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THE WARING PROBLEM FOR UPPER TRIANGULAR MATRIX ALGEBRAS

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Abstract. Our goal of the paper is to investigate the Waring problem for upper triangular matrix algebras, which gives a complete solution of a conjecture proposed by Panja and Prasad in 2023.

1. Introduction

The classical Waring problem proposed by Edward Waring in 1770 asserted that for every positive integer k there exists a positive integer $g(k)$ such that every positive integer can be expressed as a sum of $g(k)$ kth powers of nonnegative integers. In 1909, David Hilbert solved the problem. Various extensions and variations of this problem have been studied by different groups of mathematicians (see [2, 3, 4, 9, 10, 11, 14, 16, 18]).

In 2009 Shalev [18] proved that given a word $w \neq 1$, every element in any finite non-abelian simple group G of sufficiently high order can be written as the product of three elements from $w(G)$, the image of the word map induced by w. In 2011 Larsen, Shalev, and Tiep [14] proved that, under the same assumptions, every element in G is the product of two elements from $w(G)$, which gave a definitive solution of the Waring problem for finite simple groups.

Let $n \geq 2$ be an integer. Let K be a field and let $K\langle X\rangle$ be the free associative algebra over K, freely generated by the countable set $X = \{x_1, x_2, \ldots\}$ of noncommutative variables. We refer to the elements of $K\langle X\rangle$ as polynomials.

Let $p(x_1, \ldots, x_m) \in K\langle X\rangle$. Let $\mathcal{A}$ be an algebra over K. The set

$$p(\mathcal{A}) = \{p(a_1, \ldots, a_m) \mid a_1, \ldots, a_m \in \mathcal{A}\}$$

is called the image of p (on $\mathcal{A}$).

In 2020 Brešar [2] initiated the study of various Waring's problems for matrix algebras. He proved that if $\mathcal{A} = M_n(K)$, where $n \geq 2$ and K is an algebraically closed field with characteristic 0, and f is a noncommutative polynomial which is neither an identity nor a central polynomial of $\mathcal{A}$, then every trace zero matrix

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in $\mathcal{A}$ is a sum of four matrices from $f(\mathcal{A}) - f(\mathcal{A})$ [2, Corollary 3.19]. In 2023 Brešar and Šemrl [3] proved that any traceless matrix can be written as sum of two matrices from $f(M_n(\mathcal{C})) - f(M_n(\mathcal{C}))$, where $\mathcal{C}$ is the complex field and f is neither an identity nor a central polynomial for $M_n(\mathcal{C})$. Recently, they [4] have proved that if $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{C} \setminus \{0\}$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$, then any traceless matrix over $\mathcal{C}$ can be written as $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$, where $A_i \in f(M_n(\mathcal{C}))$.

By $T_n(K)$ we denote the set of all $n \times n$ upper triangular matrices over K. By $T_n(K)^{(0)}$ we denote the set of all $n \times n$ strictly upper triangular matrices over K. More generally, if $t \geq 0$, the set of all upper triangular matrices whose entries (i,j) are zero, for $j - i \leq t$, will be denoted by $T_n(K)^{(t)}$. It is easy to check that $J^t = T_n(K)^{(t-1)}$, where $t \geq 1$ and J is the Jacobson radical of $T_n(K)$ (see [1, Example 5.58]).

Let $p(x_1, \ldots, x_m)$ be a noncommutative polynomial with zero constant term over K. We define its **order** as the least positive integer r such that $p(T_r(K)) = \{0\}$ but $p(T_{r+1}(K)) \neq \{0\}$. Note that $T_1(K) = K$. We say that p has order 0 if $p(K) \neq \{0\}$. We denote the order of p by $\mathrm{ord}(p)$. For a detailed introduction of the order of polynomials we refer the reader to the book [7, Chapter 5].

In 2023 Panja and Prasad [16] discussed the image of polynomials with zero constant term and Waring type problems on upper triangular matrix algebras over an algebraically closed field, which generalized two results in [6, 19]. More precisely, they obtained the following main result:

Theorem 1.1. [16, Theorem 5.18] *Let $n \geq 2$ and $m \geq 1$ be integers. Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in non-commutative variables over an algebraically closed field K. Set $r =\mathrm{ord}(p)$. Then one of the following statements holds.*

- (i) *Suppose that $r = 0$. We have that $p(T_n(K))$ is a dense subset of $T_n(K)$ (with respect to the Zariski topology);*
- (ii) *Suppose that $r = 1$. We have that $p(T_n(K)) = T_n(K)^{(0)}$;*
- (iii) *Suppose that $1 < r < n - 1$. We have that $p(T_n(K)) \subseteq T_n(K)^{(r-1)}$, and equality might not hold in general. Furthermore, for every n and r there exists d such that each element of $T_n(K)^{(r-1)}$ can be written as a sum of d many elements from $p(T_n(K))$;*
- (iv) *Suppose that $r = n - 1$. We have that $p(T_n(K)) = T_n(K)^{(n-2)}$;*
- (v) *Suppose that $r \geq n$. We have that $p(T_n(K)) = \{0\}$.*

They proposed the following conjecture:

Conjecture 1.1. [16, Conjecture] *Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in non-commutative variables over an algebraically closed field K. Suppose ord$(p) = r$, where $1 < r < n-1$. Then $p(T_n(K)) + p(T_n(K)) = T_n(K)^{(r-1)}$.*

We note that if p is a multilinear polynomial and K is an infinite field, then $p(T_n(K)) = T_n(K)^{(r-1)}$ (see [8, 12, 15]).

In the present paper, we shall prove the following main result of the paper, which gives a complete solution of Conjecture 1.1.

Theorem 1.2. *Let $n \geq 2$ and $m \geq 1$ be integers. Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in non-commutative variables over an infinite field K. Suppose ord$(p) = r$, where $1 < r < n - 1$. We have that $p(T_n(K)) + p(T_n(K)) = T_n(K)^{(r-1)}$. Furthermore, if $r = n - 2$, we have that $p(T_n(K)) = T_n(K)^{(n-3)}$.*

We organize the paper as follows: In Section 2 we shall give some preliminaries. We shall modify some results in [5, 8, 13], which will be used in the proof of Theorem 1.2. In Section 3 we shall give the proof of Theorem 1.2 by using some new arguments (for example, compatible variables in polynomials and recursive polynomials).

2. PRELIMINARIES

Let $\mathcal{N}$ be the set of all positive integers. Let $m \in \mathcal{N}$. Let K be a field. Set $K^* = K \setminus \{0\}$. For any $k \in \mathcal{N}$ we set

$$T_m^k = \left\{(i_1, \ldots, i_k) \in \mathcal{N}^k \mid 1 \leq i_1, \ldots, i_k \leq m\right\}.$$

Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in non-commutative variables over K. We can write

$$p(x_1, \ldots, x_m) = \sum_{k=1}^{d} \left(\sum_{(i_1, i_2, \ldots, i_k) \in T_m^k} \lambda_{i_1 i_2 \cdots i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \right), \tag{1}$$

where $\lambda_{i_1 i_2 \cdots i_k} \in K$ and d is the degree of p.

We begin with the following result, which is slightly different from [5, Lemma 3.2]. We give its proof for completeness.

Lemma 2.1. *For any $u_i = (a_{jk}^{(i)}) \in T_n(K)$, $i = 1, \ldots, m$, we set*

$$\bar{a}_{jj} = (a_{jj}^{(1)}, \ldots, a_{jj}^{(m)}),$$

where $j = 1, \ldots, n$. We have that

$$p(u_1, \ldots, u_m) = \begin{pmatrix} p(\bar{a}_{11}) & p_{12} & \ldots & p_{1n} \\ 0 & p(\bar{a}_{22}) & \ldots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p(\bar{a}_{nn}) \end{pmatrix}, \tag{2}$$

where

$$p_{st} = \sum_{k=1}^{t-s} \left(\sum_{\substack{s=j_1<j_2<\cdots<j_{k+1}=t \\ (i_1,\ldots,i_k)\in T_m^k}} p_{i_1 \cdots i_k}(\bar{a}_{j_1 j_1}, \ldots, \bar{a}_{j_{k+1} j_{k+1}}) a_{j_1 j_2}^{(i_1)} \cdots a_{j_k j_{k+1}}^{(i_k)} \right)$$

for all $1 \leq s < t \leq n$, where $p_{i_1,\ldots,i_k}(z_1, \ldots, z_{m(k+1)})$, $1 \leq i_1, i_2, \ldots, i_k \leq m$, $k = 1, \ldots, n-1$, is a polynomial in commutative variables over K.

Proof. Let $u_i = (a_{jk}^{(i)}) \in T_n(K)$, where $i = 1, \ldots, m$. For any $1 \leq i_1, \ldots, i_k \leq m$, we easily check that

$$u_{i_1} \cdots u_{i_k} = \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ 0 & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_{nn} \end{pmatrix},$$

where

$$m_{st} = \sum_{s=j_1 \leq j_2 \leq \cdots \leq j_{k+1}=t} a_{j_1 j_2}^{(i_1)} \cdots a_{j_k j_{k+1}}^{(i_k)}$$

