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SOME TRACE INEQUALITIES FOR OPERATORS

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Abstract

In this paper, we obtain some trace inequalities for arbitrary finite positive definite operators. Finally an open question is presented.

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In 1978, after giving some trace inequalities for positive definite matrices, R. Bellman brought attention to two open questions [1]. One of the questions asks: Is there a matrix analogy of the arithmetic mean – geometric mean inequality (for positive definite matrices)? Y. Yang [4] proved that the answer to the above question is affirmative for two positive definite matrices. Recently Dinesh Singh [3] generalized the result of Yang to infinite-dimensional spaces.

In this paper, we generalize the trace inequality in [3] from two positive definite operators to an arbitrary finite number of positive definite operators.

Throughout, C_p $(1 \le p < \infty)$ stands for the class of all bounded operators A on an infinite-dimensional separable Hilbert space H, such that $\sum_{n=1}^{\infty} |\langle Ae_n, e_n\rangle|^p < \infty$ for each orthonormal basis $\{e_n\}_1^\infty$ in H. Let $\{e_n\}$ be any orthonormal basis in H. Let $\mathrm{Tr} : C_1 \to \mathbb{C}$ (complex numbers) be defined by

$$\mathrm{Tr}\,(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle.$$

It is easy to see that Tr is independent of $\{e_n\}$ [2, Lemma 2.2.1], and since C_1 consists of compact operators [2, Theorem 2.1.6], $\mathrm{Tr}\,(A)$ is the sum of the eigenvalues of A.

Furthermore, Tr defines an inner product on C_2 given by

$$\langle A, B\rangle = \operatorname{Tr}(B^* A)$$

where B is the adjoint of B. This inner product makes C_2 into a Hilbert space [2, Theorem 2.4.2]. Clearly C_1 is contained in C_2.

We now state and prove our results as following.

LEMMA 1. *Let A be a positive definite operator in C_1 and B be a operator in C_1. Then*

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA) \tag{1}$$

PROOF. Choose an orthonormal basis $\{e_n\}_1^\infty$ of H such that each e_n is an eigenvector for A with corresponding eigenvalue α_n. Since $A > 0$, each $\alpha_n > 0$. Let $\beta_n = \langle Be_n, e_n\rangle$. Then

$$\begin{aligned}
\operatorname{Tr}(AB) &= \sum_{n=1}^{\infty}\langle ABe_n, e_n\rangle &&= \sum_{n=1}^{\infty}\langle Be_n, Ae_n\rangle \\
&= \sum_{n=1}^{\infty}\alpha_n\langle Be_n, e_n\rangle &&= \sum_{n=1}^{\infty}\alpha_n\beta_n,
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tr}(BA) &= \sum_{n=1}^{\infty}\langle BAe_n, e_n\rangle &&= \sum_{n=1}^{\infty}\langle Ae_n, B^*e_n\rangle \\
&= \sum_{n=1}^{\infty}\alpha_n\langle e_n, B^*e_n\rangle &&= \sum_{n=1}^{\infty}\alpha_n\langle Be_n, e_n\rangle \\
&= \sum_{n=1}^{\infty}\alpha_n\beta_n.
\end{aligned}$$

Hence (1) is proved.

By the Cauchy-Schwartz inequality, we have

LEMMA 2. *Let A, B be two operators in C_1; then*

$$\operatorname{Tr}(AB) \le |\operatorname{Tr}(AB)| \le \sqrt{\operatorname{Tr}(AA^*)}\cdot\sqrt{\operatorname{Tr}(BB^*)}.$$

LEMMA 3. ([3]). *Let A, B be two positive definite operators in C_1; then*

$$\operatorname{Tr}(AB) < \operatorname{Tr}(A)\cdot\operatorname{Tr}(B).$$

LEMMA 4. *Let A_i $(1 \le i \le m)$ be positive definite operators in C_1; then*

$$\operatorname{Tr}\left\{(A_1A_2\cdots A_m)(A_1A_2\cdots A_m)^*\right\} < \prod_{i=1}^{m}\operatorname{Tr}\left(A_i^2\right) < \prod_{i=1}^{m}(\operatorname{Tr}(A_i))^2.$$

