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A NOTE ON THE RADIAL GROWTH OF BLOCH FUNCTIONS

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The radial growth of Bloch functions has been extensively studied. Using integral means estimates and the Hardy Littlewood theorem, Makarov proved the so called law of iterated logarithm for Bloch functions. This result has also been obtained using probabilistic arguments. In this paper we present another method of studying the radial growth of Bloch functions, having the integral means estimates as starting point and using certain results about normal functions.

1. INTRODUCTION

Let Δ denote the unit disc $\{z \in \mathbf{C} : |z| < 1\}$. For $0 < p < \infty$ and g analytic in Δ define

$$I_p(r, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|g\left(re^{i\theta}\right)\right|^p d\theta, \qquad 0 < r < 1.$$

A function f analytic in Δ is said to be a Bloch function if

$$\|f\|_{\mathbf{B}} = |f(0)| + \operatorname*{Sup}_{z\in\Delta} \left(1 - |z|^2\right) |f'(z)| < \infty.$$

The space of all Bloch functions will be denoted by $\mathbf{B}$.

Clunie and MacGregor in [4] and Makarov in [10] have proved the sharp estimate

$$I_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{p/2}\right), \qquad \text{as } r \to 1, \tag{1}$$

valid for $0 < p < \infty$ and $f \in \mathbf{B}$.

The radial growth of Bloch functions has been extensively studied recently. Clunie and MacGregor proved in [4] the following result.

THEOREM A. ([4, Theorem 3]). *Let f be a Bloch function and $\alpha > 1/2$; then for almost every $\theta \in (-\pi, \pi)$*

$$\left|f\left(re^{i\theta}\right)\right| = o\left(\left(\log \frac{1}{1-r}\right)^{\alpha}\right), \qquad \text{as } r \to 1. \tag{2}$$

The sharp result of this kind is the so-called law of iterated logarithm for Bloch functions proved by Makarov in [10].

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Theorem B. ([**10**, Theorem A]). *There exists an absolute constant* $C > 0$ *such that if* f *is a Bloch function then for almost every* $\theta \in (-\pi, \pi)$

$$\limsup_{r \to 1} \frac{|f(re^{i\theta})|}{\left(\log \frac{1}{1-r} \operatorname{logloglog} \frac{1}{1-r}\right)^{1/2}} \leqslant C \|f\|_{\mathbf{B}}. \tag{3}$$

A proof of Theorem B with $C = 1$ is presented by Pommerenke in [**12**] where he also proves that $C > 0.685$.

The proofs of Clunie and MacGregor, Makarov and Pommerenke are based on (1) and the Hardy-Littlewood theorem [**5**, pp.12, 235]. A similar argument was used by the author in [**6**] to obtain the analogue of Theorem B for certain subspaces of $\mathbf{B}$.

Subsequently a number of different proofs of Theorem B have been given. Let $f \in \mathbf{B}$ and set $f_r(z) = f(rz)$, $0 < r < 1$. Define

$$A_\alpha(f_r)(\theta) = \left(\iint_{\Gamma_\alpha(\theta)} |f_r'(z)|^2 \, dxdy \right)^{1/2}$$

where $0 < \alpha < 1$ and $\Gamma_\alpha(\theta)$ is the interior of the smallest convex set containing the disc $\{|z| < \alpha\}$ and $e^{i\theta}$.

$$g_*(f_r)(\theta) = \left(\frac{1}{\pi} \iint_\Delta \log \frac{1}{|z|} P_\theta(z) \, |f_r'(z)|^2 \, dxdy \right)^{1/2}$$

where $P_\theta(z) = \left(1 - |z|^2\right) / \left|e^{i\theta} - z\right|^2$ is the Poisson kernel. Then Bañuelos proved in [**1**] that

$$\|g_*(f_r)\|_\infty^2 \leqslant \frac{1}{2} \|f\|_{\mathbf{B}}^2 \log \frac{1}{1 - r^2}. \tag{4}$$

Since $A_\alpha(h) \leqslant C_\alpha g_*(h)$ almost everywhere, we have

$$\|A_\alpha(f_r)\|_\infty^2 \leqslant C_\alpha \|f\|_{\mathbf{B}}^2 \log \frac{1}{1 - r^2}. \tag{5}$$

Using (4) and probabilistic arguments, Bañuelos [**1**, Theorem 3] obtained a proof of Theorem B with $C = 1$. We remark that (3) can also be deduced from (5) and [**3**, Theorem 3.2]. Przytycki obtained in [**13**] another proof of Theorem B by approximating a sequence of trigonometric polynomials associated to a Bloch function by a martingale on $\partial\Delta$.