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for all $1 \le s \le t \le n$. It follows from (1) that

$$p(u_1, \ldots, u_m) = \sum_{k=1}^{d} \left(\sum_{(i_1,\ldots,i_k)\in T_m^k} \lambda_{i_1\cdots i_k} u_{i_1} \cdots u_{i_k} \right)$$
$$= \sum_{k=1}^{d} \left(\sum_{(i_1,\ldots,i_k)\in T_m^k} \lambda_{i_1\cdots i_k} \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ 0 & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_{nn} \end{pmatrix} \right)$$
$$= \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ 0 & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_{nn} \end{pmatrix}$$

where

$$p_{st} = \sum_{k=1}^{d} \left(\sum_{(i_1,\ldots,i_k)\in T_m^k} \lambda_{i_1\cdots i_k} m_{st} \right)$$
$$= \sum_{k=1}^{d} \left(\sum_{(i_1,\ldots,i_k)\in T_m^k} \lambda_{i_1\cdots i_k} \left(\sum_{s=j_1\le j_2\le\cdots\le j_{k+1}=t} a_{j_1j_2}^{(i_1)} \cdots a_{j_kj_{k+1}}^{(i_k)} \right) \right)$$
$$= \sum_{k=1}^{d} \left(\sum_{\substack{s=j_1\le j_2\le\cdots\le j_{k+1}=t \\ (i_1,\ldots,i_k)\in T_m^k}} \lambda_{i_1 i_2\cdots i_k} a_{j_1j_2}^{(i_1)} \cdots a_{j_kj_{k+1}}^{(i_k)} \right),$$

where $1 \le s \le t \le n$. In particular

$$p_{ss} = \sum_{k=1}^{d} \left(\sum_{(i_1,\ldots,i_k)\in T_m^k} \lambda_{i_1 i_2\cdots i_k} a_{ss}^{(i_1)} \cdots a_{ss}^{(i_k)} \right)$$
$$= p(\bar{a}_{ss})$$

for all $s = 1, \ldots, n$, and

$$p_{st} = \sum_{k=1}^{d} \left(\sum_{\substack{s=j_1\le j_2\le\cdots\le j_{k+1}=t \\ (i_1,\ldots,i_k)\in T_m^k}} \lambda_{i_1 i_2\cdots i_k} a_{j_1j_2}^{(i_1)} \cdots a_{j_kj_{k+1}}^{(i_k)} \right)$$
$$= \sum_{k=1}^{t-s} \left(\sum_{\substack{s=j_1< j_2<\cdots< j_{k+1}=t \\ (i_1,\ldots,i_k)\in T_m^k}} p_{i_1 i_2\cdots i_k}(\bar{a}_{j_1j_1}, \ldots, \bar{a}_{j_{k+1}j_{k+1}}) a_{j_1j_2}^{(i_1)} \cdots a_{j_kj_{k+1}}^{(i_k)} \right)$$

for all $1 \le s < t \le n$, where $p_{i_1,\ldots,i_k}(z_1, \ldots, z_{m(k+1)})$ is a polynomial in commutative variables over K. This proves the result. □

The following result will be used in the proof of our main result.

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Lemma 2.2. *Let $m \geq 1$ be an integer. Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in non-commutative variables over K. Let $p_{i_1,\ldots,i_k}(z_1, \ldots, z_{m(k+1)})$ be a polynomial in commutative variables over K in (2), where $1 \leq i_1, \ldots, i_k \leq m$, $1 \leq k \leq n-1$. Suppose that* $\operatorname{ord}(p) = r$, *$1 < r < n-1$. We have that*

(i) $p(K) = \{0\}$;
(ii) $p_{i_1,\ldots,i_k}(K) = \{0\}$ *for all* $1 \leq i_1, \ldots, i_k \leq m$, *where* $k = 1, \ldots, r-1$;
(iii) $p_{i'_1,\ldots,i'_r}(K) \neq \{0\}$ *for some* $1 \leq i'_1, \ldots, i'_r \leq m$.

Proof. The statement (i) is clear. We now claim that the statement (ii) holds true. Suppose on the contrary that

$$p_{i'_1\cdots i'_s}(K) \neq \{0\}$$

for some $1 \leq i'_1, \ldots, i'_s \leq m$, where $1 \leq s \leq r-1$. Then there exist $\bar{b}_j \in K^m$, where $j = 1, \ldots, s+1$ such that

$$p_{i'_1\cdots i'_s}(\bar{b}_1, \ldots, \bar{b}_{s+1}) \neq 0.$$

We take $u_i = (a^{(i)}_{jk}) \in T_{s+1}(K)$, $i = 1, \ldots, m$, where

$$\begin{cases} \bar{a}_{jj} = \bar{b}_j, & j = 1, \ldots, s+1; \\ a^{(i'_k)}_{k,k+1} = 1, & k = 1, \ldots, s; \\ a^{(i)}_{jk} = 0, & \text{otherwise}. \end{cases}$$

It follows from (2) that

$$p_{1,s+1} = p_{i'_1\cdots i'_s}(\bar{b}_1, \ldots, \bar{b}_{s+1}) \neq 0.$$

This implies that $p(T_{s+1}(K)) \neq \{0\}$, a contradiction. This proves the statement (ii).

We finally claim that the statement (iii) holds true. Note that $p(T_{1+r}(K)) \neq \{0\}$. Thus, we have that there exist $u_i = (a^{(i)}_{jk}) \in T_{1+r}(K)$, $i = 1, \ldots, m$, such that

$$p(u_1, \ldots, u_m) = (p_{st}) \neq 0.$$

In view of the statement (ii) we get that

$$p_{1,r+1} = \sum_{\substack{1=j_1<j_2<\cdots<j_{r+1}=r+1 \\ (i_1,\ldots,i_r)\in T^r_m}} p_{i_1 i_2\cdots i_r}(\bar{a}_{j_1 j_1}, \ldots, \bar{a}_{j_{r+1} j_{r+1}}) a^{(i_1)}_{j_1 j_2} \cdots a^{(i_r)}_{j_r j_{r+1}} \neq 0.$$

This implies that $p_{i'_1,\ldots,i'_r}(K) \neq \{0\}$ for some $1 \leq i'_1, \ldots, i'_r \leq m$. This proves the statement (iii). The proof of the result is complete. □

The following well-known result will be used in the proof of the rest results.

Lemma 2.3. [13, Theorem 2.19] *Let K be an infinite field. Let $f(x_1, \ldots, x_m)$ be a nonzero polynomial in commutative variables over K. Then there exist $a_1, \ldots, a_m \in K$ such that $f(a_1, \ldots, a_m) \neq 0$.*

Lemma 2.4. *Let n, s be integers with $1 \leq s \leq n$. Let $p(x_1, \ldots, x_s)$ be a nonzero polynomial in commutative variables over an infinite field K. We have that there exist $a_1, \ldots, a_n \in K$ such that*

$$p(a_{i_1}, \ldots, a_{i_s}) \neq 0$$

for all $1 \leq i_1 < \cdots < i_s \leq n$.

Proof. We set
$$f(x_1,\ldots,x_n)=\prod_{1\le i_1<\cdots<i_s\le n} p(x_{i_1},\ldots,x_{i_s}).$$
It is clear that $f\neq 0$. In view of Lemma 2.3 we have that there exist $a_1,\ldots,a_n\in K$ such that
$$f(a_1,\ldots,a_n)\neq 0.$$
This implies that
$$p(a_{i_1},\ldots,a_{i_s})\neq 0$$
for all $1\le i_1<\cdots<i_s\le n$. This proves the result. □

The following technical result is a generalized form of [8, Lemma 2.11], which discusses compatible variables in polynomials.

Lemma 2.5. *Let $t\ge 1$. Let $U_i=\{i_1,\ldots,i_s\}\subseteq\mathcal{N}$, $i=1,\ldots,t$. Let $p_i(x_{i_1},\ldots,x_{i_s})$ be a nonzero polynomial in commutative variables over an infinite field K, where $i=1,\ldots,t$. Then there exist $a_k\in K$ with $k\in\bigcup_{i=1}^t U_i$ such that*
$$p_i(a_{i_1},\ldots,a_{i_s})\neq 0$$
for all $i=1,\ldots,t$.

Proof. Without loss of generality we assume that
$$\{1,2,\ldots,n\}=\bigcup_{i=1}^t U_i.$$
We set
$$f(x_1,\ldots,x_n)=\prod_{i=1}^t p_i(x_{i_1},\ldots,x_{i_s}).$$
It is clear that $f\neq 0$. In view of Lemma 2.3 we have that there exist $a_1,\ldots,a_n\in K$ such that
$$f(a_1,\ldots,a_n)\neq 0.$$
This implies that
$$p_i(a_{i_1},\ldots,a_{i_s})\neq 0$$
for all $i=1,\ldots,t$. This proves the result. □

The following technical result will be used in the proof of the main result of the paper.

Lemma 2.6. *Let $s\ge 1$ and $t\ge 2$ be integers. Let K be an infinite field. Let $a_{ij}\in K$, where $1\le i\le t$, $1\le j\le s$ with $a_{11}\in K^*$ and $b\in K^*$. For any $2\le i\le t$, there exists a nonzero element in $\{a_{i1},\ldots,a_{is}\}$. Then there exist $c_i\in K$, $i=1,\ldots,s$, such that*
$$\begin{cases} a_{11}c_1+\cdots+a_{1s}c_s=b;\\ a_{i1}c_1+\cdots+a_{is}c_s\neq 0\end{cases}$$
for all $i=2,\ldots,t$.

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Proof. Suppose first that $s = 1$. Note that $a_{i1} \in K^*$, $i = 1, \ldots, t$. Take $c_1 = a_{11}^{-1} b$. It is clear

$$\begin{cases} a_{11} c_1 = b; \\ a_{i1} c_1 \neq 0 \end{cases}$$

for all $2 \leq i \leq t$. Suppose next that $s \geq 2$. Suppose first that $a_{i1} \neq 0$ for all $i = 2, \ldots, t$. We define the following polynomials.

$$\begin{cases} f_1(x_2, \ldots, x_s) = b - a_{12}x_2 - \cdots - a_{1s}x_s; \\ f_i(x_2, \ldots, x_s) = a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})x_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})x_s \end{cases}$$

for all $2 \leq i \leq t$. Since $b, a_{i1} \in K^*$, $i = 1, \ldots, t$, we note that $f_i \neq 0$ for all $i = 1, \ldots, t$. In view of Lemma 2.5 we get that there exist $c_2, \ldots, c_s \in K$ such that

$$f_i(c_2, \ldots, c_s) \neq 0$$

for all $i = 1, \ldots, t$. This implies that

$$\begin{cases} b - a_{12}c_2 - \cdots - a_{1s}c_s \neq 0; \\ a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})c_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})c_s \neq 0 \end{cases} \tag{3}$$

for all $2 \leq i \leq t$. We set

$$c_1 = a_{11}^{-1}(b - a_{12}c_2 - \cdots - a_{1s}c_s).$$

It follows from (3) that

$$\begin{cases} a_{11}c_1 + \cdots + a_{1s}c_s = b; \\ a_{i1}c_1 + \cdots + a_{is}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t$, as desired.

