + \sum a_{ij}\beta_i^* D_j\omega \\ &\quad - \sum a_{ij}\beta_i^*\beta_j^*\omega - 2\sum \beta_j(D_j\omega - \omega\beta_j^*) \\ &= \tau h_0.\end{aligned}$$

Consequently, we obtain

$$(\hat{l}\omega, \omega) = (l_1\omega, \omega) - (\sum a_{ij}\beta_i^*\beta_j^*\omega, \omega) - 2(\sum \beta_j\beta_j^*\omega, \omega).$$

Observe that $(l_1\omega, \omega) \geqq \delta(-\Delta\omega, \omega)$ while

$$|\beta_j^*| \leqq \frac{C(\alpha)}{(1 + |x - x_0|)} \quad \text{and} \quad |\beta_j| \leqq \frac{C}{(1 + |x|)}$$

imply (see Lemma 6 below)

$$(\sum a_{ij}\beta_i^*\beta_j^*\omega, \omega) + 2|(\sum \beta_j\beta_j^*\omega, \omega)| \leqq \frac{\delta}{2}(-\Delta\omega, \omega)$$

by choosing α small enough. We conclude that:

$$(\hat{l}\omega, \omega) > \frac{\delta}{2}(-\Delta\omega, \omega).$$

i.e., $\hat{l}$ satisfies the same structure as l_1 and once again by [**12**, p. 194], and a simple limit argument on m, we obtain

$$\left|\hat{u}(x_0) - \frac{a(1 + \sigma)}{2}\right| \leqq K_0[\, ||F_1h_0||_{\mathscr{L}_1} + ||F_2h_0||_{\mathscr{L}_2}]$$

with K_0 independent of x_0. But, by assumption,

$$||F_ih_0||_{L^q(B_2(x_0))} \leqq C(\alpha)||F_i||_{L^q(B_2(x_0))} \to 0 \quad \text{as } |x_0| \to \infty$$

for $i = 1, 2$ while a simple decomposition of $\mathbf{R}^n$ shows:

$$||F_1h_0||^2_{L_t^2(\mathbf{R}^n)} \leqq ||F_1||^2_{L_t^2(|x|>(|x_0|/2))} + \frac{C}{|x_0|^{2\alpha}}||F_1||^2_{L_t^2(\mathbf{R}^n)}$$

for some constant C. From this we conclude that

$$||F_1h_0||_{\mathscr{L}_1} \to 0 \quad \text{as } |x_0| \to \infty.$$

An analogous result applies to $||F_2h_0||_{\mathscr{L}_2}$ and we observe that

$$||F_1h_0||_{\mathscr{L}_1} + ||F_2h_0||_{\mathscr{L}_2} \to 0,$$

whence

$$\hat{u}(x_0) \to a(1 + \sigma)/2 \quad \text{as } |x_0| \to \infty.$$

This concludes the proof.

Since explicit bounds on E_1 are important for some of the examples we consider, we sketch for $a_{ij} = \delta_{ij}$ in a latter section the lengthy but straightforward calculation which leads to an explicit estimate. Of course, such estimates also show the existence of E_1 but, as mentioned above, such existence is well known. We remark that (4) indicates that no estimate on E_1 is needed if, for example,

$$\lim_{a\to\infty} \frac{M(F(x, a, a))}{a} = 0.$$

Next we consider the existence of a weak subsolution. We emphasize, as mentioned earlier, that this argument is not needed if $l_0(z) = 0$ since in this case v as given by Theorem 2 is actually a solution. We observe that the limit result $v/z \to c$ in Theorem 2 appears to be new under the conditions we consider.

Suppose there exists a neighbourhood Q of zero in $\mathbf{R}^n \times \mathbf{R}^+ \times \mathbf{R}^n$ in which:

(6) $$f(x, u, \vec{\xi}) \geqq h_1(x) + p(x)u^\gamma + q(x)|\vec{\xi}|^\theta$$

where: $0 < \gamma < 1$; $\theta \geqq 0$; $h_1(x) \geqq 0$; $q(x) \geqq 0$; $p(x) \geqq 0$; $h_1, p, q \in C$; and $h_1(x) + p(x) > \epsilon_0 > 0$. Note that this is a local estimate on f and nothing new is postulated outside Q. Observe also that (6) ensures that f is not globally nonpositive, a situation explicitly forbidden by the maximum principle as mentioned in the introduction. Finally, we assume globally that:

(7) $$f(x, u, \vec{\tau}) \geqq g(x, u, \vec{\tau})$$

for some $g \in C^1(\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n)$, $g(x, 0, \vec{0}) \geqq 0$.