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PROOF.

$$\begin{aligned}
\operatorname{Tr}\left\{(A_1A_2\cdots A_m)(A_1A_2\cdots A_m)^*\right\} &= \operatorname{Tr}\left\{A_1(A_2\cdots A_m)(A_2\cdots A_m)^* A_1^*\right\} \\
&= \operatorname{Tr}\left\{\left(A_1^*A_1\right)(A_2\cdots A_m)(A_2\cdots A_m)^*\right\} && \text{(by Lemma 1)} \\
&< \operatorname{Tr}\left(A_1^*A_1\right)\operatorname{Tr}\left\{(A_2\cdots A_m)(A_2\cdots A_m)^*\right\} && \text{(by Lemma 3)} \\
&\cdots \\
&< \operatorname{Tr}\left(A_1^2\right)\operatorname{Tr}\left(A_2^2\right)\cdots\operatorname{Tr}\left(A_m^2\right) < \prod_{i=1}^{m}[\operatorname{Tr}(A_i)]^2 . && \text{(by Lemma 3)}
\end{aligned}$$

The proof is complete.

THEOREM 5. *Let A_i $(1 \le i \le m)$ be positive definite operators in C_1; then*

$$\left|\operatorname{Tr}(A_1A_2\cdots A_m)\right| < \prod_{i=1}^{m}\operatorname{Tr}(A_i), \qquad m \ge 2.$$

PROOF. By Lemma 2 and Lemma 4, we get

$$\begin{aligned}
\left|\operatorname{Tr}(A_1A_2\cdots A_m)\right| &= \left|\operatorname{Tr}[A_1(A_2\cdots A_m)]\right| \\
&\le \sqrt{\operatorname{Tr}\left(A_1A_1^*\right)}\cdot\sqrt{\operatorname{Tr}\left[(A_2\cdots A_m)(A_2\cdots A_m)^*\right]} < \prod_{i=1}^{m}\operatorname{Tr}(A_i)
\end{aligned}$$

The proof is complete.

THEOREM 6. *Let A_i $(1 \le i \le m)$ be positive definite operators in C_1; then*

$$\frac{1}{m}\left[\sum_{i=1}^{m}\operatorname{Tr}(A_i)\right] > \left|\operatorname{Tr}(A_1A_2\cdots A_m)\right|^{\frac{1}{m}}.$$

PROOF. By the arithmetic mean – geometric mean inequality for m positive real numbers, we have

$$\frac{1}{m}\left[\sum_{i=1}^{m}\operatorname{Tr}(A_i)\right] \ge \left(\prod_{i=1}^{m}\operatorname{Tr}(A_i)\right)^{\frac{1}{m}}.$$

From Theorem 5, we get

$$\frac{1}{m}\left[\sum_{i=1}^{m}\operatorname{Tr}(A_i)\right] > \left|\operatorname{Tr}(A_1A_2\cdots A_m)\right|^{\frac{1}{m}}.$$

The proof is complete.

The above Theorem 6 generalizes the theorem in [3].

THEOREM 7. *Let* A_i $(1 \leq i \leq 2^m)$ *be positive definite operators in* C_1*; then*

$$\left|\operatorname{Tr}(A_1 A_2 \cdots A_{2^m})\right|^{2^m} \leq \prod_{i=1}^{2^m} \left[\operatorname{Tr}\left(A_i^{2^m}\right)\right] \tag{2}$$

