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ON THE ACTION OF THE SPORADIC SIMPLE BABY MONSTER GROUP ON ITS CONJUGACY CLASS 2B

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Abstract

We determine the character table of the endomorphism ring of the permutation module associated with the multiplicity-free action of the sporadic simple Baby Monster group $\mathbb{B}$ on its conjugacy class 2B, where the centraliser of a 2B-element is a maximal subgroup of shape $2^{1+22}.\mathrm{Co}_2$. This is one of the first applications of a new general computational technique to enumerate big orbits.

1. *Introduction*

The aim of the present work is to determine the character table of the endomorphism ring of the permutation module associated with the multiplicity-free action of the Baby Monster group $\mathbb{B}$, the second largest of the sporadic simple groups, on its conjugacy class 2B, where the centraliser of a 2B-element is a maximal subgroup of shape $2^{1+22}.\mathrm{Co}_2$. The final result is given in Table 4.

In general, the endomorphism ring of a permutation module reflects aspects of the representation theory of the underlying group. Its character table in particular encodes information about the spectral properties of the orbital graphs associated with the permutation action, such as distance-transitivity or distance-regularity, see [**14, 5**], or the Ramanujan property, see [**7**]. Here multiplicity-free actions, that is those whose associated endomorphism ring is commutative, have been of particular interest; for example a distance-transitive graph necessarily is an orbital graph associated with a multiplicity-free action.

The multiplicity-free permutation actions of the sporadic simple groups and the related almost quasi-simple groups have been classified in [**3, 16, 2**], and the associated character tables including the one presented here have been collected in [**4, 20**]. In particular, the Baby Monster group $\mathbb{B}$ has exactly four multiplicity-free actions. In order of increasing degree these are the actions on the cosets of a maximal subgroup of shape $2.{}^2E_6(2).2$, on the cosets of a subgroup of shape $2.{}^2E_6(2)$ which is of index 2 in $2.{}^2E_6(2).2$, on the cosets of a maximal subgroup of shape $2^{1+22}.\mathrm{Co}_2$, and on the cosets of a maximal subgroup isomorphic to the sporadic simple Fischer group Fi_{23}.

The character tables associated with the first two $\mathbb{B}$-actions have been determined in [**10**], while the remaining ones have already been computed in [**21**]. For the $\mathbb{B}$-action on the cosets of $2^{1+22}.\mathrm{Co}_2$, the sizes of the $(2^{1+22}.\mathrm{Co}_2)$-orbits are already given in [**14**], up to a typo we are going to correct first. Moreover, the intersection

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matrix of the shortest non-trivial ($2^{1+22}.\mathrm{Co}_2$)-orbit, given in Table 3, has been determined independently in [26, 27], using a 'by hand' strategy exploiting geometric arguments which yield a wealth of combinatorial data about the associated orbital graph.

Here we pursue a computational strategy aiming straightforwardly at determining intersection matrices. Due to the sheer size of the permutation domains underlying the larger two $\mathbb{B}$-actions, a new general computational technique to handle these has been devised in [21]. This technique has been elaborated and analysed fully in [23], and has now been incorporated into the GAP [9] package ORB [22]. Moreover, in [23] it is also reported on the computations concerned with the $\mathbb{B}$-action on the cosets of Fi_{23}, and in particular on the relation of this action to the conjugation action of the sporadic simple Monster group on its 6-transpositions. The aim of the present paper is to report on the computations concerned with the $\mathbb{B}$-action on the cosets of $2^{1+22}.\mathrm{Co}_2$, completing the picture for the multiplicity-free actions of $\mathbb{B}$.

The present paper is organised as follows. In Section 2 we recall the necessary facts about permutation modules, endomorphism rings and their character tables. In Section 3 we give a rough outline of the orbit enumeration technique applied, in particular explaining which input data has to be provided. In Section 4 we specify the data needed for the action of $\mathbb{B}$ on the cosets of $2^{1+22}.\mathrm{Co}_2$, and show how the results of orbit enumerations are actually used to determine the character table associated with this action.

2. *Endomorphism rings and their character tables*

We recall the necessary facts about permutation modules and their endomorphism rings; as general references see [32, 1].