Suppose next that $a_{i1} = 0$, $i = 2, \ldots, t$. Note that $a_{il(i)} \neq 0$, for some $2 \leq l(i) \leq s$ for all $i = 2, \ldots, t$. We define the following polynomials:

$$\begin{cases} f_1(x_2, \ldots, x_s) = a_{12}x_2 + \cdots + a_{1s}x_s - b; \\ f_i(x_2, \ldots, x_s) = a_{i2}x_2 + \cdots + a_{is}x_s \end{cases}$$

for all $2 \leq i \leq t$. Note that $f_i \neq 0$ for all $i = 1, \ldots, t$. In view of Lemma 2.5 we get that there exist $c_i \in K$, $i = 2, \ldots, s$, such that

$$f_i(c_2, \ldots, c_s) \neq 0$$

for all $i = 1, \ldots, t$. That is

$$\begin{cases} a_{12}c_2 + \cdots + a_{1s}c_s - b \neq 0; \\ a_{i2}c_2 + \cdots + a_{is}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t$. Since $a_{11} \neq 0$ we get that there exists $c_1 \in K$ such that

$$a_{11}c_1 = b - a_{12}c_2 - \cdots - a_{1s}c_s.$$

This implies that

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1s}c_s = b; \\ a_{i2}c_2 + \cdots + a_{is}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t$, as desired.

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We finally assume that there exist $a_{i1} \neq 0$ and $a_{j1} = 0$ for some $i, j \in \{2, \ldots, t\}$. Without loss of generality we assume that $a_{i1} \neq 0$ for all $i = 2, \ldots, t_1$ and $a_{i1} = 0$ for all $i = t_1 + 1, \ldots, t$. We define the following polynomials:

$$\begin{cases} f_1(x_2, \ldots, x_s) = b - a_{12}x_2 - \cdots - a_{1s}x_s; \\ f_i(x_2, \ldots, x_s) = a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})x_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})x_s; \\ f_j(x_2, \ldots, x_s) = a_{j2}x_2 + \cdots + a_{js}x_s \end{cases}$$

for all $2 \leq i \leq t_1$ and $t_1 + 1 \leq j \leq t$. Note that $b, a_{i1} \in K^*$, $i = 1, \ldots, t_1$, $a_{jl(j)} \neq 0$ where $2 \leq l(j) \leq s$ for all $j = t_1 + 1, \ldots t$. It is clear that $f_i \neq 0$ for all $i = 1, \ldots, t$. In view of Lemma 2.5 we get that there exist $c_i \in K$, $i = 2, \ldots, s$, such that

$$f_i(c_2, \ldots, c_s) \neq 0,$$

where $i = 1, \ldots, t$. This implies that

$$\begin{cases} b - a_{12}c_2 - \cdots - a_{1s}c_s \neq 0; \\ a_{i1}a_{11}^{-1}b + (a_{i2} - a_{i1}a_{11}^{-1}a_{12})c_2 + \cdots + (a_{is} - a_{i1}a_{11}^{-1}a_{1s})c_s \neq 0; \\ a_{j2}c_2 + \cdots + a_{js}c_s \neq 0 \end{cases} \tag{4}$$

for all $2 \leq i \leq t_1$ and $t_1 + 1 \leq j \leq t$. We set

$$c_1 = a_{11}^{-1}(b - a_{12}c_2 - \cdots - a_{1s}c_s).$$

It follows from (4) that

$$\begin{cases} a_{11}c_1 + \cdots + a_{1s}c_s = b; \\ a_{i1}c_1 + \cdots + a_{is}c_s \neq 0; \\ a_{j1}c_2 + \cdots + a_{js}c_s \neq 0 \end{cases}$$

for all $2 \leq i \leq t_1$ and $t_1 + 1 \leq j \leq t$, as desired. The proof of the result is now complete. □

3. The proof of Theorem 1.2

Let $n \geq 2$ and $m \geq 1$ be integers. Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in non-commutative variables over an infinite field K. Suppose that $1 < r < n - 1$, where $r = ord(p)$.

Take any $u_i = (a_{jk}^{(i)}) \in T_n(K)$, $i = 1, \ldots, m$. In view of both Lemma 2.1 and Lemma 2.2 we have that

$$p(u_1, \ldots, u_m) = (p_{s,r+s+t}) \tag{5}$$

where

$$p_{s,r+s+t} = \sum_{k=r}^{r+t} \left(\sum_{\substack{s=j_1<\cdots<j_{k+1}=r+s+t \\ (i_1,\ldots,i_k)\in T_m^k}} p_{i_1\cdots i_k}(\bar{a}_{j_1j_1}, \ldots, \bar{a}_{j_{k+1}j_{k+1}})a_{j_1j_2}^{(i_1)} \cdots a_{j_kj_{k+1}}^{(i_k)} \right)$$

for all $1 \leq s < r + s + t \leq n$ and

$$p_{i'_1\cdots i'_r}(K) \neq \{0\}$$

for some $1 \leq i'_1, \ldots, i'_r \leq m$. It follows from Lemma 2.4 that there exist $\bar{c}_1, \ldots, \bar{c}_n \in K^m$ such that

$$p_{i'_1\cdots i'_r}(\bar{c}_{j_1}, \ldots, \bar{c}_{j_{r+1}}) \neq 0 \tag{6}$$

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for all $1 \leq j_1 < \ldots < j_{r+1} \leq n$. We set

$$\begin{cases} \bar{a}_{jj} = \bar{c}_j, \quad j = 1, \ldots, n; \\ a^{(k)}_{i,i+1} = a^{(k)}_{i,i+1}, \quad i = 1, \ldots, r-1 \text{ and } k = 1, \ldots, m; \\ a^{(i'_k)}_{r+s-1,r+s+t} = x^{(i'_k)}_{r+s-1,r+s+t}, \quad 1 \leq s < r+s+t \leq n,\ k = 1, \ldots, r; \\ a^{(k)}_{ij} = 0, \quad \text{otherwise.} \end{cases}$$

For any $1 \leq s < r+s+t \leq n$, we set

$$U_{s,r+s+t} = \left\{ (r+u-1, r+u+w, i'_k) \mid x^{(i'_k)}_{r+u-1,r+u+w} \quad \text{in } p_{s,r+s+t} \right\}$$

and

$$\overline{U}_{s,r+s+t} = \{(r+u-1, r+u, i'_k) \mid (r+u-1, r+u, i'_k) \in U_{s,r+s+t}\}.$$

We define an order on the set

$$\{(s, r+s+t) \mid 1 \leq s < r+s+t \leq n\}$$

as follows:

(i) $(s, r+s+t) < (s_1, r+s_1+t_1)$ if $t < t_1$;
(ii) $(s, r+s+t) < (s_1, r+s_1+t_1)$ if $t = t_1$ and $s < s_1$.

That is

$$(1, r+1) < \cdots < (n-r, n) < (1, r+2) < \cdots < (n-r-1, n) < \cdots < (1, n). \quad (7)$$

For any $1 \leq s < r+s+t \leq n$, we set

$$W_{s,r+s+t} = \bigcup_{(1,r+1) \leq (i,r+i+j) \leq (s,r+s+t)} U_{i,r+i+j},$$

and

$$\overline{W}_{s,r+s+t} = \bigcup_{(1,r+1) \leq (i,r+i+j) \leq (s,r+s+t)} \overline{U}_{i,r+i+j}.$$

We begin with the following lemmas, which will be used in the proof of our main result.

Lemma 3.1. *Let $1 \leq s < r+s \leq n$. Suppose that $(s, r+s) \neq (1, r+1)$. We claim that*

$$\overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \leq k \leq r\} = \overline{W}_{s-1,r+s-1}. \quad (8)$$

Proof. We first claim that

$$\overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \leq k \leq r\} \subseteq \overline{W}_{s-1,r+s-1}.$$

Take any $(r+i-1, r+i, i'_k) \in \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \leq k \leq r\}$. We have that

$$(r+i-1, r+i, i'_k) \in \overline{U}_{s_2,r+s_2}$$

for some $(1, r+1) \leq (s_2, r+s_2) \leq (s, r+s)$. This implies that

$$r+i \leq r+s_2 \leq r+s.$$

We get that $i \leq s$. Suppose that $i = s$. It follows that

$$(r+i-1, r+i, i'_k) \in \{(r+s-1, r+s, i'_k) \mid 1 \leq k \leq r\},$$

a contradiction. Hence $i \leq s-1$. It is clear that

$$(r+i-1, r+i, i'_k) \in \overline{U}_{i,r+i},$$

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where $(1, r+1) \le (i, r+i) \le (s-1, r+s-1)$. It follows that

$$(r+i-1, r+i, i'_k) \in \overline{W}_{s-1,r+s-1}.$$

We obtain that

$$\overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\} \subseteq \overline{W}_{s-1,r+s-1},$$

as desired. We next claim that

$$\overline{W}_{s-1,r+s-1} \subseteq \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\}.$$

If $(r+s-1, r+s, i'_k) \in \overline{W}_{s-1,r+s-1}$ for $1 \le k \le r$, we have that

$$r+s \le r+s-1,$$

a contradiction. Hence

$$\{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\} \bigcap \overline{W}_{s-1,r+s-1} = \emptyset.$$

Since $\overline{W}_{s-1,r+s-1} \subseteq \overline{W}_{s,r+s}$ we get that

$$\overline{W}_{s-1,r+s-1} \subseteq \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\},$$

as desired. We obtain that

$$\overline{W}_{s-1,r+s-1} = \overline{W}_{s,r+s} \setminus \{(r+s-1, r+s, i'_k) \mid 1 \le k \le r\}.$$

This proves the result. □

Lemma 3.2. *Let* $1 \le s < r+s+t \le n$. *Suppose that* $t > 0$. *We claim that*

$$\overline{W}_{s_1,r+s_1+t_1} = \overline{W}_{s,r+s+t},$$

where

$$(s_1, r+s_1+t_1) = max\{(i, r+i+j) \mid (1, r+1) \le (i, r+i+j) < (s, r+s+t)\}.$$

Proof. We first claim that

$$\overline{W}_{s,r+s+t} = \overline{W}_{n-r,n}.$$

Since $t > 0$, we note that

$$(s, r+s+t) > (n-r, n).$$

This implies that $\overline{W}_{s,r+s+t} \supseteq \overline{W}_{n-r,n}$. Take any $(r+u-1, r+u, i'_k) \in \overline{W}_{s,r+s+t}$. It is clear that

$$(r+u-1, r+u, i'_k) \in \overline{U}_{u,r+u} \subseteq \overline{W}_{n-r,n}.$$

This implies that $\overline{W}_{s,r+s+t} \subseteq \overline{W}_{n-r,n}$. Hence, $\overline{W}_{s,r+s+t} = \overline{W}_{n-r,n}$ as desired.

