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Holomorphic 2-Forms and Vanishing Theorems for Gromov–Witten Invariants

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Abstract. On a compact Kähler manifold X with a holomorphic 2-form α, there is an almost complex structure associated with α. We show how this implies vanishing theorems for the Gromov–Witten invariants of X. This extends the approach used by Parker and the author for Kähler surfaces to higher dimensions.

Let X be a Kähler surface with a non-zero holomorphic 2-form α. Then α is a section of the canonical bundle and its zero locus Z_α, with multiplicity, is a canonical divisor. We showed in [L] that the real 2-form $\mathrm{Re}(\alpha)$ determines a (non-integrable) almost complex structure J_α that has the following remarkable "Image Localization Property": *if a J_α-holomorphic map $f\colon C \to X$ represents a non-zero $(1,1)$ class, then f is in fact holomorphic and its image lies in Z_α.* As shown in [LP], this property together with Gromov Convergence Theorem leads to the following theorem.

Theorem 1 (**[LP]**) *Let X be a Kähler surface with a non-zero holomorphic 2-form α. Then, any class A with non-trivial Gromov–Witten invariant $GW_{g,k}(X,A)$ is represented by a stable holomorphic map $f\colon C \to X$ whose image lies in the canonical divisor Z_α.*

This paper extends Theorem 1 to higher dimensions. The principle is the same: perturbing the Kähler structure to a non-integrable almost complex structure J_α forces the holomorphic maps to satisfy certain geometric conditions determined by α. This gives constraints on the Gromov–Witten invariants.

Specifically, let X be a compact Kähler manifold with a non-zero holomorphic 2-form α. Then the real part of α defines an endomorphism K_α of TX and an almost complex structure J_α, just as in the surface case (see (2.1) and (2.2)). These geometric structures lead, naturally and easily, to our main theorem.

Theorem 2 *Let X be a compact Kähler manifold with a non-zero holomorphic 2-form α. Then any class A with non-trivial Gromov–Witten invariant $GW_{g,k}(X,A)$ is represented by a stable holomorphic map $f\colon C \to X$ satisfying the equation $K_\alpha df = 0$.*

This theorem follows from Theorem 3.1 which is more suitable for applications. It generalizes Theorem 1 since, when X is a surface, the kernel of the endomorphism K_α is trivial on $X \setminus Z_\alpha$ (see Example 3.5). The equation $K_\alpha df = 0$ is a geometric fact about holomorphic maps that directly implies numerous vanishing results about Gromov–Witten invariants (see Section 3).

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Section 1 briefly describes Gromov–Witten invariants and states a vanishing principle for them. Section 2 contains the definition of the almost complex structures J_α and some of the consequences of that definition. In Section 3 we apply a stronger version of Theorem 2, which directly follows from properties of J_α, to show various vanishing results for Gromov–Witten invariants.

1 Gromov–Witten Invariants

The aim of this section is to give a brief description of the Gromov–Witten invariants and to set up notations for them. Let (X,ω) be a compact symplectic $2n$-dimensional manifold with an ω-tamed almost complex structure J, *i.e.,* $\omega(u, Ju) > 0$. A J-holomorphic map $f\colon (C, j) \to X$ from a (connected) marked nodal curve is *stable* if its automorphism group is finite (cf. [HZ]). Denote by $\overline{\mathcal{M}}_{g,k}(X, A, J)$ the moduli space of stable J-holomorphic maps from marked nodal curves of (arithmetic) genus g with k marked points that represent the homology class $A \in H_2(X)$. This moduli space carries a (virtual) fundamental homology class of real dimension

$$2[\, c_1(TX)\cdot A + (n-3)(1-g) + k] \tag{1.1}$$

(cf. [LT]) whose push-forward under the map $\mathrm{st}\times\mathrm{ev}\colon \overline{\mathcal{M}}_{g,k}(X,A,J) \to \overline{\mathcal{M}}_{g,k}\times X^k$ defined by stabilization and evaluation at the marked points is the Gromov–Witten invariant

$$GW_{g,k}(X,A) \in H_*(\overline{\mathcal{M}}_{g,k}\times X^k;\mathbb{Q}). \tag{1.2}$$

This is equivalent to the collection of "GW numbers" $GW_{g,k}(X,A)(\mu;\gamma_1,\dots,\gamma_k)$ obtained by evaluating the homology class (1.2) on the cohomology classes Poincaré dual to $\mu \in H_*(\overline{\mathcal{M}}_{g,k})$ and $\gamma_j \in H_*(X)$ whose total degree is the dimension (1.1). Standard cobordism arguments then show that these are independent of the choice of J, and depend only on the deformation class of the symplectic form ω.