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We should also notice that Korenblum proved in [8] that if f is a Bloch function then

(6) $$\|f_r\|_{\mathrm{BMO}} = O\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \qquad \text{as } r \to 1,$$

and using this result and the John-Nirenberg theorem [7] he obtained a result which is stronger than Theorem A but weaker than Theorem B.

In this paper we will present a new method of studying the radial growth of Bloch functions having the integral means estimates (1) as starting point. However, we will not use the Hardy-Littlewood maximal theorem. We will use instead a result of Bagemihl and Seidel about limits of normal functions. This method will lead us to obtain a new proof of Theorem A but unfortunately it does not seem to lead to a proof of Makarov's theorem. Consequently, it is the proof and not the result which is interesting in this paper.

In Section 2 we will state some results about normal functions which will be needed in our proof of Theorem A which will be presented in Section 3.

2. Some results about normal functions

Recall [9] (see also [11, Chapter 9]) that a function g meromorphic in Δ is said to be a normal function if

(7) $$\operatorname*{Sup}_{z\in\Delta}\left(1-|z|^2\right)\frac{|g'(z)|}{1+|g(z)|^2} < \infty.$$

It is trivial that every Bloch function is a normal function. We will need the following stronger result.

Proposition 1. *Let f be a Bloch function and $0 \leqslant \lambda \leqslant 1$. Set*

$$g(z) = f(z)\left(\log\frac{1}{1-z} + i\pi\right)^{-\lambda}, \qquad z \in \Delta.$$

Then g is a normal function.

Proof: Notice that

(8) $$\left|\log\frac{1}{1-z} + i\pi\right| > 1, \qquad z \in \Delta.$$

Computing $g'(z)$ and using (8), we easily see that

(9) $$\left(1-|z|^2\right)\frac{|g'(z)|}{1+|g(z)|^2} \leqslant A(z) + B(z)$$

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where

$$A(z) = \frac{1-|z|^2}{1+|g(z)|^2}\,|f'(z)| \leqslant \left(1-|z|^2\right)|f'(z)| \leqslant \|f\|_{\mathcal{B}} \tag{10}$$

and
$$B(z) = \frac{1-|z|^2}{1+|g(z)|^2}\lambda\,|1-z|^{-1}\,|f(z)|\left|\log\frac{1}{1-z}+i\pi\right|^{-\lambda-1}.$$

Since
$$\lambda\frac{1-|z|^2}{|1-z|} \leqslant 2, \qquad z \in \Delta,$$

we obtain
$$B(z) \leqslant \frac{2}{1+|g(z)|^2}\,|f(z)|\left|\log\frac{1}{1-z}+i\pi\right|^{-\lambda-1}.$$

Now, if $|f(z)| \leqslant 1$, using (8), we obtain trivially $B(z) \leqslant 2$.

On the other hand, if $|f(z)| \geqslant 1$, we have

$$B(z) \leqslant 2\,|g(z)|^{-2}\,|f(z)|\left|\log\frac{1}{1-z}+i\pi\right|^{-\lambda-1} = 2\,|f(z)|^{-1}\left|\log\frac{1}{1-z}+i\pi\right|^{\lambda-1} \leqslant 2;$$

the last inequality follows from (8) and the fact that $\lambda \leqslant 1$. Hence, in any case we have $B(z) \leqslant 2$ which, with (9) and (10), shows that

$$\frac{1-|z|^2}{1+|g(z)|^2}\,|g'(z)| \leqslant 2+\|f\|_{\mathcal{B}}, \qquad z \in \Delta,$$

finishing the proof of Proposition 1. □

Bagemihl and Seidel studied in [2] the following question: Let f be a function meromorphic in Δ and $\{z_n\}$ be a sequence of points in Δ that converges to a point $\xi \in \partial\Delta$. Suppose that $f(z_n) \to c$, as $n \to \infty$. Under what conditions on f or on the sequence $\{z_n\}$ can it be inferred that $f(z) \to c$, as $z \to \xi$ in some continuous manner? Among other, they proved the following result.