Since $(n-r, n) < (s, r+s+t)$ we get that

$$(n-r, n) \le (s_1, r+s_1+t_1) < (s, r+s+t).$$

This implies that

$$\overline{W}_{n-r,n} \subseteq \overline{W}_{s_1,r+s_1+t_1} \subseteq \overline{W}_{s,r+s+t}.$$

Since $\overline{W}_{s,r+s+t} = \overline{W}_{n-r,n}$ we obtain that $\overline{W}_{s_1,r+s_1+t_1} = \overline{W}_{s,r+s+t}$. This proves the result. □

The following technical result will be used in the proof of the next result.

Lemma 3.3. *Let* $1 \le s < r+s+t \le n$. *If* $(r+i-1, r+i+j, i'_k) \in U_{s,r+s+t}$, *we have that* $j \le t$.

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Proof. Suppose that $(r+i-1, r+i+j, i'_k) \in U_{s,r+s+t}$. That is, $x^{(i'_k)}_{r+i-1,r+i+j}$ appears in $p_{s,r+s+t}$. In view of (5) we note that every monomial in $p_{s,r+s+t}$ is made up of at least r elements multiplied together. This implies that

$$((r+s+t)-s)-((r+i+j)-(r+i-1)) \geq r-1.$$

We obtain that $j \leq t$. This proves the result. □

Lemma 3.4. *Let $1 \leq s < r+s+t \leq n$ and $t > 0$. We claim that*

$$W_{s_1,r+s_1+t_1} = W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\},$$

where

$$(s_1, r+s_1+t_1) = max\{(i, r+i+j) \mid (1, r+1) \leq (i, r+i+j) < (s, r+s+t)\}.$$

Proof. We first claim that

$$W_{s_1,r+s_1+t_1} \subseteq W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\}.$$

If $(r+s-1, r+s+t, i'_k) \in W_{s_1,r+s_1+t_1}$ for some $1 \leq k \leq r$, we get that

$$(r+s-1, r+s+t, i'_k) \in U_{s_2,r+s_2+t_2} \tag{9}$$

for some $(1, r+1) \leq (s_2, r+s_2+t_2) \leq (s_1, r+s_1+t_1)$. It is clear that

$$t_2 \leq t_1 \leq t.$$

In view of Lemma 3.3 we get that $t \leq t_2$. It follows that

$$t_1 = t_2 = t.$$

Since $(s_1, r+s_1+t_1) < (s, r+s+t)$ we get that $s_1 < s$. Since $(s_2, r+s_2+t_2) \leq (s_1, r+s_1+t_1)$ we get that $s_2 \leq s_1$. Thus, we obtain that $s_2 < s$. It follows from (9) that

$$r+s+t \leq r+s_2+t_2.$$

This implies that $s \leq s_2$, a contradiction. Hence, we have that

$$(r+s-1, r+s+t, i'_k) \notin W_{s_1,r+s_1+t_1}$$

for all $1 \leq k \leq r$. It is clear that $W_{s_1,r+s_1+t_1} \subseteq W_{s,r+s+t}$. We obtain that

$$W_{s_1,r+s_1+t_1} \subseteq W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\},$$

as desired. We next claim that

$$W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\} \subseteq W_{s_1,r+s_1+t_1}.$$

For any $(r+i-1, r+i+j, i'_k) \in W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\}$, we have

$$(r+i-1, r+i+j, i'_k) \in U_{s_2,r+s_2+t_2}$$

for some $(1, r+1) \leq (s_2, r+s_2+t_2) \leq (s, r+s+t)$. This implies that $t_2 \leq t$. In view of Lemma 3.3 we note that $j \leq t_2$. We have that $j \leq t$. It is clear that

$$(r+i-1, r+i+j, i'_k) \in U_{i,r+i+j}$$

where $(1, r+1) \leq (i, r+i+j) \leq (s, r+s+t)$. Note that

$$(r+i-1, r+i+j, i'_k) \notin \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\}.$$

We get that

$$(i, r+i+j) \neq (s, r+s+t).$$

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This implies that

$$(1, r+1) \leq (i, r+i+j) \leq (s_1, r+s_1+t_1) \leq (s, r+s+t).$$

It follows that $U_{i,r+i+j} \subseteq W_{s_1,r+s_1+t_1}$. We have that

$$(r+i-1, r+i+j, i'_k) \in W_{s_1,r+s_1+t_1}.$$

We obtain that

$$W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\} \subseteq W_{s_1,r+s_1+t_1},$$

as desired. Thus, we obtain that

$$W_{s_1,r+s_1+t_1} = W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k) \mid 1 \leq k \leq r\}.$$

This proves the result. □

We set

$$\hat{c}_{s,t} = (\bar{c}_s, \bar{c}_{s+1}, \ldots, \bar{c}_{r+s-1}, \bar{c}_{r+s+t}).$$

It follows from (6) that

$$p_{i'_1 \cdots i'_r}(\hat{c}_{s,t}) \neq 0. \tag{10}$$

For any $1 \leq s < r+s \leq n$ and $s \leq r-1$, we set

$$f_{s,r} = \sum_{(i_1,\ldots,i_{r-s}) \in T_m^{r-s}} p_{i_1 \cdots i_{r-s} i'_{r-s+1} \cdots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \cdots a_{r-1,r}^{(i_{r-s})}.$$

We set

$$V_{s,r} = \{(i, i+1, k) \mid i = s, \ldots, r-1, \quad k = 1, \ldots, m\},$$

where $1 \leq s < r+s \leq n$ and $s \leq r-1$. It is clear that $f_{s,r}$ is a polynomial on commutative variables indexed by elements from $V_{s,r}$.

For any $1 \leq s < r+s \leq n$ and $s \geq r$, we set

$$f_{s,r} = p_{i'_1 \cdots i'_r}(\hat{c}_{s,t}).$$

We claim that $f_{s,r}(K) \neq \{0\}$ for all $1 \leq s < r+s \leq n$. In view of (10), it suffices to prove that $f_{s,r}(K) \neq 0$, where $1 \leq s < r+s \leq n$ and $s \leq r-1$.

We take $a_{i,i+1}^{(k)} \in K$, $(i, i+1, k) \in V_{s,r}$ such that

$$\begin{cases} a_{s+i,s+i+1}^{(i'_{i+1})} = 1 & i = 0, \ldots, r-s-1; \\ a_{i,i+1}^{(k)} = 0 & \text{otherwise.} \end{cases}$$

It follows from (10) that

$$f_{s,r}(a_{i,i+1}^{(k)}) = p_{i'_1 \cdots i'_r}(\hat{c}_{s,t}) \neq 0,$$

as desired. In view of Lemma 2.5 we get that there exist $a_{i,i+1}^{(k)} \in K$, $(i, i+1, k) \in \bigcup_{s=1}^{min\{n-r,r-1\}} V_{s,r}$ such that

$$f_{s,r}(a_{i,i+1}^{(k)}) \neq 0$$

for all $1 \leq s < r+s \leq n$ and $s \leq r-1$.

For any $2 \leq s \leq r+s \leq n$, we define

$$f_{s,r+s-i} = \sum_{(i_1,\ldots,i_{r-i}) \in T_m^{r-i}} p_{i_1 \cdots i_{r-i} i'_{r-i+1} \cdots i'_r}(\hat{c}_{s,t}) a_{s,s+1}^{(i_1)} \cdots a_{r+s-i-1,r+s-i}^{(i_{r-i})} \tag{11}$$

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for all $1 \le i \le min\{s-1, r-1\}$. It is clear that $f_{s,r+s-i}$ is a polynomial over K on commutative variables indexed by elements from $\overline{W}_{s-i,r+s-i}$, where $1 \le i \le min\{s-1, r-1\}$.

The following result implies that $f_{s,r+s-i}$, where $1 \le i \le min\{s-1, r-1\}$, is a recursive polynomial.

Lemma 3.5. *For any $2 \le s < r+s \le n$, we claim that*

$$f_{s,r+s-i} = f_{s,r+s-i-1}x^{(i'_{r-i})}_{r+s-i-1,r+s-i} + \sum_{\substack{1\le k\le r\\ i'_k\neq i'_{r-i}}} \alpha_{s,r+s-i-1,k}x^{(i'_k)}_{r+s-i-1,r+s-i}$$

for all $1 \le i \le min\{s-1, r-1\}$, where both $f_{s,r+s-i-1}$ and $\alpha_{s,r+s-i-1,k}$ are polynomials over K on commutative variables indexed by elements from $\overline{W}_{s-i-1,r+s-i-1}$.

Proof. We get from (11) that

$$\begin{aligned} f_{s,r+s-i} = &\left(\sum_{(i_1,\ldots,i_{r-i-1})\in T_m^{r-i-1}} p_{i_1\cdots i_{r-i-1}i'_{r-i}\cdots i'_r}(\hat{c}_{s,t})a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-i-1})}_{r+s-i-2,r+s-i-1}\right)x^{(i'_{r-i})}_{r+s-i-1,r+s-i} \\ &+\sum_{\substack{1\le k\le r\\ i'_k\neq i'_{r-i}}}\left(\sum_{(i_1,\ldots,i_{r-i-1})\in T_m^{r-i-1}} p_{i_1\cdots i_{r-i-1}i'_k i'_{r-i+1}\cdots i'_r}(\hat{c}_{s,t})a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-i-1})}_{r+s-i-2,r+s-i-1}\right)x^{(i'_k)}_{r+s-i-1,r+s-i} \end{aligned} \tag{12}$$

for all $1 \le i \le min\{s-1, r-1\}$. It follows from (11) that

$$f_{s,r+s-i-1} = \sum_{(i_1,\ldots,i_{r-i-1})\in T_m^{r-i-1}} p_{i_1\cdots i_{r-i-1}i'_{r-i}\cdots i'_r}(\hat{c}_{s,t})a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-i-1})}_{r+s-i-2,r+s-i-1}.$$