Our subsequent discussions are based on the following vanishing principle for GW invariants.

Proposition 1.1 *Fix a compact symplectic manifold (X,ω). Suppose*

$$GW_{g,k}(X,A)(\mu;\gamma_1,\dots,\gamma_k) \neq 0.$$

Then, for any ω-tamed almost complex structure J and for any geometric representatives $M \subset \overline{\mathcal{M}}_{g,k}$ and $\Gamma_j \subset X$ of classes $\mu \in H_*(\overline{\mathcal{M}}_{g,k})$ and $\gamma_j \in H_*(X)$ there exists a stable J-holomorphic map $f\colon (C,x_1,\dots,x_k) \to X$ representing class A with $\mathrm{st}(C)\in M$ and $f(x_j)\in\Gamma_j$.

The proof is straightforward (cf. [LP]). For convenience, we will assemble all GW invariants for a class A into a single invariant by introducing a variable λ to keep track of the genus. The GW series of (X,ω) for a class A is then the formal power series

$$GW_A(X) = \sum_{g,k}\frac{1}{k!}GW_{g,k}(X,A)\lambda^g.$$

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2 The Almost Complex Structures J_α

Let (X, ω) be a compact symplectic manifold with an ω-compatible almost complex structure J, namely $\langle u, v\rangle = \omega(u, Jv)$ is a Riemannian metric. A 2-form α is then called *J-anti-invariant* if $\alpha(Ju, Jv) = -\alpha(u, v)$. As observed in [L], each J-anti-invariant 2-form α induces an almost complex structure

$$J_\alpha = (Id + JK_\alpha)^{-1} J(Id + JK_\alpha) \tag{2.1}$$

where K_α is an endomorphism of TX defined by the equation

$$\langle u, K_\alpha v\rangle = \alpha(u, v). \tag{2.2}$$

Such endomorphisms K_α are skew-adjoint and anti-commute with J. It follows that $Id + JK_\alpha$ is invertible and hence (2.1) is well-defined. A simple computation then shows that for any C^1 map $f\colon (C, j) \to X$,

$$\overline{\partial}_{J_\alpha} f = 0 \Longleftrightarrow \overline{\partial}_J f = K_\alpha \partial_J f j, \tag{2.3}$$

where

$$\overline{\partial}_J f = \tfrac{1}{2}(df + Jdf), \qquad \partial_J f = \tfrac{1}{2}(df - Jdfj).$$

Equation (2.3) implies that every J-holomorphic map f satisfying $K_\alpha df = 0$ is also J_α-holomorphic. One can also show that if f is J_α-holomorphic then

$$\int_C |\overline{\partial}_J f|^2 = \int_C |K_\alpha \partial_J f|^2 = \int_C f^*(\alpha) \tag{2.4}$$

(cf. [L]). This integral vanishes when α is closed and $\alpha(A) = 0$ where A is the class represented by f. In this case, the given J_α-holomorphic map f is J-holomorphic ($\overline{\partial}_J f = 0$) and satisfies $K_\alpha df = K_\alpha \partial_J f = 0$. Therefore, when α is closed and $\alpha(A) = 0$, a map f representing the class A is J_α-holomorphic if and only if f is J-holomorphic and satisfies the equation $K_\alpha df = 0$. Combined with Proposition 1.1, these observations lead to the following proposition.

Proposition 2.1 *Let (X, ω) be a compact symplectic manifold with an ω-compatible J and with a closed J-anti-invariant 2-form α. Then, for any class A with $GW_{g,k}(X, A) \neq 0$ we have $\overline{\mathcal{M}}_{g,k}(X, A, J_\alpha) = \{ f \in \overline{\mathcal{M}}_{g,k}(X, A, J) \mid K_\alpha df = 0\}$. Furthermore, this space is not empty.*