Conditions (6), (7) permit an elementary construction of a subsolution and we obtain:

THEOREM 3. *Assume that f satisfies the above postulated regularity conditions and that the estimates* (2), (4), (6), (7) *hold. Then equation* (1) *has a classical positive solution u such that $u \leqq Cz$ at ∞.*

Proof. Let $B_\epsilon(0) \subset\subset Q^1$ where Q^1 is the projection of Q on its first n components and let u_1 be a positive eigenfunction of the Dirichlet problem: $l_0(u_1) = \lambda u_1$ in $B_\epsilon(0)$. Since $\gamma < 1$ we choose ϵ_1 small enough and ensure that

$$l_0(\epsilon_1 u_1) \leqq f(x, \epsilon_1 u_1, \nabla(\epsilon_1 u_1))$$

and $\epsilon_1 u_1 \leqq v$ in $B_\epsilon(0)$ by (6). Finally, we extend u_1 to $\mathbf{R}^n$ by setting $u_1 = 0$ in $\mathbf{R}^n - B_\epsilon(0)$ and observe $w = \epsilon_1 u_1 \leqq v$ globally. We note that it is an immediate consequence of the Divergence Theorem, the positivity of u_1 in $B_\epsilon(0)$ and (7) that w is a subsolution of (1) in $\mathbf{R}^n$. Since Theorem 1 and Theorem 2 then show the existence of a nonnegative solution $u \in C^2$, to conclude the proof we need only show that u is positive. Assume $u(x_0) = 0$, whence x_0 is a minimum of u and $\nabla u(x_0) = \vec{0}$. Since $f(x, u, \nabla u) \geqq g(x, u, \nabla u)$ and $g \in C^1$, $g(x, 0, \vec{0}) \geqq 0$ we observe that for x near x_0:

$$g(x, u, \nabla u) \geqq \int_0^1 \frac{d}{dt}[g(x, tu(x), t\nabla u(x))]dt$$

$$= \sum_{i=1}^n \psi_i(x)D_i u + \psi_0(x)u$$

for some $\psi_i \in L^\infty$, $i = 0, \dots, n$. We conclude:

$$l_0(u) - \sum_{i=1}^n \psi_i(x)D_i u - \psi_0(x)u \geqq 0$$

and $u \geqq 0$ near x_0. But then, by e.g. [**12**, p. 194],

$$\|u\|_{L^p(B_{2R}(x_0))} \leqq C \inf_{B_R(x_0)} u = 0$$

for some R, p. We conclude that $u \equiv 0$ near x_0 and that $S = \{x | u(x) = 0\}$ is both open and closed. Since $u \geqq v$, we must have $S = \emptyset$. Theorem 3 is proved.

We observe that, apart from the various regularity and growth conditions specified in (2), (6), (7), to apply Theorem 3 to the partial differential equation (1) we need only choose the function λ, estimate E_1 and verify that the algebraic system (4) has a solution for some a, b, σ. To illustrate these results we give the following examples in which we always assume for simplicity that $p(x)$, $q(x)$, $h(x)$, $g(x) \geqq 0$, nontrivial and in $C^\infty(\mathbf{R}^n)$, while $a_{ij} = \delta_{ij}$. We let α, β denote any constants such that:

$$0 \leqq \alpha < (\sqrt{n^2 + (n-2)^2} - n)/2,$$

$$0 < \beta = [(n-2)^2/4] - \alpha n - \alpha^2.$$

Observe that the upper bound on α is monotonically increasing in n and, for n large, is asymptotic to $(n-2)/(2 + 2\sqrt{2})$. We then explicitly choose the function z given by:

$$z = \begin{cases} |x|^{-\alpha}, & |x| \geqq 1 \\ 1 + \dfrac{\alpha}{2} - \dfrac{\alpha}{2}|x|^2, & 0 \leqq |x| \leqq 1. \end{cases}$$

Observe that:

$$z \in C^1(\mathbf{R}^n) \cap H^{2,2}_{\text{loc}}(\mathbf{R}^n); \ -\Delta z \geqq 0;$$

$$\operatorname{div}(\nabla z/z) \geqq [4\beta - (n-2)^2]/4|x|^2; \ z, \nabla z \in L^\infty(\mathbf{R}^n);$$

$z \to 0$ as $r \to \infty$ if $\alpha > 0$, whence z satisfies all the needed conditions. We heuristically observe that if $(a_{ij}) \neq I$, then direct substitution shows that such a z can still be chosen (possibly with different α, β) if (a_{ij}) and its derivatives behave suitably. We were unable to adapt our procedure, however, to the case of more general (a_{ij}), e.g. to $a_{ij} \in L^\infty$.