We set

$$\alpha_{s,r+s-i-1,k} = \sum_{(i_1,\ldots,i_{r-i-1})\in T_m^{r-i-1}} p_{i_1\cdots i_{r-i-1}i'_k i'_{r-i+1}\cdots i'_r}(\hat{c}_{s,t})a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-i-1})}_{r+s-i-2,r+s-i-1}$$

for all $1 \le i \le min\{s-1, r-1\}$ and $k = 1, \ldots, r$. It follows from both (11) and (12) that

$$f_{s,r+s-i} = f_{s,r+s-i-1}x^{(i'_{r-i})}_{r+s-i-1,r+s-i} + \sum_{\substack{1\le k\le r\\ i'_k\neq i'_{r-i}}} \alpha_{s,r+s-i-1,k}x^{(i'_k)}_{r+s-i-1,r+s-i}$$

for all $1 \le i \le min\{s-1, r-1\}$. It is clear that both $f_{s,r+s-i-1}$ and $\alpha_{s,r+s-i-1,k}$ are polynomials over K on commutative variables indexed by elements from

$$\overline{W}_{s-i,r+s-i} \setminus \{(r+s-i-1, r+s-i, i'_k) \mid k = 1, \ldots r\}.$$

In view of Lemma 3.1 we note that

$$\overline{W}_{s-i-1,r+s-i-1} = \overline{W}_{s-i,r+s-i} \setminus \{(r+s-i-1, r+s-i, i'_k) \mid k = 1, \ldots r\}.$$

We have that both $f_{s,r+s-i-1}$ and $\alpha_{s,r+s-i-1,k}$ are polynomials over K on commutative variables indexed by elements from $\overline{W}_{s-i-1,r+s-i-1}$. This proves the result. □

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Lemma 3.6. *For any $1 \leq s < r+s \leq n$, we have that*

$$p_{s,r+s+t} = f_{s,r+s-1} x^{(i'_r)}_{r+s-1,r+s+t} + \sum_{\substack{1\leq k\leq r\\ i'_k\neq i'_r}} \beta_{s,r+s-1,k} x^{(i'_k)}_{r+s-1,r+s+t} + \beta_{s,r+s+t},$$

where $f_{1,r} \in K^*$, $\beta_{1,r,k} \in K$, $k = 1,\ldots,r$ with $i'_k \neq i'_r$, $f_{s,r+s-1}, \beta_{s,r+s-1,k}$, $s \geq 2$, $1 \leq k \leq r$ with $i'_k \neq i'_r$ are polynomials on some commutative variables in $\overline{W}_{s_1,r+s_1+t_1}$ and $\beta_{s,r+s+t}$, where $t > 0$, is a polynomial over K in some commutative variables in $W_{s_1,r+s_1+t_1}$, where

$$(s_1, r+s_1+t_1) = max\{(i, r+i+j) \mid (1, r+1) \leq (i, r+i+j) < (s, r+s+t)\}.$$

Moreover, $\beta_{s,r+s} = 0$.

Proof. It follows from (5) that

$$\begin{aligned} p_{s,r+s+t} = &\left(\sum_{(i_1,\ldots,i_{r-1})\in T_m^{r-1}} p_{i_1\cdots i_{r-1}i'_r}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-1})}_{r+s-2,r+s-1}\right) x^{(i'_r)}_{r+s-1,r+s+t} \\ &+ \sum_{\substack{1\leq k\leq r\\ i'_k\neq i'_r}} \left(\sum_{(i_1,\ldots,i_{r-1})\in T_m^{r-1}} p_{i_1\cdots i_{r-1}i'_k}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-1})}_{r+s-2,r+s-1}\right) x^{(i'_k)}_{r+s-1,r+s+t} \\ &+ \sum_{k=r}^{r+t} \left(\sum_{\substack{s=j_1<\cdots<j_{k+1}=r+s+t\\ (j_k,j_{k+1})\neq(r+s-1,r+s+t)\\ (i_1,\ldots,i_k)\in T_m^k}} p_{i_1\cdots i_k}(\bar{c}_{j_1},\ldots,\bar{c}_{j_{k+1}}) a^{(i_1)}_{j_1j_2}\cdots a^{(i_k)}_{j_kj_{k+1}}\right). \end{aligned} \tag{13}$$

It follows from (11) that

$$f_{s,r+s-1} = \sum_{(i_1,\ldots,i_{r-1})\in T_m^{r-1}} p_{i_1\cdots i_{r-1}i'_r}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-1})}_{r+s-2,r+s-1}.$$

We set

$$\beta_{s,r+s-1,k} = \sum_{(i_1,\ldots,i_{r-1})\in T_m^{r-1}} p_{i_1\cdots i_{r-1}i'_k}(\hat{c}_{s,t}) a^{(i_1)}_{s,s+1}\cdots a^{(i_{r-1})}_{r+s-2,r+s-1}$$

for $k = 1,\ldots,r$ with $i'_k \neq i'_r$, and

$$\beta_{s,r+s+t} = \sum_{k=r}^{r+t} \left(\sum_{\substack{s=j_1<\cdots<j_{k+1}=r+s+t\\ (j_k,j_{k+1})\neq(r+s-1,r+s+t)\\ (i_1,\ldots,i_k)\in T_m^k}} p_{i_1\cdots i_k}(\bar{c}_{j_1},\ldots,\bar{c}_{j_{k+1}}) a^{(i_1)}_{j_1j_2}\cdots a^{(i_k)}_{j_kj_{k+1}}\right).$$

It follows from (13) that

$$p_{s,r+s+t} = f_{s,r+s-1} x^{(i'_r)}_{r+s-1,r+s+t} + \sum_{\substack{1\leq k\leq r\\ i'_k\neq i'_r}} \beta_{s,r+s-1,k} x^{(i'_k)}_{r+s-1,r+s+t} + \beta_{s,r+s+t}, \tag{14}$$

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where $f_{1,r} \in K^*, \beta_{1,r,k} \in K$, $k = 1, \dots, r$ with $i'_k \neq i'_r$, $f_{s,r+s-1}, \beta_{s,r+s+t,k}$, where $s \geq 2$, $1 \leq k \leq r$ with $i'_k \neq i'_r$, are polynomials on some commutative variables indexed by elements from

$$\overline{W}_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k), \quad k = 1, \dots, r\} \tag{15}$$

and $\beta_{s,r+s+t}$, where $t > 0$, is a polynomial over K in some commutative variables indexed by elements from

$$W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k), \quad k = 1, \dots, r\}\,. \tag{16}$$

Suppose first that $t = 0$. In view of Lemma 3.1 we note that

$$\overline{W}_{s-1,r+s-1} = \overline{W}_{s,r+s+t} \setminus \{(r+s-1, r+s, i'_k), \quad k = 1, \dots, r\}\,.$$

We get from (15) that $f_{s,r+s-1}, \beta_{s,r+s+t,k}$, where $s \geq 2$, $1 \leq k \leq r$ with $i'_k \neq i'_r$, are polynomials on some commutative variables indexed by elements from $\overline{W}_{s-1,r+s-1}$. It is clear that $\beta_{s,r+s} = 0$. Suppose next that $t > 0$. In view of Lemma 3.2 we note that

$$\overline{W}_{s_1,r+s_1+t_1} = \overline{W}_{s,r+s+t}.$$

We get from (15) that $f_{s,r+s-1}, \beta_{s,r+s+t,k}$, where $s \geq 2$, $1 \leq k \leq r$ with $i'_k \neq i'_r$, are polynomials on some commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1+t_1}$. In view of Lemma 3.4 we note that

$$W_{s_1,r+s_1+t_1} = W_{s,r+s+t} \setminus \{(r+s-1, r+s+t, i'_k), \quad k = 1, \dots, r\}\,.$$

We get from (16) that $\beta_{s,r+s+t}$ is a polynomial over K in some commutative variables indexed by elements from $W_{s_1,r+s_1+t_1}$. This proves the result. □

The following result is crucial for the proof of the main result.

Lemma 3.7. *Let $p(x_1, \dots, x_m)$ be a polynomial with zero constant term in noncommutative variables over an infinite field K. Suppose $ord(p) = r$, where $1 < r < n-1$. For any $A' = (a'_{s,r+s+t}) \in T_n(K)^{(r-1)}$, where $a'_{s,r+s} \neq 0$ for all $1 \leq s < r+s+t \leq n$, we have that $A' \in p(T_n(K))$.*

Proof. Take any $A' = (a'_{s,r+s+t}) \in T_n(K)^{(r-1)}$, where $a'_{s,r+s} \neq 0$ for all $1 \leq s < r+s \leq n$. For any $1 \leq s < r+s+t \leq n$, we claim that there exist $c^{(i'_k)}_{r+u-1,r+u+w} \in K$ with

$$(r+u-1, r+u+w, k) \in W_{s,r+s+t}$$

such that

$$p_{i,r+i+j}(c^{(i'_k)}_{r+u-1,r+u+w}) = a_{i,r+i+j}$$

for all $(1, r+1) \leq (i, r+i+j) \leq (s, r+s+t)$ and

$$f_{s',r+s'-v}(c^{(i'_k)}_{r+u-1,r+u}) \neq 0$$

for all $f_{s',r+s'-v}$ on commutative variables in $\overline{W}_{s,r+s+t}$, where $s' \geq 2$ and $1 \leq v \leq min\{s'-1, r-1\}$.

We prove the claim by induction on $(s, r+s+t)$. Suppose first that $(s, r+s+t) = (1, r+1)$. Note that

$$W_{1,r+1} = \overline{W}_{1,r+1} = \{(r, r+1, i'_k) \mid k = 1, \dots, r\}.$$

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In view of Lemma 3.6 we get that

$$p_{1,r+1} = f_{1,r} x_{r,r+1}^{(i'_r)} + \sum_{\substack{1 \le k \le r \\ i_k \neq i'_r}} \beta_{1,r,k} x_{r,r+1}^{(i'_k)}, \tag{17}$$

where $f_{1,r} \in K^*$, $\beta_{1,r,k} \in K$, $k = 1, \ldots, r$ with $i'_k \neq i'_r$.