Proof By the above discussion, it suffices to show that $\alpha(A) = 0$ and $\overline{\mathcal{M}}_{g,k}(X, A, J_\alpha) \neq \varnothing$. Proposition 1.1 shows that there is a J-holomorphic map $h\colon (D, j) \to X$ representing the class A. Fix a point $p \in D$ and choose an orthonormal basis $\{e_1, e_2 = je_1\}$ of T_pD. Then, $h^*\alpha(e_1, e_2) = \alpha(h_*e_1, h_* je_1) = \alpha(h_*e_1, Jh_*e_1)$. Since α is J-anti-invariant, this vanishes and hence $\alpha(A) = \int_D h^*(\alpha) = 0$. On the other hand, for any sufficiently small $t > 0$ the almost complex structure $J_{t\alpha}$ is ω-tamed since ω-tamed is an open condition. Proposition 1.1 then asserts that there exists a $J_{t\alpha}$-holomorphic map f representing the class A. By (2.4) and the fact $K_{t\alpha} = tK_\alpha$, this map f is J-holomorphic and satisfies $K_\alpha df = 0$. Thus, f is also J_α-holomorphic by (2.3). ■

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Below, we will show some basic properties of the zero locus Z_α of α and $\ker K_\alpha$, which will be frequently used in subsequent arguments. One can use J to decompose $\Omega^2(X) \otimes \mathbb{C}$ as

$$\Omega^2(X) \otimes \mathbb{C} = \Omega_J^{2,0}(X) \oplus \Omega_J^{1,1}(X) \oplus \Omega_J^{0,2}(X).$$

Every J-anti-invariant 2-form α can then be written as $\alpha = \beta + \overline{\beta}$ for some $\beta \in \Omega_J^{2,0}(X)$. The next lemma simply follows from the definitions and the properties of K_α.

Lemma 2.2 *Let $\dim X = 2n$, and α and β be as above. Then,*

- (i) *α and β have the same zero locus,*
- (ii) *if n is odd then $\alpha^n = 0$, and if $n = 2m$ then $\alpha^n = c\beta^m \wedge \overline{\beta}^m$ where $c = \binom{n}{m}$,*
- (iii) *the (real) dimension of $\ker K_\alpha$ is at most $2n - 4$ at every point in $X \setminus Z_\alpha$,*
- (iv) *$u \in \ker K_\alpha$ if and only if $\alpha(u, w) = 0$ for any w. Thus, $\ker K_\alpha$ is trivial if and only if α is non-degenerate.*

A foliation $\mathcal{F}$ of dimension m on n-dimensional manifold M is a decomposition $\mathcal{F} = (L_i)_{i \in I}$ of M into pairwise disjoint connected subsets L_i, which are called leaves of the foliation $\mathcal{F}$, with the following property: for each $p \in M$ there exists a foliation chart $\varphi\colon U \to W_1 \times W_2$, where W_1 and W_2 are open disks in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively, such that for each point $q \in W_2$ the preimage $\varphi^{-1}(W_1 \times \{q\})$ is a connected component of $U \cap L_i$ for some leaf L_i. We refer to [CN] and [Ho] for more details on foliations.

Lemma 2.3 *Let (X, ω) be a six-dimensional symplectic manifold with ω-compatible J. If α is a closed J-anti-invariant 2-form, then $\ker K_\alpha$ gives a foliation on $X \setminus Z_\alpha$ of (real) dimension two whose leaves are all J-invariant.*

Proof Since K_α is anti-commutative with J, Lemma 2.2(ii),(iii) implies that on $X \setminus Z_\alpha$ the dimension of $\ker K_\alpha$ is two. On the other hand, $d\alpha(u, v, w) = 0$ gives

$$L_u(\alpha(v, w)) - L_v(\alpha(u, w)) + L_w(\alpha(u, v)) - \alpha([u, v], w) + \alpha([u, w], v) - \alpha([v, w], u) = 0,$$

where L denotes the Lie derivative. This and Lemma 2.2(iv) imply that if $u, v \in \ker K_\alpha$ then $[u, v] \in \ker K_\alpha$. Therefore, by Frobenius' Theorem $\ker K_\alpha$ gives a foliation on $X \setminus Z_\alpha$ of dimension two. Since K_α is anti-commute with J, every leaf is J-invariant. ∎

3 Vanishing Results

Let (X, J) be a compact Kähler manifold with a non-zero holomorphic 2-form α. By the Hodge Theorem α is closed and hence its real part $\mathrm{Re}(\alpha)$ is also closed. Moreover, the real 2-form $\mathrm{Re}(\alpha)$ is J-anti-invariant and its zero locus is Z_α by Lemma 2.2(i). Throughout this section, we will denote by K_α the endomorphism of TX defined by $\mathrm{Re}(\alpha)$ as in (2.2).