2.1.
Let G be a finite group, let $H \leqslant G$ and let $n := [G\colon H]$. Let $X \neq \emptyset$ be a transitive G-set such that $\mathrm{Stab}_G(x_1) = H$, for some $x_1 \in X$, hence we have $n = |X|$. Let $X = \coprod_{i=1}^r X_i$ be its decomposition into H-orbits, where $r \in \mathbb{N}$ is called the *rank* of X. For all $i \in \{1, \ldots, r\}$ we choose $x_i \in X_i$ and $g_i \in G$ such that $x_1 g_i = x_i$, where we assume $g_1 = 1$ and $X_1 = \{x_1\}$, and we let $H_i := \mathrm{Stab}_H(x_i) \leqslant H$ and $k_i := |X_i| = \frac{|H|}{|H_i|}$.

For $i \in \{1, \ldots, r\}$, the orbits $\Gamma_i := [x_1, x_i]G \subseteq X \times X$ of the diagonal action of G on $X \times X$ are called *orbitals*. If $i^* \in \{1, \ldots, r\}$ is defined by $\Gamma_{i^*} = [x_i, x_1]G \subseteq X \times X$, then $X_{i^*} \subseteq X$ is called the H-orbit *paired* to X_i; in particular we have $k_{i^*} = k_i$. Let $A_i = [a_{i,x,y}] \in \{0,1\}^{n \times n}$, with row index $x \in X$ and column index $y \in X$, be defined by $a_{i,x,y} = 1$ if and only if $[x, y] \in \Gamma_i$.

Let $\mathbb{Z}X$ be the permutation $\mathbb{Z}G$-module associated with the G-set X, and let $E := \mathrm{End}_{\mathbb{Z}G}(\mathbb{Z}X)$ be its endomorphism ring. By [28], see also [15, Ch.II.12], the set $\{A_i; i \in \{1, \ldots, r\}\} \subseteq E$ is a basis of E, called its *Schur basis*. It can also be considered as a basis of $E_{\mathbb{C}} := E \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{End}_{\mathbb{C}G}(\mathbb{C}X)$, which is a split semisimple $\mathbb{C}$-algebra. Moreover, E is commutative if and only if the permutation character $1_H^G \in \mathbb{Z}\mathrm{Irr}(G)$ associated with the G-set X is *multiplicity-free*, that is all the constituents of 1_H^G occur with multiplicity 1, where $\mathrm{Irr}(G)$ denotes the set of irreducible $\mathbb{C}$-valued characters of G.

For $i \in \{1, \dots, r\}$ let $P_i = [p_{h,i,j}] \in \mathbb{Z}^{r \times r}$, with row index $h \in \{1, \dots, r\}$ and column index $j \in \{1, \dots, r\}$, be the representing matrix of A_i for its right regular action on E, with respect to the Schur basis, thus we have $A_h A_i = \sum_{j=1}^{r} p_{h,i,j} A_j$. Hence $E \to \mathbb{Z}^{r \times r}$: $A_i \mapsto P_i$, for $i \in \{1, \dots, r\}$, is a faithful representation of E. The matrices P_i, whose entries are given as $p_{h,i,j} = |X_h \cap X_{i^*} g_j| \in \mathbb{N}_0$, are called *intersection matrices*.

The first row and the first column of P_i are given as $p_{1,i,j} = \delta_{i,j}$ and $p_{h,i,1} = k_h \delta_{h,i^*}$, where $\delta_{\cdot,\cdot} \in \{0, 1\}$ denotes the Kronecker function, and the column sums of P_i are given as $\sum_{h=1}^{r} p_{h,i,j} = \sum_{h=1}^{r} |X_h \cap X_{i^*} g_j| = k_i$, for all $j \in \{1, \dots, r\}$. Moreover, we have $k_j \cdot |X_h \cap X_{i^*} g_j| = k_h \cdot |X_j \cap X_i g_h|$, implying $k_j p_{h,i,j} = k_h p_{j,i^*,h}$. Thus from $\sum_{j=1}^{r} |X_j \cap X_i g_h| = k_i$ depending on $h \in \{1, \dots, r\}$ we get the weighted row sums of P_i as $\sum_{j=1}^{r} k_j p_{h,i,j} = k_h k_i$.

2.2.