We begin by assuming in the next three examples that the functions in Range J which arise are all of type $J([(v_1, 0)])$. In such a case,

$$M(J[(v_1, 0)]) \leqq \|v_1\|_{\mathcal{L}_1}$$

whence explicit criteria can be obtained by replacing $M(\ )$ by $\|\ \|_{\mathcal{L}_1}$ in the conditions which follow. Despite this simplification, each example is an improvement/extension of earlier results as we explicitly indicate. We observe, however, that an examination of our references shows that sharper estimates are known, but only for some radially symmetric problems of related type.

Example 1. Consider

$$-\Delta u = q(x)u^{\delta} + p(x)u^{\gamma}$$

in $\mathbf{R}^n$, $n \geqq 3$, with $0 < \gamma < 1 < \delta$. Let

$$P = M(pz^{\gamma-1}), \quad Q = M(qz^{\delta-1}) \quad \text{and}$$

$$\eta = P^{(\delta-1)/(\delta-\gamma)}Q^{(1-\gamma)/(\delta-\gamma)}\left[\left(\frac{1-\gamma}{\delta-1}\right)^{(\delta-1)/(\delta-\gamma)} + \left(\frac{\delta-1}{1-\gamma}\right)^{(1-\gamma)/(\delta-\gamma)}\right].$$

Then if $E_1 \cdot \eta < 1/2$, the problem has a positive solution $u \in C^2(\mathbf{R}^n)$ such that $0 < u \leqq cz$; c is a constant. This is a case left open in [**22**].

Example 2. Consider

$$-\Delta u = p(x)u^{\gamma} + q(x)u^{\delta} + h(x)|\nabla u|^{\gamma} + g(x)|\nabla u|^{\delta}$$

in $\mathbf{R}^n$, $n \geqq 3$, with $0 < \gamma < 1 < \delta \leqq 2$. Let

$$H_1 = 2^{\gamma}M(h|\nabla z|^{\gamma}z^{-1}), \quad H_2 = 2^{\gamma}M(hz^{\gamma-1}),$$

$$G_1 = 2^{\delta}M(g|\nabla z|^{\delta}z^{-1}), \quad G_2 = 2^{\delta}M(gz^{\delta-1}), \quad \text{and}$$

$$\pi = (P + H_1 + H_2)^{(\delta-1)/(\delta-\gamma)}(Q + G_1 + G_2)^{(1-\gamma)/(\delta-\gamma)} \times \left[\left(\frac{1-\gamma}{\delta-1}\right)^{(\delta-1)/(\delta-\gamma)} + \left(\frac{\delta-1}{1-\gamma}\right)^{(1-\gamma)/(\delta-\gamma)}\right].$$

Then if $\pi < 1/2E_1$, the same conclusion holds. This is an extension of the results in [**22**], [**21**] and [**8**].

Example 3. Consider

$$-\Delta u = p(x)u^{\gamma} + q(x)u^{\delta}|\nabla u|^{\mu}$$

in $\mathbf{R}^n$, $n \geqq 3$, $0 < \gamma < 1 < \delta + \mu$, $0 \leqq \mu \leqq 2$.
Let

$$Q_1 = 2^{\mu}M(q|\nabla z|^{\mu}z^{\delta-1}), \quad Q_2 = 2^{\mu}M(qz^{\mu+\delta-1}) \quad \text{and}$$

$$\sigma = P^{(\delta+\mu-1)/(\delta+\mu-\gamma)}(Q_1 + Q_2)^{(1-\gamma)/(\delta+\mu-\gamma)} \times \left[\left(\frac{1-\gamma}{\delta+\mu-1}\right)^{(\delta+\mu-1)/(\delta+\mu-\gamma)} + \left(\frac{\delta+\mu-1}{1-\gamma)}\right)^{(1-\gamma)/(\delta+\mu-\gamma)}\right].$$

Then if $\sigma < 1/2E_1$, the same conclusion holds. We note that Kusano and Oharu [**19**] considered some other combinations of γ, δ and μ, and obtained the existence of infinitely many positive solutions which are bounded and bounded away from zero, cf [**19**, p. 131]. By the same ideas we could also consider

$$-\Delta u = p(x)u^{\gamma}(1 + |\nabla u|^{\lambda}) + q(x)u^{\delta}(1 + |\nabla u|^{\mu})$$

in $\mathbf{R}^n$, $n \geqq 3$, with $0 < \gamma < 1 < \delta$, $0 \leqq \lambda, \mu \leqq 2$. In fact, let

$$P_1 = 2^{\lambda}M(p|\nabla z|^{\lambda}z^{\delta-1}), \quad P_2 = 2^{\lambda}M(pz^{\lambda+\delta-1}),$$

if there exists a positive solution α of

$$P\alpha^{\gamma-1} + (P_1 + P_2)^{\gamma+\lambda-1} + Q\alpha^{\delta-1} + (Q_1 + Q_2)\alpha^{\delta+\mu-1} < \frac{1}{2E_1},$$

then the same conclusion holds.