A holomorphic 2-form α is called *non-degenerate* if $\mathrm{Re}(\alpha)$ is non-degenerate, or equivalently $\ker K_\alpha$ is trivial. The next theorem directly follows from Proposition 1.1 and Proposition 2.1.

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Theorem 3.1 *Fix a compact Kähler manifold X with a non-zero holomorphic 2-form α. If for a non-zero class A*

$$GW_{g,k}(X,A)(\mu;\gamma_1,\dots,\gamma_k)\neq 0,$$

then for any geometric representatives $M\subset\overline{\mathcal{M}}_{g,k}$ and $\Gamma_j\subset X$ of classes $\mu\in H_*(\overline{\mathcal{M}}_{g,k})$ and $\gamma_j\in H_*(X)$ there exists a stable holomorphic map $f\colon(C,x_1,\dots,x_k)\to X$ representing the class A with $\mathrm{st}(C)\in M$ and $f(x_j)\in\Gamma_j$ and satisfying the equation $K_\alpha\, df=0$. Consequently, if α is non-degenerate on an open set $U\subset X$ then the image of f lies in $X\setminus U$.

Using this theorem, one can obtain various vanishing results about GW invariants.

Example 3.2 Given a compact hyperkähler manifold X of (complex) dimension $2m$, there exists a holomorphic symplectic 2-form α, *i.e.*, α^m is nowhere vanishing (cf. [BDL]). The 2-form α is non-degenerate on X and hence Theorem 3.1 implies that the series $GW_A(X)$ vanishes unless $A=0$.

Example 3.3 Let $X=E_1\times\cdots\times E_n$ where each E_i is an elliptic curve and $n\geq 2$. For $i\neq j$, denote by α_{ij} the pull-back 2-form $\pi_i^*(\lambda_i)\wedge\pi_j^*(\lambda_j)$ where $\pi_i\colon X\to E_i$ is the i-th projection and λ_i is a nowhere vanishing holomorphic 1-form on E_i. Now suppose $GW_A(X)\neq 0$. Theorem 3.1 then shows that there is a holomorphic map $f\colon C\to X$ representing the class A with $K_{\alpha_{ij}}df=0$. Since α_{ij} has no zeros and $\ker K_{\alpha_{ij}}$ consists of vectors tangent to fibers of the projection $\pi_i\times\pi_j\colon X\to E_i\times E_j$, we have $(\pi_i\times\pi_j)_*df=0$ for each $i\neq j$. This implies $A=0$. The same arguments also apply to show that when $X=X_1\times\cdots\times X_n$ where each X_i is a hyperkähler manifold or a complex torus of (complex) dimension at least two the series $GW_A(X)$ vanishes unless $A=0$.

Remark 3.4 There are well-known proofs for the above two examples (cf. [BL]). For instance, if X is a compact hyperkähler manifold, then every Kähler structure J in the twistor family is deformation equivalent to $-J$ through Kähler structures (cf. [BDL]). This directly implies $GW_A(X)=0$ unless $A=0$. The product formula of [B] for GW invariants applies to give the same vanishing results as in Example 3.3.

The following example appears in [LP].

Example 3.5 ([LP]) Let X be a Kähler surface with a non-zero holomorphic 2-form α. Then, α is non-degenerate on $X\setminus Z_\alpha$ by Lemma 2.2(iii),(iv). Note that since α is a section of the canonical bundle the zero locus Z_α is a support of a canonical divisor. Theorem 3.1 thus shows that for any non-zero class A and for any genus g,

$$GW_{g,k}(X,A)(\,\cdot\,;\gamma,\dots)=0,\tag{3.1}$$

where γ lies in $H_i(X)$ for $i=0,1$. On the other hand, if X is a minimal surface of general type, then every canonical divisor is connected (cf. [BHPV]). We further assume that the zero locus Z_α is a smooth (reduced) canonical divisor. Then, any holomorphic map f whose image lies in Z_α represents a (non-negative) multiple of the canonical class K. Therefore, Theorem 3.1 implies that the series $GW_A(X)$ vanishes unless $A=mK$ for some non-negative integer m.