From now on suppose E is commutative. Letting $\mathrm{Irr}(E)$ be the set of irreducible $\mathbb{C}$-valued characters of $E_{\mathbb{C}}$, we have $|\mathrm{Irr}(E)| = r$, and $\lambda(A_1) = 1$ for all $\lambda \in \mathrm{Irr}(E)$. The *character table* of E is defined as the matrix $\Phi_E := [\lambda(A_i)] \in \mathbb{C}^{r \times r}$, with row index $\lambda \in \mathrm{Irr}(E)$ and column index $i \in \{1, \dots, r\}$; hence in particular Φ_E is invertible. There is a natural bijection, called the *Fitting correspondence*, between $\mathrm{Irr}(E)$ and the constituents of 1_H^G; the Fitting correspondent of $\lambda \in \mathrm{Irr}(E)$ is denoted by $\chi_\lambda \in \mathrm{Irr}(G)$. We have $\frac{n}{\chi_\lambda(1)} = \sum_{i=1}^{r} \frac{\|\lambda(A_i)\|^2}{k_i}$, where $\|\cdot\|$ denotes the complex absolute value; thus degrees of Fitting correspondents are easily computed from Φ_E.

Let $\mathbb{Q} \subseteq K$ be the algebraic number field generated by the character values $\{\chi_\lambda(g) \in \mathbb{C}; \lambda \in \mathrm{Irr}(E), g \in G\}$. Hence by [8, La.IV.9.1] the χ_λ are realisable over K. Thus by Schur's Lemma the $A_i \in E$ are simultaneously diagonalisable over K. Hence K is a splitting field of E, the eigenvalues of A_i are the character values $\lambda(A_i)$, which are algebraic integers in K, and we have $\Phi_E \in K^{r \times r}$.

The character table Φ_E and the intersection matrices P_i are related as follows. If Φ_E is given, we have $P_i = \Phi_E^{\mathrm{tr}} \cdot \mathrm{diag}[\lambda(A_i); \lambda \in \mathrm{Irr}(E)] \cdot \Phi_E^{-\mathrm{tr}}$, where $\mathrm{diag}[\cdot] \in \mathbb{C}^{r \times r}$ denotes the diagonal matrix having the indicated entries. Hence the P_i are easily computed from Φ_E. Conversely, if all the P_i are given, the set of rows $\{[\lambda(A_1), \dots, \lambda(A_r)] \in \mathbb{C}^r; \lambda \in \mathrm{Irr}(E)\}$ of Φ_E is the unique basis of $\mathbb{C}^r$ consisting of simultaneous row eigenvectors of all the $P_i^{\mathrm{tr}} \in \mathbb{C}^{r \times r}$ and being normalised to have 1 as their first entry. Hence Φ_E can already be determined from a subset of the P_i^{tr}, as soon as the associated set of simultaneous normalised row eigenvectors is uniquely determined. Actually, we will pursue the extreme strategy to compute Φ_E from a single non-identity intersection matrix.

3. *Enumeration of big orbits*

To handle a finite G-set X, where G is a finite group acting from the right, using standard orbit enumeration techniques, see [11], every point in X eventually has to

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be stored. If X is too big to be stored completely, this is no longer feasible. We give a rough outline of the new orbit enumeration technique remedying this; for more details see [**21**, **23**, **22**].

3.1.
The basic idea, invented independently in [**24**, **17**], is not to store single points in X, but to enumerate X by enumerating the U-orbits contained in X, where $U \leqslant G$ is a suitable *helper subgroup*, and only storing suitable representatives of each U-orbit. To this end, let Y be another finite U-set admitting a homomorphism of U-sets $\overline{}\colon X \to Y$. The most common case for this setting is that $X \subseteq M$, where M is an FG-module for some field F, such that there is an FU-module homomorphism $\pi\colon M_U \to M'$, where M_U is the restriction of M to U and M' is a suitable FU-module, such that we may let $Y := X^{\pi} \subseteq M'$ and let $\overline{}$ be the restriction of π to $X \subseteq M$.

Now, for any U-orbit in Y we arbitrarily designate a *U-minimal* point in it, and a point $x \in X$ is called *U-minimal* if $\overline{x} \in Y$ is U-minimal. To enumerate X we only store the U-minimal points in X. More precisely, to perform an orbit-stabiliser algorithm for a G-orbit $x_1G \subseteq X$, in a way eventually facilitating iteration in 3.2, we devise the following procedures. For any point $x \in X$ the procedure $\mathsf{Minimaliser}_U(\cdot)$ computes an element $u \in U$ such that $xu \in X$ is U-minimal, and for any U-minimal point $x \in X$ the procedure $\mathsf{BarStabiliser}_U(\cdot)$ computes $\mathrm{Stab}_U(\overline{x}) \leqslant U$ and its order. These are used as follows.