PROOF. We will prove the above inequality by induction on m. If $m = 1$, inequality (2) is obvious by Lemma 2. Now suppose that, for $m < p$, inequality (2) is true. If $m = p$, let

$$\begin{aligned} B_i &= A_{2^{p-i}} A_{2^{p-i}-1} \cdots A_2 A_1 A_1 A_2 \cdots A_{2^{p-i}-1} A_{2^{p-i}} \\ \text{and} \quad C_i &= A_{2^{p-i}+1} A_{2^{p-i}} \cdots A_{2^{p-(i-1)}} A_{2^{p-(i-1)}} \cdots A_{2^{p-i}} A_{2^{p-i}+1} \\ u_i &= \left|\operatorname{Tr}\left(B_i^{2^{i-1}}\right)\right|^{2^{p-i}}, \quad v_i = \left|\operatorname{Tr}\left(C_i^{2^{i-1}}\right)\right|^{2^{p-i}}, \quad 1 \leq i \leq p. \end{aligned}$$

We have

$$\begin{aligned} u_i &= \left|\operatorname{Tr}\left(B_i^{2^{i-1}}\right)\right|^{2^{p-i}} \\ &= \Big|\operatorname{Tr}\Big[\left(A_{2^{p-i}} \cdots A_1^2 \cdots A_{2^{p-i}}\right)\left(A_{2^{p-i}} \cdots A_1^2 \cdots A_{2^{p-i}}\right) \cdots \\ &\qquad \left(A_{2^{p-i}} \cdots A_1^2 \cdots A_{2^{p-i}}\right)\Big]\Big|^{2^{p-i}} \qquad \text{(with } 2^{i-1} \text{ bracketed factors)} \\ &= \Big|\operatorname{Tr}\Big[\left(A_{2^{p-i-1}} \cdots A_1^2 \cdots A_{2^{p-i-1}}\right)\left(A_{2^{p-i-1}+1} \cdots A_{2^{p-i}}^2 \cdots A_{2^{p-i-1}+1}\right) \cdots \\ &\qquad \left(A_{2^{p-i-1}} \cdots A_1^2 \cdots A_{2^{p-i-1}}\right)\left(A_{2^{p-i-1}+1} \cdots A_{2^{p-i}}^2 \cdots A_{2^{p-i-1}+1}\right)\Big]\Big|^{2^{p-i}} \\ &\qquad \text{(by Lemma 1)} \qquad (2^i \text{ factors}) \\ &= \left|\operatorname{Tr}(B_{i+1} C_{i+1} \cdots B_{i+1} C_{i+1})\right|^{2^{p-i}}, \quad (i \leq p-1) \qquad (2^{i-1} \text{ factors } B_{i+1} C_{i+1}) \\ &\leq \left|\left[\prod_{j=1}^{2^{i-1}} \left\{\operatorname{Tr}\left(B_{i+1}^{2^i}\right) \operatorname{Tr}\left(C_{i+1}^{2^i}\right)\right\}^{\frac{1}{2^i}}\right]\right|^{2^{p-i}} \qquad \text{(by inductive hypothesis)} \\ &= \left|\operatorname{Tr}\left(B_{i+1}^{2^i}\right)\right|^{2^{p-i-1}} \cdot \left|\operatorname{Tr}\left(C_{i+1}^{2^i}\right)\right|^{2^{p-i-1}} \\ &\leq u_{i+1} \cdot v_{i+1} \quad (1 \leq i < p) \end{aligned}$$

that is, $u_i \leq u_{i+1} \cdot v_{i+1}, \ 1 \leq i < p$.