We conclude the examples by illustrating the advantages obtained by our method.

Example 4. Again consider the semilinear problem:

$$-\Delta u = q(x)u^{\delta} + p(x)u^{\gamma}$$

in $\mathbf{R}^3$ with $0 < \gamma < 1 < \delta$. Assume that, for some α,

$$0 < z^{\delta-1}q(x) \in L^{\xi} \cap L^1(\mathbf{R}^3), \quad \xi > 3/2 \quad \text{and}$$

$$p(x)z^{\gamma-1} \in L_t^2(\mathbf{R}^3) \cap L^{\infty}(\mathbf{R}^3).$$

Here we select $\lambda^{-1} \triangleq z^{\delta-1}q(x)$ and observe that λ^{-1} is admissible since the problem is semilinear and $\xi > n/2$. Furthermore, note that

$$F(x, a) = q(x)z^{\delta-1}a^{\delta-1} + p(x)z^{\gamma-1}a^{\gamma-1} = J([(v_1, v_2)])$$

with

$$v_1 = p(x)z^{\gamma-1}a^{\gamma-1}, \quad v_2 = q(x)z^{\delta-1}a^{\delta-1}.$$

We can now formally repeat the calculations of Example 1 and, since we do not assume that $z^{\delta-1}q(x) \in L_t^2$, actually obtain a new criterion. To explicitly illustrate this remark, suppose $\alpha = 0$ (i.e., $z \equiv 1$) and $p \equiv 0$. In such a case the calculation of E_1 is irrelevant. Then, as earlier noted, the supersolution of Theorem 2 is actually a solution and we conclude that if $q(x) \in L^{\xi} \cap L^1(\mathbf{R}^3)$ then $-\Delta u = q(x)u^{\delta}$ has infinitely many bounded positive solutions which tend to positive constants at ∞. This is an improvement over a result given in [**1**], where it was assumed that

$$|x|q(x) \in L^{\infty} \cap L^1(\mathbf{R}^3)$$

and no conclusion was obtained about the convergence of the solutions at ∞.

Finally, we observe that our procedures can also deal with the case:

$$l_0'(u) = -\sum D_i(a_{ij}D_ju) + m^2u = f(x, u, \nabla u)$$

where m^2 is a positive constant. Indeed, one need now only choose z such that $l_0'(z) \geqq 0$, and follow an identical procedure. Observe that such a z need not be radial.

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4. Estimates. As we have seen in the examples of the previous section, explicit bounds on E_1 play a critical role in the formulation of concrete results from Theorems 2, 3.

Bounds on E_1 are obtained in this section by following the ideas of [**1**], [**12**], [**23**] where the existence of E_1 is shown. We are interested in reducing the calculations involved to a reasonable length and thus find it convenient to blend some of the procedures of [**1**], [**12**], [**23**]. In what follows we shall assume that vectors and matrices are normed by the standard Hilbert Space norm: if $\vec{v} = (v_0, \dots, v_m)^T$ then $|\vec{v}|^2 = \sum_{i=0}^m v_i^2$, etc. Furthermore, we assume $a_{ij} = \delta_{ij}$. The more general case is handled identically.

LEMMA 4. *Let $\vec{u} = (u_0, \dots, u_m)^T$ be a solution of the system*

$$-\Delta\vec{u} - 2\sum_{j=1}^n b_j D_j \vec{u} + C\vec{u} = \sum_{i=1}^n D_i(\vec{f}_i) + \vec{g}$$

in a ball $B_2(x_0)$. Suppose: $\vec{u} \in C^\alpha \cap H^{1,2}(B_2(x_0))$; the scalars b_j^2, the vector $\vec{g}$ and the $(m+1) \times (m+1)$ matrix C belong to $L^{q/2}(B_2(x_0))$ while the vectors $\vec{f}_i$ are in $L^q(B_2(x_0))$ for some $q > n$. Then:

$$|\vec{u}|_{L^\infty(B_1(x_0))} \leqq K_0\Big[\,\|\,|\vec{u}|\,\|_{L^2(B_2(x_0))} + \sum_{i=1}^n \|\,|\vec{f}_i|^2\|^{1/2}_{L^{q/2}(B_2(x_0))} + \|\,|\vec{g}|\,\|_{L^{q/2}(B_2(x_0))}\Big]$$