Given a point $x' \in X$, applying $u := \mathsf{Minimaliser}_U(x') \in U$ yields the U-minimal point $x := x'u \in X$. Hence by looking up whether x has already been stored, we decide whether the U-orbit $xU = x'U \subseteq X$ has been encountered earlier. If xU is a new U-orbit, the U-minimal points in xU and the stabiliser $\mathrm{Stab}_U(x) \leqslant U$ are computed by a standard orbit-stabiliser algorithm using $\mathrm{Stab}_U(\overline{x}) = \mathsf{BarStabiliser}_U(x) \leqslant U$. If xU has been touched upon before, we collect a Schreier generator of $\mathrm{Stab}_G(x_1) \leqslant G$.

To perform this we assume that orders of subgroups of G, given by sets of generators, can be determined, for example by using a suitable permutation representation of G. Moreover, the $\mathrm{Stab}_U(\overline{x})$-orbits occurring have to be small enough to be enumerable by a standard orbit-stabiliser algorithm.

The helper subgroup $U \leqslant G$ is chosen optimally if it only has regular orbits in Y. In this case, storing only the U-minimal points in X, compared to storing all points in X, yields a memory saving factor of $\sim |U|$, and since for enumeration the generators of G essentially have to be applied to the U-minimal points only we also get a time saving factor of $\sim |U|$; moreover, we have $\mathrm{Stab}_U(\overline{x}) = \{1\}$ for all $x \in X$, hence the $\mathrm{Stab}_U(\overline{x})$-orbits in X are as small as possible anyway.

Typically Y cannot be chosen to consist of regular U-orbits only, but just to have many U-orbits $yU \subseteq Y$ such that $|\mathrm{Stab}_U(y)|$ is small. These U-sets in practice turn out to be very effective as well, in particular if we are content with enumerating only the usually large part of X consisting of those U-orbits $xU \subseteq X$ such that $|\mathrm{Stab}_U(\overline{x})|$ is small.

3.2.

The idea in [**21, 23**] now is to iterate the helper subgroup trick. Let $V \leqslant U \leqslant G$ be helper subgroups, such that the index $[U\colon V]$ is small enough such that a left transversal $\mathcal{L}$ of V in U can be computed explicitly. Moreover, let Z be a V-set, let $\widetilde{\ }\colon Y \to Z$ be a homomorphism of V-sets, and assume that we already given procedures $\mathsf{Minimaliser}_V(\cdot)$ and $\mathsf{BarStabiliser}_V(\cdot)$ with respect to the map $\widetilde{\ }$.

Hence the U-orbits in Y can be enumerated by V-orbits, and we have a notion of V-minimal points in Y. For any U-orbit in Y we designate a *U-minimal* point $y \in Y$ amongst the V-minimal points in it, and still a point $x \in X$ is called *U-minimal* if $\overline{x} \in Y$ is U-minimal. Moreover, for any V-minimal point $y' \in yU \setminus yV$ we store an element $u \in \mathcal{L} \subseteq U$ such that $y'u \in yV \subseteq Y$, and for any V-minimal point $y' \in yV \subseteq Y$ we store an element $v \in \mathrm{Stab}_V(\widetilde{y}) = \mathsf{BarStabiliser}_V(y) \leqslant V$ such that $y'v = y \in Y$ is the U-minimal point in yU. With these preparations done, we are able to devise procedures $\mathsf{Minimaliser}_U(x)$ and $\mathsf{BarStabiliser}_U(x)$ with respect to the map $\overline{\ }$.

Given a point $x \in X$, let $\overline{x}U = yU \subseteq Y$, where $y \in Y$ is the U-minimal point in yU. Let $v' := \mathsf{Minimaliser}_V(\overline{x}) \in V$, hence $y' := \overline{x}v' \in yU \subseteq Y$ is V-minimal. Thus we have stored an element $u \in \mathcal{L} \subseteq U$ such that $y'' := y'u \in yV \subseteq Y$. Let $v'' := \mathsf{Minimaliser}_V(y'') \in V$, hence $y''' := y''v'' \in yV \subseteq Y$ is V-minimal. Thus we have stored an element $v \in \mathrm{Stab}_V(\widetilde{y'''}) = \mathsf{BarStabiliser}_V(y''') \leqslant V$ such that $y'''v = y \in Y$. Hence in conclusion we have $\overline{x}v'uv''v = y \in Y$ being U-minimal, and we let $\mathsf{Minimaliser}_U(x) := v'uv''v \in U$. Finally, if $x \in X$ already is U-minimal, then $y := \overline{x} \in Y$ is U-minimal as well, hence $\mathsf{BarStabiliser}_U(x) = \mathrm{Stab}_U(y) \leqslant U$ is found by enumerating the U-orbit $yU \subseteq Y$ by V-orbits.

