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REMARK ON IRREDUCIBLE FACTORIZATIONS OF POLYNOMIALS IN SEVERAL VARIABLES*

BY

LEIBA RODMAN

ABSTRACT. It is proved that a polynomial on several complex variables, whose coefficients depend analytically on a parameter ε, admits a factorization which is irreducible for every value of the parameter, with the possible exception of an analytic set of points. Moreover, the coefficients of the irreducible factors can be chosen to depend analytically on ε in a neighborhood of every point not belonging to this analytic set.

Let $\varphi(x)$ be a monic polynomial on the complex variable x with complex coefficients. It is well known that the zeroes of $\varphi(x)$ are analytic functions of its coefficients (with the exception of branching points). In other words, $\varphi(x)$ admits factorization of type

$$\varphi(x)=\prod_{i=1}^{m}(x-x_i),$$

where $x_1,\ldots,x_m$ are analytic functions on the coefficients of $\varphi(x)$, with possible branching points. It is the purpose of this note to extend this observation for the case of polynomials in several complex variables $z_1,\ldots,z_n$. Such polynomial $\varphi(z_1,\ldots,z_n)$ obviously can be factored into a product of irreducible polynomials. A question we deal with here is how the factors of the irreducible factorization depend on the coefficients of $\varphi(z_1,\ldots,z_n)$. It will be convenient for us to assume that the coefficients of $\varphi(z_1,\ldots,z_n)$ are analytic functions of variable $\varepsilon\in\Omega$, where Ω is an analytic manifold. In this case we prove that the coefficients of the factors in an irreducible factorization of $\varphi(z_1,\ldots,z_n)$ depend analytically on ε, provided an analytic set of points in the domain of definition of ε is neglected.

We introduce the necessary definitions and notations: $\mathbb{C}^n$ stands for the n-dimensional complex linear space, $\mathbb{N}_+^n$ stands for the set of all n-tuples $k=(k_1,\ldots,k_n)$, where k_j are non-negative integers ($k\in\mathbb{N}_+^n$ will be referred to

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as multi-index). For a given $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ and given multi-index $k=(k_1,\ldots,k_n)\in\mathbb{N}_+^n$, denote $z^k=\Pi_{i=1}^n z_i^{k_i}$ and $|k|=k_1+\cdots+k_n$. By Ω we denote an analytic manifold.

By an analytic set $V\subset\Omega$ we mean the subset in Ω with the following property: for every $\varepsilon_0\in\Omega$ there exists a neighborhood $U(\varepsilon_0)$ and analytic functions $\{g_{ij}(\varepsilon)\}$, $\varepsilon\in U(\varepsilon_0)$; $i=1,\ldots,p$; $j=1,\ldots,q$, such that

$$U\cap U(\varepsilon_0)=\bigcup_{j=1}^{q}\{\varepsilon\in U(\varepsilon_0)\mid g_{1j}(\varepsilon)=\cdots=g_{pj}(\varepsilon)=0\}.$$

In this note we prove the following theorem.

THEOREM. *Let $\Phi(z,\varepsilon)\not\equiv 0$ be a polynomial in complex variables $z=(z_1,\ldots,z_r)\in\mathbb{C}^n$, with coefficients depending analytically on $\varepsilon\in\Omega$:*

$$\Phi(z,\varepsilon)=\sum_{|k|\le r} q_k(\varepsilon)z^k,$$

where $q_k(\varepsilon)$ depend analytically in $\varepsilon\in\Omega$. Then there exists an analytic set $V\subset\Omega$ and polynomials $\Psi_1(z,\varepsilon),\ldots,\Psi_m(z,\varepsilon)$ of variables $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ such that

$$(1)\qquad \Phi(z,\varepsilon)=\Psi_1(z,\varepsilon),\ldots,\Psi_m(z,\varepsilon),\qquad \varepsilon\in\Omega\backslash V$$

the coefficients of $\Psi_i(z,\varepsilon)$ are analytic in $\Omega\backslash V$, and $\Psi_i(z,\varepsilon)$ is an irreducible polynomial for every $\varepsilon\in\Omega\backslash V$, $i=1,\ldots,m$.