Take any $f_{s',r+s'-v}$ on $x_{r,r+1}^{(i'_k)}$, where $k = 1, \ldots, r$, $s' \geq 2$, and $1 \leq v \leq min\{s' - 1, r - 1\}$, we get from Lemma 3.5 that

$$r + s' - v - 1 = r$$

and so $v = s' - 1$. It follows that

$$f_{s',r+s'-v} = f_{s',r} x_{r,r+1}^{(i'_{r-v})} + \sum_{\substack{1 \le k \le r \\ i'_k \neq i'_{r-v}}} \alpha_{s',r,k} x_{r,r+1}^{(i'_k)}. \tag{18}$$

Note that $f_{s',r} \in K^*$ and $\alpha_{s',r,k} \in K$, $k = 1, \ldots, r$ with $i'_k \neq i_{r-v}$. Note that $a'_{1,r+1} \in K^*$. In view of Lemma 2.6, we get from both (17) and (18) that there exist $c_{r,r+1}^{(i'_k)} \in K$, $k = 1, \ldots, r$, such that

$$\begin{cases} p_{1,r+1}(c_{r,r+1}^{(i'_k)}) = a'_{1,r+1}; \\ f_{s',r+s'-v}(c_{r,r+1}^{(i'_k)}) \neq 0 \end{cases}$$

where $2 \leq s' \leq r$ and $v = s' - 1$, as desired.

Suppose next that $(s, r + s + t) \neq (1, r + 1)$. We rewrite (7) as follows.

$$(1, r + 1) < \cdots < (s_1, r + s_1 + t_1) < (s, r + s + t) < \cdots < (1, n),$$

where

$$(s_1, r + s_1 + t_1) = max\{(i, r + i + j) \mid (1, r + 1) \leq (i, r + i + j) < (s, r + s + t)\}.$$

By induction on $(s_1, r + s_1 + t_1)$ we have that there exist $c_{r+u-1,r+u+w}^{(i'_k)} \in K$ with

$$(r + u - 1, r + u + w, k) \in W_{s_1,r+s_1+t_1}$$

such that

$$p_{i,r+i+j}(c_{r+u-1,r+u+w}^{(i'_k)}) = a'_{i,r+i+j}$$

for all $(1, r + 1) \leq (i, r + i + j) \leq (s_1, r + s_1 + t_1)$ and

$$f_{s',r+s'-v}(c_{r+u-1,r+u}^{(i'_k)}) \neq 0$$

for any $f_{s',r+s'-v}$ with commutative variables in $\overline{W}_{s_1,r+s_1+t_1}$, where $s' \geq 2$, and $1 \leq v \leq min\{s' - 1, r - 1\}$. We now divide the proof into the following two cases.

Suppose first that $t = 0$. Note that

$$(s_1, r + s_1 + t_1) = (s - 1, r + s - 1).$$

That is, $s_1 = s - 1$ and $t_1 = 0$. In view of Lemma 3.6 we get that

$$p_{s,r+s} = f_{s,r+s-1} x_{r+s-1,r+s}^{(i'_r)} + \sum_{\substack{1 \le k \le r \\ i'_k \neq i'_r}} \beta_{s,r+s-1,k} x_{r+s-1,r+s}^{(i'_k)}, \tag{19}$$

where $f_{s,r+s-1}, \beta_{s,r+s-1,k}$, where $k = 1, \ldots, r$ with $i'_k \neq i'_r$, are polynomials in commutative variables in $\overline{W}_{s_1,r+s_1}$. By induction hypothesis we get that $f_{s,r+s-1} \in K^*$ and $\beta_{s,r+s-1,k} \in K$.

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Take any $f_{s',r+s'-v}$ on commutative variables indexed by elements from $\overline{W}_{s,r+s}$, where $s' \geq 2$ and $1 \leq v \leq min\{s'-1, r-1\}$. Suppose first that $f_{s',r+s'-v}$ is a polynomial on commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1}$. By induction hypothesis we have that $f_{s',r+s'-v} \in K^*$. Suppose next that $f_{s',r+s'-v}$ is not a polynomial on commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1}$. In view of Lemma 3.1 we note that

$$\overline{W}_{s,r+s} \setminus \overline{W}_{s-1,r+s-1} = \{(r+s-1, r+s, i'_k) \mid k = 1, \ldots, r\}.$$

This implies that $x^{(i'_k)}_{r+s-1,r+s}$ appears in $f_{s',r+s'-v}$ for $k = 1, \ldots, r$. In view of Lemma 3.5 we get that

$$(r+s'-v-1, r+s'-v) = (r+s-1, r+s)$$

and so $v = s'-s$. We get that

$$f_{s',r+s'-v} = f_{s',r+s'-v-1} x^{(i'_{r-v})}_{r+s-1,r+s} + \sum_{\substack{1 \leq k \leq r \\ i'_k \neq i'_{r-v}}} \alpha_{s',r+s'-v-1,k} x^{(i'_k)}_{r+s-1,r+s}, \tag{20}$$

where $f_{s',r+s'-v-1}$ and $\alpha_{s',r+s'-v-1,k}$, $k = 1, \ldots, r$ with $i'_k \neq i'_{r-v}$, are polynomials over K on commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1}$. By induction hypothesis we have that $f_{s',r+s'-v-1} \in K^*$ and $\alpha_{s',r+s'-v-1,k} \in K$, where $k = 1, \ldots, r$ with $i'_k \neq i'_{r-v}$.

Note that $a'_{s,r+s} \in K^*$. In view of Lemma 2.6, we get from both (19) and (20) that there exist $c^{(i'_k)}_{r+s-1,r+s} \in K$, $k = 1, \ldots, r$, such that

$$\begin{cases} p_{s,r+s}(c^{(i'_k)}_{r+s-1,r+s}) = a'_{s,r+s}; \\ f_{s',r+s'-v}(c^{(i'_k)}_{r+s-1,r+s}) \neq 0, \end{cases}$$

as desired.

Suppose next that $t > 0$. It follows from Lemma 3.6 that

$$p_{s,r+s+t} = f_{s,r+s-1} x^{(i'_r)}_{r+s-1,r+s+t} + \sum_{\substack{1 \leq k \leq r \\ i'_k \neq i'_r}} \beta_{s,r+s-1,k} x^{(i'_k)}_{r+s-1,r+s+t} + \beta_{s,r+s+t}, \tag{21}$$

where $f_{s,r+s-1}, \beta_{s,r+s-1,k}$, where $k = 1, \ldots, r$ with $i'_k \neq i'_r$, are polynomials over K in commutative variables indexed by elements from $\overline{W}_{r+s_1+t_1}$, and $\beta_{s,r+s+t}$ is a polynomial over K in commutative variables indexed by elements from $W_{s_1,r+s_1+t_1}$. By induction hypothesis we have that $f_{s,r+s-1} \in K^*$, $\beta_{s,r+s-1,k} \in K$ for all $k = 1, \ldots, r$ with $i'_k \neq i'_r$, and $\beta_{s,r+s+t} \in K$.

Take $c^{(i'_k)}_{r+s-1,r+s+t} \in K$, where $k = 1, \ldots, r$ in (21) such that

$$\begin{cases} c^{(i'_r)}_{r+s-1,r+s+t} = f^{-1}_{s,r+s-1}(a'_{s,r+s+t} - \beta_{s,r+s+t}); \\ c^{(i'_k)}_{r+s-1,r+s+t} = 0 \quad \text{for all } 1 \leq k \leq r \text{ with } i'_k \neq i'_r. \end{cases}$$

We get that

$$p_{s,r+s+t}(c^{(i'_k)}_{r+s-1,r+s+t}) = a'_{s,r+s+t}.$$

Take any $f_{s',r+s'-v}$ on commutative variables indexed by elements from $\overline{W}_{s,r+s+t}$, where $s' \geq 2$ and $1 \leq v \leq min\{s'-1, r-1\}$. In view of Lemma 3.2 we note that

$$\overline{W}_{s,r+s+t} = \overline{W}_{s_1,r+s_1+t_1}.$$

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This implies that $f_{s',r+s'-v}$ is a commutative polynomial over K on some commutative variables indexed by elements from $\overline{W}_{s_1,r+s_1+t_1}$. By induction hypothesis we get that

$$f_{s',r+s'-v} \in K^*$$

where $s' \geq 2$ and $1 \leq v \leq min\{s'-1, r-1\}$, as desired. This proves the claim.

Let $(s, r+s+t) = (1, n)$. We have that there exist $c^{(i'_k)}_{r+u-1,r+u+w} \in K$, $k = 1, \ldots, r$, with

$$(r+u-1, r+u+w, k) \in W_{1,n},$$

such that

$$p_{i,r+i+j}(c^{(i'_k)}_{r+u-1,r+u+w}) = a'_{i,r+i+j} \tag{22}$$

for all $(1, r+1) \leq (i, r+i+j) \leq (1, n)$ and

$$f_{s',r+s'-v}(c^{(i'_k)}_{r+u-1,r+u}) \neq 0$$

for all $f_{s',r+s'-v}$ on commutative variables indexed by elements from $\overline{W}_{1,n}$, where $s' \geq 2$ and $1 \leq v \leq min\{s'-1, r-1\}$. It follows from both (5) and (22) that

$$p(u_1, \ldots, u_m) = (p_{s,r+s+t}) = (a'_{s,r+s+t}) = A'.$$

This implies that $A' \in p(T_n(K))$. The proof of the result is complete. □

Lemma 3.8. *Let $n \geq 4$ and $m \geq 1$ be integers. Let $p(x_1, \ldots, x_m)$ be a polynomial with zero constant term in non-commutative variables over an infinite field K. Suppose that $ord(p) = n-2$. We have that $p(T_n(K)) = T_n(K)^{(n-3)}$.*

Proof. In view of Lemma 2.2(ii) we note that $p(T_n(K)) \subseteq T_n(K)^{(n-3)}$. It suffices to prove that $T_n(K)^{(n-3)} \subseteq p(T_n(K))$.