3.3.

Hence this may be iterated along chains $\{1\} =: U_0 \leqslant U_1 \leqslant U_2 \leqslant \cdots \leqslant U_k \leqslant U_{k+1} := G$ of helper subgroups, for some $k \in \mathbb{N}$, admitting U_i-sets Y_i and homomorphisms of U_i-sets $Y_{i+1} \to Y_i$, for $i \in \{1, \ldots, k\}$, where we let $Y_{k+1} := X$. Here, while $[G\colon U_k]$ is allowed to be arbitrary, we assume that all the indices $[U_i\colon U_{i-1}]$, for $i \in \{1, \ldots, k\}$, are small enough such that left transversals of U_{i-1} in U_i can be computed explicitly.

Letting Y_0 be the singleton U_0-set, each point in Y_1 is U_0-minimal anyway, and $\mathsf{Minimaliser}_{U_0}(\cdot)$ and $\mathsf{BarStabiliser}_{U_0}(\cdot)$ are trivial procedures always returning $1 \in U_0$ and $\{1\} \leqslant U_0$, respectively. Hence we may proceed by induction along the subgroup chain as described in 3.2. Again the most common case is that $Y_i \subseteq M_i$, for $i \in \{1, \ldots, k+1\}$, where M_i is an FU_i-module for some field F, such that the homomorphisms of U_i-sets $Y_{i+1} \to Y_i$ are restrictions of FU_i-module homomorphisms $\pi_i\colon (M_{i+1})_{U_i} \to M_i$.

Note that, for example if we already know the sizes of the G-orbits in X, we might want to restrict ourselves to a simple orbit algorithm for the G-set X without determining stabilisers in G. In this case, stabiliser computations only take place in U_k, hence we only have to be able to determine orders of subgroups of U_k, which can be done for example by specifying a suitable permutation representation of U_k only, or just by sifting through the subgroup chain using the left transversals available anyway.

Table 1: Conjugacy classes in G and H-orbits.

i	C	k_C	splits into	$\dim_{\mathbb{F}_2}(\mathrm{Fix}_M(\cdot))$
1	1A	1		
2, 3	2B	7 379 550	93 150 + 7 286 400	2 322
4	2D	262 310 400		2 202
6	3A	9 646 899 200		
5	4B	4 196 966 400		1 256
8	4E	537 211 699 200		1 114
7	4G	470 060 236 800		1 166
9	5A	4 000 762 036 224		
10	6C	6 685 301 145 600		

4. *Determining the character table*

We are now prepared to consider the action of the Baby Monster group $\mathbb{B}$ on the cosets of $2^{1+22}.\mathrm{Co}_2$. The group theoretical and representation theoretic data concerning the groups involved is available in [**6**] and [**13**], and also accessible in the character table library of GAP. Computations with characters and with permutation and matrix representations are done with GAP and the MeatAxe [**25**], in particular we make use of the algorithms to compute submodule lattices described in [**18**], and those to compute socle series described in [**19**].

4.1.

From now on let $G = \mathbb{B}$ and $2^{1+22}.\mathrm{Co}_2 \cong H < G$, and let X be the set of right cosets of H in G. We have $|X| = 11\,707\,448\,673\,375 \sim 1.1 \cdot 10^{13}$, and by [**3**] the permutation character 1_H^G it is multiplicity-free of rank $r = 10$, its constituents have pairwise distinct degrees and hence are $\mathbb{Q}$-valued. The H-orbit sizes k_i, for $i \in \{1, \ldots, 10\}$, are stated without explicit proof in [**14**], where unfortunately the values given there do not sum up to $|X|$. Hence we just compute the k_i anew.