Proof. The set of all polynomials of degree v in n variables can be identified with $\mathbb{C}^{N(v)}$, where $N(v)=\{k=(k_1,\ldots,k_n)\mid k_i$ are non-negative integers and $k_1+\cdots+k_n\le v\}^{\#}$. Let $S_m(v)=\{(v_1,\ldots,v_m)\mid v_i$ are positive integers and $v_1+\cdots+v_m=v\}$. For every m-tuple $(v_1,\ldots,v_m)\in S_m(v)$, the multiplication of polynomials $\varphi_1(z),\ldots,\varphi_m(z)$ with $\deg\varphi_i=v_i$ gives rise to a mapping $\pi:\mathbb{C}^{N(v_1)}\times\cdots\times\mathbb{C}^{N(v_m)}\to\mathbb{C}^{N(v)}$:

$$\pi(\varphi_1(z),\ldots,\varphi_m(z))=\varphi_1(z)\cdots\varphi_m(z).$$

Since π is multilinear and product of non-zero polynomials is again a non-zero polynomial, the restriction of π to non-zero polynomials can be regarded as mapping $\tilde{\pi}:\mathbb{P}^{M(v_1)}\times\cdots\times\mathbb{P}^{M(v_m)}\to\mathbb{P}^{M(v)}$, where $M(w)=N(w)-1$ and $\mathbb{P}^w$ is complex projective space of dimension w. According to well-known result in algebraic geometry (see for instance the proposition on p. 47 in [1]), the image of $\tilde{\pi}$ is an algebraic variety $W=W(v_1,\ldots,v_m)$ in $\mathbb{P}^{M(v)}$.

Now the polynomial $\Phi(z,\varepsilon)$ can be viewed as analytic mapping $\gamma:\Omega\to\mathbb{P}^{M(v)}$. Let $(v_1^0,\ldots,v_m^0)\in S_m(v)$ be such that $W_0=W(v_1^0,\ldots,v_m^0)$ (defined above) is a minimal variety (among all $W(v_1,\ldots,v_m)$ with $(v_1,\ldots,v_m)\in S_m(v)$) containing $\gamma(\Omega)$. For every $(v_1,\ldots,v_m)\in S_m(v)$ such that $W(v_1,\ldots,v_m)\not\supset\gamma(\Omega)$ let $V(v_1,\ldots,v_m)$ be the analytic set of Ω consisting of all $\varepsilon \in \Omega$ with the property that $W(v_1, \ldots, v_m) \ni \gamma(\varepsilon)$. Put

$$V_1 = \bigcup V(v_1, \ldots, v_m),$$

where the union is over all $(v_1, \ldots, v_m) \in S_m(v)$ such that $W(v_1, \ldots, v_m) \not\ni \gamma(\Omega)$.

Consider the mapping

$$\tilde{\pi}_0 : \mathbb{P}^{M(v_1{}^0)} \times \cdots \times \mathbb{P}^{M(v_m{}^0)} \to \mathbb{P}^{M(v)}$$

defined by $v_1^0, \ldots, v_m^0$. For $y \in \operatorname{Im} \tilde{\pi}_0 = \{\tilde{\pi}_0(x) \mid x \in \mathbb{P}^{M(v_1{}^0)} \times \cdots \times \mathbb{P}^{M(v_m{}^0)}\}$, the set $\tilde{\pi}_0^{-1}(y)$ consists of finite number of different points $(y_1^{(i)}, \ldots, y_m^{(i)}) \in \mathbb{P}^{M(w_1{}^0)} \times \cdots \times \mathbb{P}^{M(v_m{}^0)}$, $i = 1, \ldots, r$, such that each point $(y_1^{(i)}, \ldots, y_m^{(i)})$ can be obtained from $(y_1^{(1)}, \ldots, y_m^{(1)})$ by some permutation of the coordinates $y_1^{(1)}, \ldots, y_m^{(1)}$. (This follows from the uniqueness of the factorization of a polynomial in several variables into a product of irreducible factors, up to permutation of the factors and multiplication by a non-zero constant.) The number of preimages $r = \{\tilde{\pi}_0^{-1}(y)\}^{\#}$ may depend on $y \in \operatorname{Im} \tilde{\pi}_0$ (for instance, if $v_1^0 = v_2^0$ and $v_j^0 \neq v_1^0$ for $j = 3, \ldots, m$, then $r = 2$ if $y = \tilde{\pi}_0(y_1^{(1)}, \ldots, y_m^{(1)})$ with $y_2^{(1)} \neq y_2^{(1)}$, and $r = 1$ if $y = \pi_0(y_1^{(1)}, \ldots, y_m^{(1)})$ with $y_1^{(1)} = y_2^{(1)}$). Let

$$r_0 = \max\{r(y) \mid y \in \gamma(\Omega)\},$$

and let $V_2 = \{\varepsilon \in \Omega \mid r(\gamma(\varepsilon)) < r_0\}$. Using once more the proposition on p. 47 in [1], it is easy to see that V_2 is an analytic set.