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The following example extends both the vanishing result (3.1) and Theorem 1 of the introduction to Kähler manifolds of even complex dimension. It is an immediate consequence of Theorem 3.1.

Example 3.6 Fix a compact Kähler manifold X of complex dimension $2m$ with a holomorphic 2-form α. If α^m is not identically zero, then the zero locus Z_m of α^m, with multiplicities, is a canonical divisor of X and α is non-degenerate on $X \setminus Z_m$. Theorem 3.1 implies that:

(i) if $GW_{g,k}(X, A) \neq 0$ for a non-zero class A, then A is represented by a stable holomorphic map $f: C \to X$ whose image lies in the canonical divisor Z_m, and
(ii) for any non-zero class A and for any genus g we have $GW_{g,k}(X, A)(\,\cdot\,; \gamma, \dots) = 0$ where γ lies in $H_i(X)$ for $i = 0, 1$.

Compact Kähler Threefolds Let X be a compact Kähler threefold with a non-zero holomorphic 2-form α. It then follows from Lemma 2.3 that $\ker K_\alpha$ induces a foliation on $X \setminus Z_\alpha$ of (real) dimension two. We will denote this foliation by $\mathcal{F}_\alpha$.

Lemma 3.7 *Fix a compact Kähler threefold X with a non-zero holomorphic 2-form α. If a (non-constant) stable holomorphic map $f: C \to X$ satisfies the equation $K_\alpha df = 0$, then the image of each irreducible component of C either lies in Z_α or lies in one leaf of the foliation $\mathcal{F}_\alpha$ on $X \setminus Z_\alpha$ union finitely many points of Z_α.*

Consequently, if α has no zeros then the image of f lies in one leaf of the foliation $\mathcal{F}_\alpha$ on X.

Proof Collapse all irreducible components of C whose image is a point. The resulting map still has the same image $f(C)$, so we can assume that the image of each irreducible component is not a point. Fix an irreducible component C_i of C and suppose $f(C_i)$ is not contained in Z_α. Then the intersection $f(C_i) \cap Z_\alpha$ is finite since f is holomorphic and Z_α is an analytic subvariety. Denote by D_i the set of critical points of f in C_i. This set D_i is finite and hence $C_i \setminus (D_i \cup f^{-1}(Z_\alpha))$ is open and connected. Therefore, the equation $K_\alpha df = 0$ asserts that $f(C_i \setminus D_i) \setminus Z_\alpha \subset L_i$ for some leaf L_i of the foliation $\mathcal{F}_\alpha$ on $X \setminus Z_\alpha$. It then remains to show that for each $p \in D_i$ either $f(p) \in Z_\alpha$ or $f(p) \in L_i$. Suppose $f(p)$ does not lie in Z_α. Let (U, φ) be a foliation chart around $f(p)$, namely $U \subset X \setminus Z_\alpha$ is a neighborhood of $f(p)$ and $\varphi(U) = W_1 \times W_2 \subset \mathbb{R}^2 \times \mathbb{R}^4$, where W_1 and W_2 are open disks in $\mathbb{R}^2$ and $\mathbb{R}^4$ respectively, such that for each point $t \in W_2$ the pre-image $\varphi^{-1}(W_1 \times \{t\})$ is a connected component of $U \cap L_t$ for some leaf L_t of $\mathcal{F}_\alpha$. Then for any small neighborhood $V \subset C_i$ of p there exists a point $t_i \in W_2$ such that $\varphi \circ f(V \setminus \{p\}) \subset W_1 \times \{t_i\}$. Consequently, we have $\varphi \circ f(p) \in W_1 \times \{t_i\}$. Since the pre-image $\varphi^{-1}(W_1 \times \{t_i\})$ is a connected component of $U \cap L_i$, we have $f(p) \in L_i$. ∎

Example 3.8 Fix a surface of general type S with a holomorphic 2-form γ whose zero locus is a smooth canonical divisor D. Let $\pi: X = \mathbb{P}(TS) \to S$ be the projective bundle with a pull-back 2-form $\alpha = \pi^*\gamma$. The zero locus Z_α is then the ruled surface $\pi^{-1}(D) \to D$ and every leaf of the foliation $\mathcal{F}_\alpha$ on $X \setminus \pi^{-1}(D)$ is a fiber of $\pi: X \to S$. Thus, Theorem 3.1 and Lemma 3.7 together imply that $GW_A(X) = 0$ unless $A = aD_0 + bF$ for some integers a and b, where D_0 is the section class of the ruled surface $\pi^{-1}(D)$ and F is the fiber class of X.