Theorem C. ([2, Theorem 2]). *Let $\{z_n\}$ be a sequence of points in Δ which converges to a point $\xi \in \partial\Delta$ and is such that $\rho_n \to 0$ where*

$$\rho_n = \rho(z_n, z_{n+1}) = \frac{1}{2}\log\frac{|1-\bar{z}_n z_{n+1}| + |z_n - z_{n+1}|}{|1-\bar{z}_n z_{n+1}| - |z_n - z_{n+1}|}$$

denotes the hyperbolic distance from z_n to z_{n+1}. Let f be a meromorphic normal function in Δ such that $f(z_n) \to c$, as $n \to \infty$, where c is finite or infinite. Then f has the non-tangential limit c at ξ.

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3. Proof of Theorem A

Let f be a Bloch function and $\alpha > 1/2$. We may assume without loss of generality that $\alpha \leqslant 1$. Take and fix β such that

$$\frac{1}{2} < \beta < \alpha \leqslant 1 \tag{11}$$

and let p be defined by

$$p\left(\beta - \frac{1}{2}\right) = 4. \tag{12}$$

Set

$$r_n = 1 - \exp\left(-\sqrt{n}\right), \qquad n = 1, 2, 3, \ldots \tag{13}$$

and

$$E_n = \left\{\theta \in (-\pi, \pi) : \left|f\left(r_n e^{i\theta}\right)\right| > \left(\log \frac{1}{1 - r_n}\right)^{\beta}\right\}. \tag{14}$$

Using (1), we deduce that there exists $C = C(p, f) > 0$ such that

$$\int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right|^p \, d\theta \leqslant C\left(\log \frac{1}{1-r}\right)^{p/2}, \qquad r_1 < r < 1. \tag{15}$$

Then, with $|E_n|$ denoting the Lebesgue measure of E_n, (15), the definition of E_n, (12), and (13) show that

$$|E_n| \leqslant C\left(\log \frac{1}{1 - r_n}\right)^{-p(\beta - 1/2)} = Cn^{-2}, \qquad n = 1, 2, 3, \ldots .$$

Hence

$$\sum_{n=1}^{\infty} |E_n| \leqslant C \sum_{n=1}^{\infty} n^{-2} < \infty. \tag{16}$$

Let $E = \{\theta \in (-\pi, \pi) : \theta \text{ belongs to infinitely many of the sets } E_n\}$. Then

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

and hence, using (16), we see that

$$|E| = 0.$$

Let

$$F = (-\pi, \pi) - E. \tag{18}$$

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Notice that $|F| = 2\pi$.

We will prove the following result.

$$\text{(19)} \qquad \textit{Let } \theta \in F. \textit{ Then } f(z)\left(\log \frac{1}{1-|z|}\right)^{-\alpha} \to 0 \textit{ as } z \to e^{i\theta}, \textit{ non-tangentially.}$$

Since $|F| = 2\pi$, Theorem A follows from (19). Hence it only remains to prove (19).

Proof of (19): . We may assume without loss of generality that $\theta = 0$. Set

$$\text{(20)} \qquad g(z) = f(z)\left(\log \frac{1}{1-z} + i\pi\right)^{-\alpha}, \qquad z \in \Delta.$$

Since $1/2 < \alpha \leqslant 1$, Proposition 1 shows that g is a normal function.

It follows from the definition of F that there exists N such that

$$|f(r_n)| \leqslant \left(\log \frac{1}{1-r_n}\right)^{\beta}, \qquad n \geqslant N,$$

and hence, using (11) and (20), we deduce that

$$\text{(21)} \qquad g(r_n) \to 0, \qquad \text{as } n \to \infty.$$

Let $\rho_n = \rho(r_n, r_{n+1})$ denote the hyperbolic distance from r_n to r_{n+1}. It is a simple exercise to show that $\rho_n \to 0$, as $n \to \infty$. Then, using (21), Theorem C yields

$$g(z) \to 0, \text{ as } z \to 1, \text{ non-tangentially,}$$

and, consequently

$$f(z)\left(\log \frac{1}{1-|z|}\right)^{-\alpha} \to 0, \qquad \text{as } z \to 1, \text{ non-tangentially.}$$

This finishes the proof. □

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