Using the notation in [**6**], let $2\mathrm{B} \subseteq G$ denote the associated conjugacy class in G, and picking $c \in 2\mathrm{B}$ suitably we have $H = C_G(c)$. Hence the conjugation action of G on 2B is equivalent to its action on X. For any conjugacy class $C \subseteq G$ in G let $(2\mathrm{B})_C := \{d \in 2\mathrm{B}; cd \in C\}$. Hence $(2\mathrm{B})_C \subseteq 2\mathrm{B}$ is a union of H-orbits with respect to the conjugation action. We have $k_C := |(2\mathrm{B})_C| = \frac{|C| \cdot m_{2\mathrm{B},2\mathrm{B},C}}{|2\mathrm{B}|} \in \mathbb{N}_0$, where $m_{2\mathrm{B},2\mathrm{B},C} := |\{(c,d) \in 2\mathrm{B} \times 2\mathrm{B}; cd = e\}| \in \mathbb{N}_0$ is the corresponding class multiplication coefficient and $e \in C$ is fixed. The class multiplication coefficients are easily determined from the character table of G, and we find $k_C \neq 0$ precisely for the conjugacy classes $C \in \{1\mathrm{A}, 2\mathrm{B}, 2\mathrm{D}, 3\mathrm{A}, 4\mathrm{B}, 4\mathrm{E}, 4\mathrm{G}, 5\mathrm{A}, 6\mathrm{C}\}$, where the associated sizes k_C are given in Table 1.

As we have $r = 10$, but only find nine conjugacy classes $C \subseteq G$ such that $k_C \neq 0$, we conclude that precisely one of the non-empty sets $(2\mathrm{B})_C \subseteq 2\mathrm{B}$ consists of two H-orbits, while the others each consist of a single H-orbit. As $k_{2\mathrm{B}}$ is the only of the $k_C \neq 0$ not dividing $|H|$, we conclude that $(2\mathrm{B})_{2\mathrm{B}}$ splits into two H-orbits.

Table 2: The subgroup chain.

i	U_i	$\lvert U_i \rvert$	$[U_i : U_{i-1}]$
5	$\mathbb{B}$	4 154 781 481 226 426 191 177 580 544 000 000	$\sim 1.1 \cdot 10^{13}$
4	$2^{1+22}.\mathrm{Co}_2$	354 883 595 661 213 696 000	$\sim 3.9 \cdot 10^{11}$
3	$2^{11}.M_{22}$	908 328 960	1 024
2	$2.M_{22}$	887 040	1 344
1	$L_2(11)$	660	660

The sizes of the latter are also indicated in Table 1, and are determined in 4.4. After all, it turns out that in [**14**] the value of $k_7 = k_{4G}$ is erroneously stated as '4 700 602 368', obviously just a typo.

4.2.

In order to to place ourselves into the situation described in Section 3, we look for an FG-module containing an H-invariant but not G-invariant vector. Let $\mathbb{F}_2$ be the field of order 2, and let M be the absolutely irreducible $\mathbb{F}_2G$-module of dimension 4370; by [**12**] this is the smallest faithful representation of G over fields of characteristic 2. Representing matrices for standard generators of G, in the sense of [**29**], have been constructed in [**30**] and are available in [**31**], where also words in the standard generators giving generators for H are available. Using a random search, from the latter we find generators of H being preimages of standard generators of the sporadic simple Conway group Co_2, with respect to the natural epimorphism $H \to \mathrm{Co}_2$.

We find that the subspace $\mathrm{Fix}_H(M) \leqslant M$, consisting of the vectors fixed by H, is 1-dimensional. Thus picking the non-trivial vector $0 \neq x_1 \in \mathrm{Fix}_H(M)$, the G-orbit $x_1G \subseteq M$ is as a G-set equivalent to X, and hence we may identify x_1G and X. Note that to store a vector in M we need $\lceil \frac{4370}{8} \rceil = 547$ Bytes, thus to store all of X we would need $6\,403\,974\,424\,336\,125 \sim 6.4 \cdot 10^{15}$ Bytes. Hence we are indeed tempted to apply a better strategy.

4.3.

We choose the following chain of subgroups, see Table 2:

$$G = \mathbb{B} > H = 2^{1+22}.\mathrm{Co}_2 > U_3 := 2^{11}.M_{22} > U_2 := 2.M_{22} > U_1 := L_2(11).$$

Generators of U_i, for $i \in \{1, \ldots, 3\}$, are found as follows. Words in the standard generators of Co_2 giving standard generators of the maximal subgroup $M_{23} < \mathrm{Co}_2$, and words in the latter giving standard generators of the maximal subgroup $M_{22} < M_{23}$ are available in [**31**]. Applying these to the chosen generators of H indeed yields a subgroup $2^{1+22}.M_{22} < H$. Let $2^{1+22} \cong N \trianglelefteq H$ be the maximal normal 2-subgroup of H. Hence N is an extraspecial group, and Co_2 acts absolutely irreducibly on the $\mathbb{F}_2$-vector space $N/Z(N)$ of dimension 22. It turns out that $(N/Z(N))_{M_{22}}$ is an uniserial $\mathbb{F}_2M_{22}$-module with ascending composition series $[1a, 10a, 10b, 1a]$, where

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the constituents are absolutely irreducible $\mathbb{F}_2M_{22}$-modules having the indicated dimensions.