Now, suppose X is a compact threefold with a holomorphic 2-form α without zeros. The foliation $\mathcal{F}_\alpha$ is then a foliation on the whole X. In fact, $\mathcal{F}_\alpha$ is a holomorphic foliation; in holomorphic local coordinates, the foliation $\mathcal{F}_\alpha$ is given locally by the holomorphic vector field

$$Y = f_{23}\frac{\partial}{\partial z_1} - f_{13}\frac{\partial}{\partial z_2} + f_{12}\frac{\partial}{\partial z_3},$$

where $\alpha = f_{12}\, dz_1 \wedge dz_2 + f_{13}\, dz_1 \wedge dz_3 + f_{23}\, dz_2 \wedge dz_3$. After a suitable change of local coordinates, we can write $Y = \frac{\partial}{\partial z_1}$. Such local coordinates give the required holomorphic foliation chart.

Corollary 3.9 *Let X be a compact Kähler threefold with a holomorphic 2-form α without zeros. Suppose X is not a $\mathbb{P}^1$-bundle over a K3 or an abelian surface. Then for any non-zero class A the invariant*

$$GW_{g,k}(X, A)(\,\cdot\,; \gamma, \dots) \tag{3.2}$$

vanishes if the genus g is 0 or if one constraint γ lies in $H_i(X)$ for $0 \le i \le 3$.

Proof Assume that for some $A \neq 0$ the invariant (3.2) is not zero with either $g = 0$ or $\gamma \in H_i(X)$ for $0 \le i \le 3$. We will show that X is a $\mathbb{P}^1$-bundle over a K3 or an abelian surface. Since every leaf of $\mathcal{F}_\alpha$ is a smooth connected holomorphic curve, by Theorem 3.1 and Lemma 3.7 there exists a stable holomorphic map $f : C \to X$ representing the non-zero class A with $f(C) = L$ for some leaf L of $\mathcal{F}_\alpha$. The leaf L is thus compact and $A = m[L]$ for some integer $m \ge 1$. If the genus g is 0 then obviously $L = \mathbb{P}^1$.

On the other hand, if γ lies in $H_i(X)$ for $0 \le i \le 3$ and the invariant (3.2) is non-zero, then the formal dimension (1.1) of the moduli space $\mathcal{M}_{g,0}(X, A)$ is strictly positive, so $c_1(X) \cdot A \ge 2$. But by Theorem 2 of [Le], the normal bundle N to L satisfies $c_1(N) = 0$, so

$$2 \le c_1(X)A = mc_1(X)[L] = m(c_1(L) + c_1(N))[L] = mc_1(L).$$

Hence $c_1(L) = 2$ and therefore $L = \mathbb{P}^1$ in this case also.

Now, by the proof of Corollary 2.8 of [H], the fact that one leaf L of $\mathcal{F}_\alpha$ is a rational curve $\mathbb{P}^1$ implies that every leaf of $\mathcal{F}_\alpha$ is rational. It follows that the leaf space $S = X/\mathcal{F}_\alpha$ is a (smooth) compact Kähler surface and the quotient map $\pi : X \to S$ is holomorphic. Consequently, X is a $\mathbb{P}^1$-bundle over S and α descends to a holomorphic 2-form γ on S with $\pi^*\gamma = \alpha$. Since the holomorphic 2-form γ has no zeros, $c_1(S) = 0$ and hence S is a K3 or an abelian surface (cf. [BHPV]). ■

Remark 3.10 Let X be a projective threefold with a holomorphic 2-form α. Suppose that for some non-zero class A the invariant $GW_{g,k}(X, A)(\,\cdot\,; \gamma, \dots)$ with the constraint $\gamma \in H_i(X)$ for $0 \le i \le 3$ does not vanish. Then the canonical divisor K_X is not nef by a dimension count. In this case, there is a contraction $\pi : X \to S$ of an extremal ray such that X is a $\mathbb{P}^1$-bundle over a surface S and α descends to a holomorphic 2-form γ on S with $\pi^*\gamma = \alpha$ (see Section 3 of [CP]). Consequently, if α has isolated zeros then S is a K3 or an abelian surface and, in fact, α has no zeros. This observation motivated Corollary 3.9.

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