where: $K_0 = K_1[\mu(B_2)^{1/2} + 1]$; with

$$K_1 = [4H2^{3/2(q/q-n)}]^{n/2} \cdot [2(n/(n-2))^{3/2(q/q-n)}]^{[n(n-2)/4]},$$

$$H = T^2(4 + E(\beta_1)) + (2T^2E(\beta_1)\{\,\|\,|C|\,\|_{L^{q/2}(B_2(x_0))} + \|\textstyle\sum b_j^2\|_{L^{q/2}(B_2(x_0))} + 2\})^{(q/q-n)},$$

$$E(\beta) = \frac{3}{2} + \frac{16}{\beta(\beta+2)},$$

$$T = \textit{optimum embedding constant} = \frac{1}{n\sqrt{\pi}}\left(\frac{n!}{2\Gamma(1+n/2)}\right)^{1/n}\left(\frac{n}{n-2}\right)^{1/2},$$

$$\beta_1 = 4/(n-2).$$

Proof. We follow the procedures of [**12**] with a test function motivated by arguments in [**23**]. Specifically, set

$$\vec{\varphi} = \vec{u}v^\beta\eta^2$$

where $v = (|\vec{u}| + k)$ and $k \geqq 0$, $\beta > 0$, $\eta \in C_0^\infty(B_2(x_0))$ to be chosen below. Observe that $\vec{\varphi}$ is a suitable test function (see, e.g., [**12**, p. 151]). We find:

$$(8) \qquad \int \sum_j \langle D_j\vec{u}, D_j\vec{\varphi}\rangle - 2\sum b_j\langle D_j\vec{u}, \vec{\varphi}\rangle + \langle C\vec{u}, \vec{\varphi}\rangle = \int \sum_j \langle f_j, D_j\vec{\varphi}\rangle + \int \langle \vec{g}, \vec{\varphi}\rangle$$

where $\langle\ ,\ \rangle$ denotes the $\mathbf{R}^{m+1}$ inner product. Note that, a.e.,

$$D_jv = \begin{cases} (\langle D_j\vec{u}, \vec{u}\rangle)/|\vec{u}| & |\vec{u}| > 0 \\ 0 & |\vec{u}| = 0 \end{cases}$$

whence $|D_jv|^2 \leqq |D_j\vec{u}|^2$ a.e. We expand (8) using the definition of $\vec{\varphi}$ and apply Cauchy's and Hölder's Inequalities repeatedly in an analogous manner to the arguments of [**12**, p. 195-196], and find:

$$\int \sum_j |D_j\vec{u}|^2 v^\beta \eta^2 + \frac{|\vec{u}|(|\nabla v|^2 \beta v^{\beta-1}\eta^2)}{2} \leqq \int v^{\beta+2}\left\{\eta^2\left[\frac{\sum|\vec{f}_i|^2}{v^2}\frac{(3+\beta)}{2} + \frac{|\vec{g}|}{v} + |C| + \frac{8\sum b_j^2}{\beta} + |\nabla\eta|^2\left(1+\frac{8}{\beta}\right)\right]\right\}.$$

We set:

$$\bar{b} = \left[\frac{\sum|\vec{f}_i|^2}{v^2} + \frac{|\vec{g}|}{v} + |C| + \sum b_j^2\right],$$

$$w = v^{(\beta+2)/2},$$

and conclude:

$$(9) \qquad \int |\nabla w|^2\eta^2 \leqq \frac{(\beta+2)^3}{4}C(\beta)\int w^2[\eta^2\bar{b} + |\nabla\eta|^2]$$

where:

$$C(\beta) = \frac{3}{2} + \frac{16}{\beta(\beta+2)}.$$

Inequality (9) has the same structure as the first of inequalities (8.53) of [**12**, p. 196] with $(\beta + 2)^3$ replacing $(\beta + 1)^2$. We can thus follow directly the steps on [**12**, p. 196-197] and obtain:

$$(10) \qquad \sup_{B_1(x_0)} [v] \leqq K_1\|v\|_{L^2(B_2(x_0))}$$

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where:

$$\sigma = \frac{n}{q-n}; \quad T = \frac{1}{n\sqrt{\pi}}\left(\frac{n!}{2\Gamma(1+n/2)}\right)^{1/n}\left(\frac{n}{n-2}\right)^{1/2};$$

$$\chi = n/(n-2); \quad \beta_1 = 4/(n-2) \quad \text{and:}$$

$$H = T^2(4 + E(\beta_1)) + (2T^2E(\beta_1)\|\bar{b}\|_{L^{q/2}(B_2(x_0))})^{1+\sigma};$$

$$K_1 = [4H2^{3/2(1+\sigma)}]^{n/2} \cdot [2\chi^{3/2(1+\sigma)}]^{[(n)(n-2)/4]}.$$