By a random search we find a subgroup $U_3 := 2^{11}.M_{22} < 2^{1+22}.M_{22}$, whose maximal normal 2-subgroup is as an $\mathbb{F}_2M_{22}$-module isomorphic to the unique submodule of $(N/Z(N))_{M_{22}}$ of dimension 11. Similarly, we find a subgroup $U_2 := 2.M_{22} < 2^{11}.M_{22} = U_3$, being a non-split central extension of M_{22}. Finally, words in the standard generators of M_{22} giving standard generators of the maximal subgroup $L_2(11) < M_{22}$ are available in [**31**], and applying these straightforwardly yields a subgroup $U_1 := L_2(11) < 2.M_{22} = U_2$.

To specify $\mathbb{F}_2U_i$-modules M_i, for $i \in \{1,\dots,3\}$, we proceed as follows. Let $M_4 := M$ be the absolutely irreducible $\mathbb{F}_2G$-module of dimension 4370. Letting $\text{rad}^5(M_{U_3}) < M_{U_3}$ be the fifth layer of the radical series of the restriction M_{U_3} of M to $\mathbb{F}_2U_3$, we first find a suitable quotient M_3 of $M_{U_3}/\text{rad}^5(M_{U_3})$ of dimension 78. It is easy then to find suitable quotients M_2 of $(M_3)_{U_2}$, and M_1 of $(M_2)_{U_1}$, having dimensions 31 and 21, respectively. The associated $\mathbb{F}_2U_i$-homomorphisms $\pi_i\colon (M_{i+1})_{U_i} \to M_i$ are just the natural maps.

4.4.
To find H-orbit representatives $x_i \in X_i \subseteq X$ and elements $g_i \in G$ such that $x_i = x_1g_i$, for $i \in \{2,\dots,10\}$, we use the G-set $2\text{B} \subseteq G$ equivalent to X. By a random search we pick a few elements $g \in G$, and check to which conjugacy class in G the commutator $[c,g] := c \cdot (g^{-1}cg) \in G$ belongs, where $c \in 2\text{B}$ is as chosen in 4.1. This is done by computing the order of $[c,g] \in G$, and the dimension of the subspace $\text{Fix}_M([c,g]) \leqslant M$, consisting of the vectors fixed by $[c,g]$; the relevant dimensions are given in Table 1. This yields suitable elements $g_i \in G$ for $i \notin \{3,5\}$; in particular we are lucky to find a representative for the small H-orbit $X_4 \subseteq X$ already at this stage. Summing up the k_i for $i \notin \{3,5\}$, and dividing by $|X|$, we obtain a fraction of $\sim \frac{9996}{10000}$. Hence it is rather improbable to find further H-orbits in X by a random search.

To proceed we concentrate on $X_2 \subseteq X$. If we had $k_2 = 93150$, then there might be an element $d \in (2\text{B})_{2\text{B}} \cap N$, where $N \trianglelefteq H$ is as in 4.3, such that $C_H(d) = 2^{1+21}.(2^{10}\colon M_{22}\colon 2) < H$, where $2^{10}\colon M_{22}\colon 2 < \text{Co}_2$ is a maximal subgroup and $C_H(d) \cap N = 2^{1+21}$. Words in the standard generators of Co_2 giving generators of $2^{10}\colon M_{22}\colon 2 < \text{Co}_2$ are available in [**31**], and it turns out that $(N/Z(N))_{2^{10}\colon M_{22}\colon 2}$ is uniserial with ascending composition series $[1a, 10a, 10b, 1a]$. Applying these words to the chosen generators of H indeed yields a subgroup $2^{1+21}.(2^{10}\colon M_{22}\colon 2) < H$, where the normal subgroup 2^{1+21} is a preimage of the unique submodule of $(N/Z(N))_{2^{10}\colon M_{22}\colon 2}$ of dimension 21, with respect to the natural epimorphism $N \to N/Z(N)$.