Consider $\Phi(z, \varepsilon)$ for $\varepsilon \in \Omega \setminus V_3$, where $V_3 = V_1 \cup V_2 \cup \{\varepsilon_0 \in \Omega \mid \Phi(z, \varepsilon_0) \equiv 0\}$ is an analytic set; then $\Phi(z, \varepsilon)$ can be regarded as analytic mapping $\Phi : \Omega \setminus V_3 \to \mathbb{C}^{N(v)} \setminus \{0\}$, so $\gamma = \sigma_{N(v)} \circ \Phi$, where $\sigma_w : \mathbb{C}^w \setminus \{0\} \to \mathbb{P}^{w-1}$ is the natural mapping.

Consider the r_0-valued mapping

$$\tilde{\pi}_0^{-1}\gamma : \Omega \setminus V_3 \to \mathbb{P}^{M(v_1{}^0)} \times \cdots \times \mathbb{P}^{M(v_m{}^0)}.$$

We choose a certain value of $\tilde{\pi}_0^{-1}\gamma$ in such a way that $\tilde{\pi}_0^{-1}\gamma$ becomes analytic (this can be done by using the local description of the graph of $\tilde{\pi}_0^{-1}\gamma$; see, for instance, pp. 93–100 in [2]).

Pick $\varepsilon_0 \in \Omega \setminus V_3$; let $k(i)$ $(1 \leq k(i) \leq N(v_i^0))$ be a non-zero coordinate in $\rho_i(\tilde{\pi}_0^{-1}\gamma(\varepsilon_0))$, where $\rho_i : \mathbb{P}^{M(v_1{}^0)} \times \cdots \times \mathbb{P}^{M(v_m{}^0)} \to \mathbb{P}^{M(v_i{}^0)}$ be the projection on the ith component; $i = 1, \ldots, m$. Let $V_{4,i} = \{\varepsilon \in \Omega \mid \gamma(\varepsilon) = \tilde{\pi}_0(y)$ for some $y \in \mathbb{P}^{M(v_1{}^0)} \times \cdots \times \mathbb{P}^{M(v_m{}^0)}$ such that the $k(i)$th coordinate of $\rho_i(y)$ is zero$\}$. Again, $V_{4,i}$ is an analytic set in Ω. Let $V_5 = V_3 \cup \bigcup_{i=1}^m V_{4i}$. Let

$Q_i = \{y \in \mathbb{P}^{M(v_i{}^0)} \mid k(i)$th coordinate of y is non-zero$\}$; define $\tau_i : Q_i \to \mathbb{C}^{N(v_i{}^0)}$ by the formula

$$\tau_i(y_1, \ldots, y_{M(v_i{}^0)}) = \left(\frac{y_1}{y_{k(i)}}, \frac{y_2}{y_{k(i)}}, \ldots, \frac{y_{M(v_i{}^0)}}{y_{k(i)}}\right).$$

For $i=1,\ldots,m$, put $\tilde{\Psi}_i(z,\varepsilon)=\tau_i\rho_i\tilde{\pi}_0^{-1}\gamma(\varepsilon)$, $\varepsilon\in\Omega\backslash V_5$. (Here we identify the set of all polynomials of degree v_i^0 in n variables with $\mathbb{C}^{N(v_i^0)}$.) From this construction it is clear that the coefficients of $\tilde{\Psi}_i(z,\varepsilon)$ are analytic functions on $\Omega\backslash V_5$, and

$$\Phi(z,\varepsilon)=a(\varepsilon)\tilde{\Psi}_1(z,\varepsilon),\ldots,\tilde{\Psi}_m(z,\varepsilon), \tag{2}$$

where $a(\varepsilon)$ is a non-zero function (which is necessarily analytic on $\Omega\backslash V_5$). From the choice of $(v_1^0,\ldots,v_m^0)$, and since $V_2\subset V_5$, it follows that factorization (2) is irreducible for every $\varepsilon\in\Omega\backslash V_5$. So formula (1) follows.

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REFERENCES

1. I. R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, New York–Heidelberg–Berlin, 1974.

2. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

DEPARTMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY OF CALGARY
2500 UNIVERSITY DRIVE N.W.
CALGARY, ALBERTA T2N 1N4, CANADA