Finally, the choice

$$k = \sum_j \| |\vec{f}_j|^2 \|^{1/2}_{L^{q/2}(B_2(x_0))} + \| |\vec{g}| \|_{L^{q/2}(B_2(x_0))}$$

yields

$$\sup_{B_1(x_0)} |\vec{u}| \leqq K_1[\mu(B_2(x_0))^{1/2} + 1] \cdot \Big[\| |\vec{u}| \|_{L^2(B_2(x_0))} + \sum_j \| |\vec{f}_j|^2 \|^{1/2}_{L^{q/2}(B_2(x_0))} + \| |\vec{g}| \|_{L^{q/2}(B_2(x_0))}\Big]$$

with $\|\bar{b}\|_{L^{q/2}(B_2(x_0))}$ majorized by

$$\| |C| \|_{L^{q/2}(B_2(x_0))} + \|\Sigma\, b_j^2\|_{L^{q/2}(B_2(x_0))} + 2$$

in H. Setting

$$K_0 = K_1[\mu(B_2(x_0))^{1/2} + 1]$$

gives the result.

We remark that our choice of test function in Lemma 4 appears to lead to more restrictive L^p norm results than those obtained by the choice made in [**12**] for the scalar case. Since we are only interested in the L^2 norm on the right hand side of (10), our approach suffices for our purposes.

COROLLARY 5. *Let* $v_0 \in \mathring{H}^{1,2}(|x| < t_m)$ *be a solution of*

$$l_1 v_0 = g = g_1 + g_2$$

where $g \in L^q(|x| < t_m)$, $q > n$ *and* l_0 *is as given in Theorem* 2. *Assume*

$$(l_1\varphi, \varphi) > \delta(-\Delta\varphi, \varphi) \quad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^n).$$

Then:

$$\sup_{|x| \leqq t_m} |v_0| \leqq E_1 M(g)$$

$$\sup_{|x| \leqq t_m - 2} \|\nabla v_0\| \leqq E_1 M(g)$$

with:

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$$E_1 = K_0 \cdot \max\left\{\frac{2}{\delta(n-2)}[(\mu(B_2))^{1/n}T + 1]; n + \mu(B_2)^{1/q}\right\},$$

and the notation of Lemma 4 is used for the constant K_0; with the $(n + 1) \times (n + 1)$ matrix $C = (c_{ij})$ given by:

$$c_{ij} = \begin{cases} 0 & ij = 0 \\ D_i(\beta_j) & ij > 0 \end{cases}$$

and the $L^{q/2}(B_2(x_0))$ norms in H replaced by the sup *of such norms over all $x_0 \in \mathbf{R}^n$.*

Proof. Let $\vec{v}_1 = (v_1, \nabla v_1)^T$, $\vec{v}_2 = (v_2, \nabla v_2)^T$ where $l_1(v_i) = g_i$, $i = 1, 2$ and set $\vec{v} = (v_0, \nabla v_0)^T$. We consider $l_1(v_1) = g_1$ first and, we observe:

$$l_1(\vec{v}_1) + C\vec{v}_1 = \vec{g} + \sum_{i=1}^{n} D_i(\vec{f}_i),$$

where: $\vec{g} = (g, 0, \ldots, 0)^T$; $\vec{f}_i = g\vec{e}_i$ (with $\vec{e}_i$ denoting the vector with 1 in the i^{th} position and all other entries zero); $C = (c_{ij})$ as given in the statement of this corollary.

Let $B_2(x_0) \subset \{ |x| < t_m\}$. We apply Lemma 4 and conclude:

$$(11) \quad |\vec{v}_1|(x_0) \leqq K_0\Big[\| |\vec{v}_1| \|_{L^2(B_2(x_0))} + \sum_{i=1}^{n} \| |\vec{f}_i|^2\|^{1/2}_{L^{q/2}(B_2(x_0))} + \| |\vec{g}| \|_{L^{q/2}(B_2(x_0))}\Big]$$

$$\leqq K_0[\| |\vec{v}_1| \|_{L^2(B_2(x_0))} + (n + \mu(B_2)^{1/q}) \|g\|_{L^q(B_2(x_0))}].$$