Since

$$u_{p-1} = \left|\operatorname{Tr}\left(B_{p-1}^{2^{p-2}}\right)\right|^2 = \left|\operatorname{Tr}\left\{\left(A_2 A_1^2 A_2\right)^{2^{p-2}}\right\}\right|^2$$

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$$= \left|\operatorname{Tr}\left(A_1^2A_2^2A_1^2A_2^2\cdots A_1^2A_2^2\right)\right|^2 \qquad \text{(by Lemma 1)} \qquad (2^{p-2} \text{ factors } A_1^2A_2^2)$$

$$\leq \left|\prod_{i=1}^{2^{p-2}}\left[\operatorname{Tr}\left\{\left(A_1^2\right)^{2^{p-1}}\right\}\cdot\operatorname{Tr}\left\{\left(A_2^2\right)^{2^{p-1}}\right\}\right]^{\frac{1}{2^{p-1}}}\right|^2 \qquad \text{(by inductive hypothesis)}$$

$$= \operatorname{Tr}\left(A_1^{2^p}\right)\operatorname{Tr}\left(A_2^{2^p}\right),$$

and

$$v_{p-1} = \left|\operatorname{Tr}\left(C_{p-1}^{2^{p-2}}\right)\right|^2 = \left|\operatorname{Tr}\left(A_3A_4^2A_3^2A_4^2\cdots A_4^2A_3\right)\right|^2$$

$$= \left|\operatorname{Tr}\left(A_3^2A_4^2\cdots A_3^2A_4^2\right)\right|^2 \qquad \text{(by Lemma 1)}$$

$$\leq \left|\prod_{i=1}^{2^{p-2}}\left[\operatorname{Tr}\left\{\left(A_3^2\right)^{2^{p-1}}\right\}\cdot\operatorname{Tr}\left\{\left(A_4^2\right)^{2^{p-1}}\right\}\right]^{\frac{1}{2^{p-1}}}\right|^2 \qquad \text{(by inductive hypothesis)}$$

$$= \operatorname{Tr}\left(A_3^{2^p}\right)\operatorname{Tr}\left(A_4^{2^p}\right),$$

we have

$$u_{p-2} \leq u_{p-1}\cdot v_{p-1} \leq \operatorname{Tr}\left(A_1^{2^p}\right)\operatorname{Tr}\left(A_2^{2^p}\right)\operatorname{Tr}\left(A_3^{2^p}\right)\operatorname{Tr}\left(A_4^{2^p}\right).$$

In exactly the same way, we can establish the following inequality.

$$v_{p-2} \leq \prod_{i=5}^{8}\operatorname{Tr}\left(A_i^{2^p}\right)$$

$$u_{p-3} \leq u_{p-2}v_{p-2} \leq \prod_{i=1}^{8}\operatorname{Tr}\left(A_i^{2^p}\right)$$

$$\cdots$$

$$u_1 \leq \prod_{i=1}^{2^{p-1}}\operatorname{Tr}\left(A_i^{2^p}\right)$$

$$v_1 \leq \prod_{i=2^{p-1}+1}^{2^p}\operatorname{Tr}\left(A_i^{2^p}\right).$$

Therefore we obtain

$$\left|\operatorname{Tr}\left(A_1A_2\cdots A_{2^p}\right)\right|^{2^p} = \left|\operatorname{Tr}\left[\left(A_1A_2\cdots A_{2^{p-1}}\right)\left(A_{2^{p-1}+1}\cdots A_{2^p}\right)\right]\right|^{2^p}$$

$$\leq \left|\operatorname{Tr}\left(A_{2^{p-1}}\cdots A_1^2\cdots A_{2^{p-1}}\right)\cdot\operatorname{Tr}\left(A_{2^{p-1}+1}\cdots A_{2^p}^2\cdots A_{2^{p-1}+1}\right)\right|^{2^{p-1}} \qquad \text{(by Lemma 2)}$$

$$= u_1\cdot v_1 \leq \prod_{i=1}^{2^p}\operatorname{Tr}\left(A_i^{2^p}\right).$$

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Finally, we present an open question:

Let A_i $(1 \le i \le m)$ be positive definite operators in C_1. Does the following inequality hold:

$$\left|\operatorname{Tr}(A_1 A_2 \cdots A_m)\right|^m \le \prod_{i=1}^{m} \left[\operatorname{Tr}\left(A_i^m\right)\right] \quad ?$$

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