For any $u_i = (a^{(i)}_{jk}) \in T_n(K)$, $i = 1, \ldots, m$, in view of Lemma 2.2(ii) we get from (2) that

$$p(u_1, \ldots, u_m) = \begin{pmatrix} 0 & 0 & \ldots & p_{1,n-1} & p_{1n} \\ 0 & 0 & \ldots & 0 & p_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{23}$$

where

$$\begin{cases} p_{1,n-1} = \displaystyle\sum_{(i_1,\ldots,i_{n-2}) \in T_m^{n-2}} p_{i_1\cdots i_{n-2}}(\bar{a}_{11}, \ldots, \bar{a}_{n-1,n-1}) a^{(i_1)}_{12} \cdots a^{(i_{n-2})}_{n-2,n-1}; \\ p_{2n} = \displaystyle\sum_{(i_1,\ldots,i_{n-2}) \in T_m^{n-2}} p_{i_1\cdots i_{n-2}}(\bar{a}_{22}, \ldots, \bar{a}_{n,n}) a^{(i_1)}_{23} \cdots a^{(i_{n-2})}_{n-1,n}; \\ p_{1n} = \displaystyle\sum_{(i_1,\ldots,i_{n-1}) \in T_m^{n-1}} p_{i_1\cdots i_{n-1}}(\bar{a}_{11}, \ldots, \bar{a}_{nn}) a^{(i_1)}_{12} \cdots a^{(i_{n-1})}_{n-1,n} \\ \qquad + \displaystyle\sum_{\substack{1=j_1<\cdots<j_{n-1}=n \\ (i_1,\ldots,i_{n-2}) \in T_m^{n-2}}} p_{i_1\cdots i_{n-2}}(\bar{a}_{j_1j_1}, \ldots, \bar{a}_{j_{n-1}j_{n-1}}) a^{(i_1)}_{j_1j_2} \cdots a^{(i_{n-2})}_{j_{n-2}j_{n-1}}. \end{cases}$$

In view of Lemma 2.2(iii) we have that

$$p_{i'_1,\ldots,i'_{n-2}}(K) \neq \{0\},$$

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for some $i'_1, \ldots, i'_{n-2} \in \{1, \ldots, m\}$. It follows from Lemma 2.4 that there exist $\bar{b}_1, \ldots, \bar{b}_n \in K^m$ such that

$$p_{i'_1,\ldots,i'_{n-2}}(\bar{b}_{j_1}, \ldots, \bar{b}_{j_{n-1}}) \neq 0$$

for all $1 \leq j_1 < \cdots < j_{n-1} \leq n$.

For any $A' = (a'_{s,n-2+s+t}) \in T_n(K)^{(n-3)}$, where $1 \leq s < n-2+s+t \leq n$, we claim that there exist $u_i = (a^{(i)}_{jk}) \in T_n(K)$, $i = 1, \ldots, m$, such that

$$p(u_1, \ldots, u_m) = (p_{s,n-2+s+t}) = A'.$$

That is

$$\begin{cases} p_{1,n-1} = a'_{1,n-1}; \\ \quad p_{2n} = a'_{2n}; \\ \quad p_{1n} = a'_{1n}. \end{cases}$$

We prove the claim by the following two cases:

Case 1. Suppose that $a'_{1,n-1} \neq 0$. We take

$$\begin{cases} \bar{a}_{jj} = \bar{b}_j, \quad \text{for all } j = 1, \ldots, n; \\ a^{(i'_1)}_{12} = x^{(i'_1)}_{12}; \\ a^{(k)}_{12} = 0 \quad \text{for all } k = 1, \ldots, m \text{ with } k \neq i'_1; \\ a^{(i'_{n-2})}_{n-1,n} = x^{(i'_{n-2})}_{n-1,n}; \\ a^{(k)}_{n-1,n} = 0 \quad \text{for all } k = 1, \ldots, m \text{ with } k \neq i'_{n-2}; \\ a^{(i'_{n-2})}_{n-2,n} = x^{(i'_{n-2})}_{n-2,n}; \\ a^{(i)}_{j,j+2} = 0 \quad \text{for all } 1 \leq i \leq m,\ 3 \leq j+2 \leq n \text{ with } (j, j+2, i) \neq (n-2, n, i'_{n-2}). \end{cases}$$

It follows from (23) that

$$\begin{cases} p_{1,n-1} = \left(\sum_{(i_2,\ldots,i_{n-2}) \in T_m^{n-3}} p_{i'_1 i_2 \cdots i_{n-2}}(\bar{b}_1, \ldots, \bar{b}_{n-1}) a^{(i_2)}_{23} \cdots a^{(i_{n-2})}_{n-2,n-1} \right) x^{(i'_1)}_{12}; \\ p_{2n} = \left(\sum_{(i_1,\ldots,i_{n-3}) \in T_m^{n-3}} p_{i_1 \cdots i_{n-3} i'_{n-2}}(\bar{b}_2, \ldots, \bar{b}_n) a^{(i_1)}_{23} \cdots a^{(i_{n-3})}_{n-2,n-1} \right) x^{(i'_{n-2})}_{n-1,n}; \\ p_{1n} = \left(\sum_{(i_2,\ldots,i_{n-2}) \in T_m^{n-3}} p_{i'_1 i_2 \cdots i_{n-2} i'_{n-2}}(\bar{b}_1, \ldots, \bar{b}_n) a^{(i_2)}_{23} \cdots a^{(i_{n-2})}_{n-2,n-1} \right) x^{(i'_1)}_{12} x^{(i'_{n-2})}_{n-1,n} \\ \qquad \left(\sum_{(i_2,\ldots,i_{n-3}) \in T_m^{n-4}} p_{i'_1 i_2 \cdots i_{n-3} i'_{n-2}}(\bar{b}_1, \ldots, \bar{b}_{n-2}, \bar{b}_n) a^{(i_2)}_{23} \cdots a^{(i_{n-3})}_{n-3,n-2} \right) x^{(i'_1)}_{12} x^{(i'_{n-2})}_{n-2,n}. \end{cases} \tag{24}$$

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We set

$$\begin{cases} f_{1,n-1} = \sum_{(i_2,\ldots,i_{n-2})\in T_m^{n-3}} p_{i_1' i_2\cdots i_{n-2}}(\bar{b}_1,\ldots,\bar{b}_{n-1}) a_{23}^{(i_2)}\cdots a_{n-2,n-1}^{(i_{n-2})} \\ f_{2n} = \sum_{(i_1,\ldots,i_{n-3})\in T_m^{n-3}} p_{i_1\cdots i_{n-3} i_{n-2}'}(\bar{b}_2,\ldots,\bar{b}_n) a_{23}^{(i_1)}\cdots a_{n-2,n-1}^{(i_{n-3})} \\ f_{1n} = \sum_{(i_2,\ldots,i_{n-3})\in T_m^{n-4}} p_{i_1' i_2\cdots i_{n-3} i_{n-2}'}(\bar{b}_1,\ldots,\bar{b}_{n-2},\bar{b}_n) a_{23}^{(i_2)}\cdots a_{n-3,n-2}^{(i_{n-3})} \end{cases} \tag{25}$$

and

$$\begin{aligned} V_{1,n-1} &= \{(i,i+1,k) \mid i=2,\ldots,n-2, k=1,\ldots,m\}; \\ V_{2n} &= V_{1,n-1}; \\ V_{1n} &= \{(i,i+1,k) \mid i=2,\ldots,n-3, k=1,\ldots,m\}. \end{aligned}$$

Note that $f_{1,n-1}, f_{2n}, f_{1n}$ are polynomials over K on commutative variables indexed by elements from $V_{1,n-1}, V_{2n}, V_{1n}$, respectively.

We claim that $f_{1,n-1}, f_{2n}, f_{1n} \neq 0$. Indeed, we take $a_{jk}^{(i)} \in K$, $(j,k,i) \in V_{1,n-1}$ such that

$$\begin{cases} a_{s,s+1}^{(i_s')} = 1 & \text{for all } s=2,\ldots,n-2; \\ a_{jk}^{(i)} = 0 & \text{otherwise.} \end{cases}$$

It follows from (25) that

$$f_{1,n-1}(a_{jk}^{(i)}) = p_{i_1'\cdots i_{n-2}'}(\bar{b}_1,\ldots,\bar{b}_{n-1}) \neq 0$$

as desired. Next, we take $a_{jk}^{(i)} \in K$, $(j,k,i) \in V_{2n}$ such that

$$\begin{cases} a_{s,s+1}^{(i_{s-1}')} = 1 & \text{for all } s=2,\ldots,n-2; \\ a_{jk}^{(i)} = 0 & \text{otherwise.} \end{cases}$$

It follows from (25) that

$$f_{2n}(a_{jk}^{(i)}) = p_{i_1'\cdots i_{n-2}'}(\bar{b}_2,\ldots,\bar{b}_n) \neq 0$$

as desired. Finally, we take $a_{jk}^{(i)} \in K$, $(j,k,i) \in V_{1n}$ such that

$$\begin{cases} a_{s,s+1}^{(i_s')} = 1 & \text{for all } s=2,\ldots,n-3; \\ a_{jk}^{(i)} = 0 & \text{otherwise.} \end{cases}$$

It follows from (25) that

$$f_{1n}(a_{jk}^{(i)}) = p_{i_1'\cdots i_{n-2}'}(\bar{b}_1,\ldots,\bar{b}_{n-2},\bar{b}_n) \neq 0$$

as desired. In view of Lemma 2.5 we get that there exist $a_{jk}^{(i)} \in K$, where $(j,k,i) \in V_{1,n-1} \cup V_{2n} \cup V_{1n}$ such that

$$\begin{cases} f_{1,n-1}(a_{jk}^{(i)}) \neq 0; \\ f_{2n}(a_{jk}^{(i)}) \neq 0; \\ f_{1n}(a_{jk}^{(i)}) \neq 0. \end{cases}$$

We set

$$\alpha = \sum_{(i_2,\ldots,i_{n-2})\in T_{n-3}} p_{i'_1 i_2\cdots i_{n-2} i'_{n-2}}(\bar{b}_1,\ldots,\bar{b}_n)a_{23}^{(i_2)}\cdots a_{n-2,n-1}^{(i_{n-2})}.$$

It follows from (24) that

$$\begin{cases} p_{1,n-1} = f_{1,n-1}x_{12}^{(i'_1)};\\ \quad p_{2n} = f_{2n}x_{n-1,n}^{(i'_{n-2})};\\ \quad p_{1n} = f_{1n}x_{12}^{(i'_1)}x_{n-2,n}^{(i'_{n-2})} + \alpha x_{12}^{(i'_1)}x_{n-1,n}^{(i'_{n-2})}. \end{cases} \tag{26}$$

We take

$$\begin{cases} \quad x_{12}^{(i'_1)} = f_{1,n-1}^{-1}a'_{1,n-1};\\ x_{n-1,n}^{(i'_{n-2})} = f_{2n}^{-1}a'_{2n};\\ x_{n-2,n}^{(i'_{n-2})} = f_{1n}^{-1}f_{1,n-1}(a'_{1,n-1})^{-1}\left(a'_{1n} - \alpha f_{1,n-1}^{-1}a'_{1,n-1}f_{2n}^{-1}a'_{2n}\right). \end{cases}$$

It follows from (26) that

$$\begin{cases} p_{1,n-1} = a'_{1,n-1};\\ \quad p_{2n} = a'_{2n};\\ \quad p_{1n} = a'_{1n}, \end{cases}$$

as desired.