But $-\Delta v_1 - 2 \sum \beta_j D_j v_1 = g_1$, whence:

$$\begin{aligned} \|v_1\|_{L^2(B_2(x_0))} &\leqq [\mu(B_2)]^{1/n} \|v_1\|_{L^{2n/(n-2)}(B_2(x_0))} \\ &\leqq [\mu(B_2)]^{1/n} \|v_1\|_{L^{2n/(n-2)}(|x|<t_m)} \\ &\leqq [\mu(B_2)]^{1/n} T \| |\nabla v_1| \|_{L^2(|x|<t_m)} \end{aligned}$$

where T denotes the embedding constant introduced earlier. We note that, following [**1**], the inequality

$$\delta(-\Delta\varphi, \varphi) \leqq (l_1\varphi, \varphi)$$

implies:

$$(12) \quad \| |\nabla v_1| \|_{L^2(|x|<t_m)} \leqq \frac{2}{\delta(n-2)} \|g_1\|_{L^2_t(\mathbf{R}^n)}.$$

Substituting these estimates into (11) yields:

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$$(13)\quad |\vec{v}_1|(x_0) \leqq K_0\frac{2}{\delta(n-2)}[\,[\mu(B_2)\,]^{1/n}T + 1]\|g_1\|_{L^2_t(\mathbf{R}_n)} + K_0(n + \mu(B_2)^{1/q})\|g_1\|_{L^q(B_2(x_0))}.$$

We immediately conclude that

$$|v_1|(x_0) \leqq E_1M(g_1) \quad \text{if } B_2(x_0) \subset (\,|x| < t_m)$$

and, by coerciveness (see [**1**]),

$$|v_1(x)\,| \leqq E_1M(g_1) \quad \text{if } x \in (\,|x| < t_m).$$

If we consider next $l_1(v_2) = g_2$ we observe that inequality (11) is still valid, but inequality (12) is replaced by the following arguments: If $g \in L^2_\lambda(\mathbf{R}^n)$ for some λ, $0 < \lambda^{-1} \in L^{n/2}(\mathbf{R}^n)$ then Sobolev's Inequality yields:

$$\|v_2\|^2_{L^2_{1/\lambda}(\mathbf{R}^n)} \leqq T\|\lambda^{-1}\|_{L^{n/2}_2(\mathbf{R}^n)}\|\,|\nabla v_2|\,\|^2_{L^2(\mathbf{R}^n)}$$
$$\leqq T\|\lambda^{-1}\|_{L^{n/2}(\mathbf{R}^n)}\|v_2\|_{L^2_{\lambda^{-1}}(\mathbf{R}^n)}\|g_2\|_{L^2_\lambda(\mathbf{R}^n)}/\delta.$$

Whence:

$$(12')\quad \|\,|\nabla v_2|\,\|^2 \leqq T\|\lambda^{-1}\|_{L^{n/2}(\mathbf{R}^n)}\|g_2\|^2_{L^2_\lambda(\mathbf{R}^n)}/\delta^2.$$

Substituting (12′) into (11) yields

$$(14)\quad |\vec{v}_2|(x_0) \leqq \frac{K_0}{\delta}[\,[\mu(B_2)\,]^{1/n}T + 1]T^{1/2}\|\lambda^{-1}\|^{1/2}_{L^{n/2}(\mathbf{R}^n)}\|g_2\|_{L^2_\lambda(\mathbf{R}^n)} + K_0(n + \mu(B_2)^{1/q})\|g_2\|_{L^q(B_2(x_0))}.$$

Since $\vec{v}_0 = \vec{v}_1 + \vec{v}_2$, by adding (13) and (14) and noting that the constant E_1 is independent of the specific decomposition $g = g_1 + g_2$, we obtain the desired estimates.

We illustrate our estimate with the following example: Suppose $n = 3$, $q = 4$, $g_2 = 0$. Observe that:

$$\left(\frac{z'}{z}\right)^2 = \sum b_j^2 \leqq \alpha^2,$$

$$|c_{ij}| \leqq 2\alpha^2 + \alpha(1 + 2\delta_{ij}) \quad (ij > 0).$$

A computer programme gives the following bounds: If $\alpha = 0$ then $1/E_1 \approxeq 1.625 \times 10^{-5}$, while if $\alpha = .05$ then $1/E_1 \approxeq 2.069 \times 10^{-7}$. The maximum allowable α in this case is $\alpha \approxeq .08113$.

We conclude this section with a proof of the following estimate used in Theorem 2.