Case 2. Suppose that $a'_{1,n-1} = 0$. We take

$$\begin{cases} \quad \bar{a}_{jj} = \bar{b}_j, \quad \text{for all } j = 1,\ldots,n;\\ \quad a_{12}^{(k)} = 0 \quad \text{for all } k = 1,\ldots,m;\\ \quad a_{23}^{(i'_1)} = x_{23}^{(i'_1)};\\ \quad a_{23}^{(k)} = 0 \quad \text{for all } k = 1,\ldots,m \text{ with } k \neq i'_1;\\ \quad a_{13}^{(i'_1)} = x_{13}^{(i'_1)};\\ a_{j,j+2}^{(k)} = 0 \quad \text{for all } 1 \le j < j+2 \le n \text{ with } (j,j+2,k) \neq (1,3,i'_1). \end{cases}$$

It follows from (23) that

$$\begin{cases} p_{1,n-1} = 0;\\ \quad p_{2n} = \left(\displaystyle\sum_{(i_2,\ldots,i_{n-2})\in T_m^{n-3}} p_{i'_1 i_2\cdots i_{n-2}}(\bar{b}_2,\ldots,\bar{b}_n)a_{34}^{(i_2)}\cdots a_{n-1,n}^{(i_{n-2})}\right)x_{23}^{(i'_1)};\\ \quad p_{1n} = \left(\displaystyle\sum_{(i_2,\ldots,i_{n-2})\in T_m^{n-3}} p_{i'_1 i_2\cdots i_{n-2}}(\bar{b}_1,\bar{b}_3,\ldots,\bar{b}_n)a_{34}^{(i_2)}\cdots a_{n-1,n}^{(i_{n-2})}\right)x_{13}^{(i'_1)}. \end{cases} \tag{27}$$

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We set

$$\begin{cases} g_{2n} = \sum_{(i_2,\ldots,i_{n-2})\in T_m^{n-3}} p_{i_1' i_2\cdots i_{n-2}}(\bar{b}_2,\ldots,\bar{b}_n)a_{34}^{(i_2)}\cdots a_{n-1,n}^{(i_{n-2})}; \\ g_{1n} = \sum_{(i_2,\ldots,i_{n-2})\in T_m^{n-3}} p_{i_1' i_2\cdots i_{n-2}}(\bar{b}_1,\bar{b}_3,\ldots,\bar{b}_n)a_{34}^{(i_2)}\cdots a_{n-1,n}^{(i_{n-2})} \end{cases} \tag{28}$$

and

$$V = \{(i,i+1,k) \mid i = 3,\ldots,n-1, k = 1,\ldots,m\}.$$

Note that both g_{2n} and g_{1n} are polynomials over K on some commutative variables indexed by elements from V. We claim that $g_{2n}, g_{1n} \neq 0$. Indeed, we take $a_{jk}^{(i)} \in K$, $(j,k,i) \in V$ such that

$$\begin{cases} a_{s,s+1}^{(i'_{s-1})} = 1 & \text{for all } s = 3,\ldots,n-1; \\ a_{jk}^{(i)} = 0 & \text{otherwise.} \end{cases}$$

It follows from (28) that

$$g_{2n} = p_{i'_1\cdots i'_{n-2}}(\bar{b}_2,\ldots,\bar{b}_n) \neq 0;$$
$$g_{1n} = p_{i'_1\cdots i'_{n-2}}(\bar{b}_1,\bar{b}_3,\ldots,\bar{b}_n) \neq 0.$$

as desired. It follows from (27) that

$$\begin{cases} p_{1,n-1} = 0; \\ p_{2n} = g_{2n}x_{23}^{(i'_1)}; \\ p_{1n} = g_{1n}x_{13}^{(i'_1)}. \end{cases} \tag{29}$$

We take

$$\begin{cases} x_{23}^{(i'_1)} = g_{2n}^{-1}a'_{2n}; \\ x_{13}^{(i'_1)} = g_{1n}^{-1}a'_{1n}. \end{cases}$$

It follows from (29) that

$$\begin{cases} p_{1,n-1} = 0; \\ p_{2n} = a'_{2,n}; \\ p_{1n} = a'_{1n}, \end{cases}$$

as desired. We obtain that

$$p(u_1,\ldots,u_m) = (p_{s,n-2+s+t}) = (a'_{s,n-2+s+t}) = A'.$$

This implies that $T_n(K)^{(n-3)} \subseteq p(T_n(K))$. Hence $p(T_n(K)) = T_n(K)^{(n-3)}$. □

We are ready to give the proof of the main result of the paper.

The proof of Theorem 1.2. For any $A = (a_{s,r+s+t}) \in T_n(K)^{(r-1)}$, we set

$$\begin{cases} f_{s,r+s}(x_{s,r+s}) = a_{s,r+s} - x_{s,r+s}; \\ g_{s,r+s}(x_{s,r+s}) = x_{s,r+s} \end{cases}$$

for all $1 \leq s < r + s \leq n$. It is clear that both $f_{s,r+s}$ and $g_{s,r+s}$ are nonzero polynomials in commutative variables over K, where $1 \leq s < r+s \leq n$. It follows from Lemma 2.5 that there exist $b_{s,r+s} \in K$, $1 \leq s < r+s \leq n$, such that

$$\begin{cases} f_{s,r+s}(b_{s,r+s}) \neq 0; \\ g_{s,r+s}(b_{s,r+s}) \neq 0 \end{cases}$$

for all $1 \leq s < r+s \leq n$. That is

$$\begin{cases} a_{s,r+s} - b_{s,r+s} \neq 0; \\ \qquad\qquad b_{s,r+s} \neq 0 \end{cases}$$

for all $1 \leq s < r+s \leq n$. We set

$$b_{s,r+s+t} = a_{s,r+s+t}$$

for all $1 \leq s < r+s+t \leq n$ and $t > 0$ and

$$\begin{cases} c_{s,r+s} = a_{s,r+s} - b_{s,r+s} & \text{for all } 1 \leq s < r+s \leq n; \\ c_{s,r+s+t} = 0 & \text{for all } 1 \leq s < r+s+t \leq n \text{ and } t > 0. \end{cases}$$

We set

$$B = (b_{s,r+s+t}) \quad \text{and} \quad C = (c_{s,r+s+t}).$$

It is clear that

$$A = B + C$$

where $B, C \in T_n(K)^{(r-1)}$ with $b_{s,r+s}, c_{s,r+s} \in K^*$ for all $1 \leq s < r+s \leq n$. In view of Lemma 3.7, we get that there exist $u_i, v_i \in T_n(K)$, $i = 1, \ldots, m$, such that

$$p(u_1, \ldots, u_m) = B \quad \text{and} \quad p(v_1, \ldots, v_m) = C.$$

It follows that

$$p(u_1, \ldots, u_m) + p(v_1, \ldots, v_m) = A.$$

This implies that

$$T_n(K)^{(r-1)} \subseteq p(T_n(K)) + p(T_n(K)).$$

In view of Lemma 2.2(ii) we note that $p(T_n(K)) \subseteq T_n(K)^{(r-1)}$. Since $T_n(K)^{(r-1)}$ is a subspace of $T_n(K)$ we get that

$$p(T_n(K)) + p(T_n(K)) \subseteq T_n(K)^{(r-1)}.$$

We obtain that

$$p(T_n(K)) + p(T_n(K)) = T_n(K)^{(r-1)}.$$

In particular, if $r = n-2$ we get from Lemma 3.8 that

$$p(T_n(K)) = T_n(K)^{(n-3)}.$$

The proof of the result is complete. □

We conclude the paper with following example.

Example 3.1. *Let $n \geq 5$ and $1 < r < n-2$ be integers. Let K be an infinite field. Let*

$$p(x, y) = [x, y]^r.$$

We have that $ord(p) = r$ and $p(T_n(K)) \neq T_n(K)^{(r-1)}$.

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Proof. It is easy to check that $p(T_r(K)) = \{0\}$. Set

$$f(x, y) = [x, y].$$

Note that f is a multilinear polynomial over K. It is clear that $\mathrm{ord}(f) = 1$. In view of [12, Theorem 4.3] or [15, Theorem 1.1] we have that

$$f(T_{r+1}(K)) = T_{r+1}(K)^{(0)}.$$

It implies that there exist $A, B \in T_{r+1}(K)$ such that

$$[A, B] = e_{12} + e_{23} + \cdots + e_{r,r+1}.$$

We get that

$$p(A, B) = [A, B]^r = e_{1,r+1} \neq 0.$$

This implies that $p(T_{r+1}(K)) \neq \{0\}$. We obtain that $\mathrm{ord}(p) = r$.

Suppose on contrary that $p(T_n(K)) = T_n(K)^{(r-1)}$ for some $n \geq 5$ and $1 < r < n-2$. For $e_{1,r+1} + e_{3,r+3} \in T_n(K)^{(r-1)}$, we get that there exists $B, C \in T_n(K)$ such that

$$p(B, C) = [B, C]^r = e_{1,r+1} + e_{3,r+3}.$$

It is clear that $[B, C] \in T_n(K)^{(0)}$. We set

$$[B, C] = (a_{s,1+s+t}).$$

It follows that

$$[B, C]^r = e_{1,r+1} + e_{3,r+3}.$$

We get from the last relation that

$$\begin{cases} (a_{12}a_{23}\cdots a_{r,r+1})e_{1,r+1} = e_{1,r+1}; \\ (a_{23}a_{34}\cdots a_{r+1,r+2})e_{2,r+2} = 0; \\ (a_{34}a_{45}\cdots a_{r+2,r+3})e_{3,r+3} = e_{3,r+3}. \end{cases}$$

This is a contradiction. We obtain that $p(T_n(K)) \neq T_n(K)^{(r-1)}$ for all $n \geq 5$ and $1 < r < n-2$. This proves the result. □

We remark that [16, Example 5.7] is a special case of Example 3.1 ($r = 2$ and $n = 5$).

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