LEMMA 6. *Let $x_0 \in \mathbf{R}^n$, $\varphi \in C_0^\infty(\mathbf{R}^n)$. Then there exists a constant $K > 0$, independent of x_0 such that*

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$$(-\Delta\varphi, \varphi) \geqq K \int_{\mathbf{R}^n} \frac{1}{(1 + |x|)(1 + |x - x_0|)} \varphi^2 dx.$$

Proof. Let $\Omega = \mathbf{R}^n - B_{\epsilon^*}(x_0) - B_{\epsilon^*}(0)$ for $\epsilon^* > 0$, small. Note that Picone's identity shows for $v > 0$:

$$\int_\Omega \sum (D_i\varphi)^2 \geqq \int_\Omega \sum D_i\left(\frac{\varphi^2}{v}\right)D_i v$$

$$\geqq \int_{\partial\Omega} \frac{\varphi^2}{v}\frac{\partial v}{\partial n} ds + \int_\Omega \frac{\varphi^2}{v}(-\Delta v).$$

Choose

$$v = |x|^\alpha |x - x_0|^\beta \quad \text{with } \alpha = -\frac{(n-2)}{2}, \quad \beta = -\epsilon,$$

$\epsilon > 0$ to be determined later.

Direct substitution yields:

$$\frac{-\Delta v}{v} = \left(\frac{n-2}{2}\right)^2 \frac{1}{|x|^2} + \frac{\epsilon(n - 2 - \epsilon)}{|x - x_0|^2}$$

$$- 2\frac{\left(\frac{n-2}{2}\right)\epsilon}{|x|\,|x - x_0|} \sum \frac{x_i}{|x|}\frac{(x_i - x_{0i})}{|x - x_0|}.$$

Since

$$\left(\frac{n-2}{2}\right)^2 \frac{1}{|x|^2} + \frac{\epsilon(n-2-\epsilon)}{|x - x_0|^2} \geqq 2\left(\frac{n-2}{2}\right)\frac{1}{|x|}\frac{\sqrt{\epsilon(n-2-\epsilon)}}{|x - x_0|},$$

we conclude in Ω:

$$\frac{-\Delta v}{v} \geqq 2\left(\frac{n-2}{2}\right)\frac{1}{|x|\,|x - x_0|}[\sqrt{\epsilon(n-2-\epsilon)} - \epsilon].$$

Choosing ϵ such that $0 < \epsilon < 1/2$, we have, for some $K > 0$

$$-\frac{\Delta v}{v} \geqq K\frac{1}{|x|\,|x - x_0|}.$$

Observe also that

$$\left|\int_{\partial\Omega} \frac{\varphi^2}{v}\frac{\partial v}{\partial n} ds\right| \leqq K_1 \frac{\|\varphi\|_\infty^2}{\epsilon^*}(\epsilon^*)^{n-1} \to 0 \quad \text{as } \epsilon^* \to 0,$$

whence letting $\epsilon^* \to 0$ gives the result.

Note that the same procedures also give a direct proof of the Hardy inequality:

$$(-\Delta\varphi, \varphi) \geqq \left(\frac{n-2}{2}\right)^2 \int_{\mathbf{R}^n} \frac{1}{(1+|x|^2)}\varphi^2$$

used in the paper.

5. Concluding remarks. We conclude with the following remarks. It is unreasonable to expect that our methods yield comparable results to those obtained by radial or variational arguments in cases where such arguments are applicable. We mention explicitly two shortcomings of our approach: First, we only know that $u \leqq C|x|^{-\alpha}$ at ∞ and we cannot guarantee $u \sim |x|^{-\alpha}$. Second, the maximum allowable α is less than $n - 2$, the value of α often used in results which hinge on radial comparison. We have observed that different choices of z greatly influence the resulting allowable values of α. Possibly, "better" choices of z could be made to allow an increase in possible α. It is not clear how this is to be done or what constitutes an optimum choice for z. Analogously, for a given z, the optimum value of E_1 is not known.

Finally, we observe heuristically that our methods are immediately applicable to problems where $\mathbf{R}^n$ is replaced by a domain Ω. Indeed, one need only assume $f \equiv 0$ if $x \in \mathbf{R}^n - \Omega$. We construct a positive supersolution in $\mathbf{R}^n$ which is clearly also a supersolution in Ω. Since the subsolution is constructed locally, we repeat exactly the steps above and find a solution $u \in H^{1,2}_{\text{loc}}(\Omega)$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$. Of course, one can obtain different estimates on E_1 and replace $M(f)$ by other norms by taking into account special properties of Ω. This will clearly happen if, for example Ω is sufficiently "thin" at ∞ so that Poincaré Inequality-eigenvalue arguments can be used in place of e.g. Hardy's Inequality. Note that arguments based on radial comparison for such a problem would appear to need the construction of a supersolution in some radially symmetric domain $\hat{\Omega} \supset \Omega$. It is not difficult to construct examples where Ω is "thin" at ∞ and yet $\hat{\Omega} = \mathbf{R}^n$.

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