Indeed we find a vector $0 \neq x_2 \in \text{Fix}_M(2^{1+21}.(2^{10}\colon M_{22}\colon 2))$ such that $x_2 \neq x_1$. Since $|x_2H| \mid [H\colon (2^{1+21}.(2^{10}\colon M_{22}\colon 2))] = 93150$, it is straightforward to enumerate $x_2H \subseteq M$ completely by a standard orbit algorithm, which shows $|x_2H| = 93150$. Moreover, by applying a few random elements of G we find a point in $x_2G \subseteq M$ being in an H-orbit in X we have encountered earlier, showing that indeed $x_2 \in X \subseteq M$, and hence $X_2 := x_2H \subseteq X$. This also yields $g_2 \in G$ such that $x_1g_2 = x_2$, and proves that $k_2 = 93150$ and $k_3 = 7286400$, as asserted in Table 1.

Finally, by checking a few random points in $X_2 g_2 \subseteq X$, we find representatives of the H-orbits $X_3 \subseteq X$ and $X_5 \subseteq X$.

4.5.

Since $X_1 = \{x_1\}$ and $X_2 \subseteq X$ has already been enumerated explicitly, we consider the H-orbits $X_i \subseteq X$ for $i \in \{3, \ldots, 10\}$. It turns out that $(X_3 \dot\cup X_4)^{\pi_3} = \{0\} \subseteq M_3$, hence for all $x \in X_3 \dot\cup X_4$ we have $\mathrm{Stab}_{U_3}(x^{\pi_3}) = U_3$, rendering orbit enumeration by U_3-orbits ineffective. Hence we do not enumerate $X_3 \subseteq X$ and $X_4 \subseteq X$ at all, and provide an alternative treatment in 4.6. But for $i \in \{5, \ldots, 10\}$ we are prepared to apply the strategy described in 3.3 to enumerate a substantial part of $X_i \subseteq X$.

For example, for the largest H-orbit $X_{10} \subseteq X$, where $k_{10} = 6\,685\,301\,145\,600 \sim 6.7 \cdot 10^{12}$, we enumerate $2\,000\,251\,387\,904 \sim 2 \cdot 10^{12}$ points, hence a fraction of $\sim \frac{3}{10}$ of the whole of X_{10}. These points are comprised into 2603 U_3-orbits, having a total of 4305 U_3-minimal points, hence we obtain a memory saving factor of $\sim 464\,634\,468 \sim 4.6 \cdot 10^8$, which indeed is of the same order of magnitude as $|U_3| = 908\,328\,960 \sim 9.1 \cdot 10^8$. Here, we just ignore those U_3-orbits $xU_3 \subseteq X_{10}$ such that $|\mathrm{Stab}_{U_3}(x^{\pi_3})| \geqslant 500$. To do this using the GAP package ORB we need $\sim 1.3 \cdot 10^9$ Bytes of memory space and ~ 7000 s of CPU time on a 3 GHz Pentium IV processor, where both figures include the time and space required to enumerate the appropriate parts of the helper sets M_i, for $i \in \{1, \ldots, 3\}$.

4.6.

Having the H-orbits $X_i \subseteq X$ under control, the aim now is to compute the intersection matrix $P_2 = [p_{h,2,j}] \in \mathbb{Z}^{10 \times 10}$ for the smallest non-trivial H-orbit $X_2 \subseteq X$, having size $k_2 = 93150$. Since it is the only H-orbit having this size X_2 is self-paired, hence we have $p_{h,2,j} = |X_2 g_j \cap X_h|$. Since we are done for $j = 1$ anyway, for all $j \in \{2, \ldots, 10\}$ we compute $X_2 g_j \subseteq X$ explicitly, and determine which H-orbits $X_h \subseteq X$, for $h \in \{1, \ldots, 10\}$, the various points in $X_2 g_j$ belong to. This is straightforward for $h \in \{1, 2\}$, and for $h \in \{5, \ldots, 10\}$ we proceed as follows.

As we have enumerated only parts but not all of the H-orbits X_h, we not only test a given point $x \in X_2 g_j$ for membership in X_h, but do the same with several points in $xH \subseteq X$. Still, this only allows to prove membership of x in a given X_h, but not to disprove it. Hence we let $h \in \{5, \ldots, 10\}$ vary, and in a first run we test a very few points in $xH \subseteq X$, at most 5 say, for membership in the various H-orbits X_h. If x cannot be proven to belong to a particular H-orbit, we launch a second run where we test some more points in $xH \subseteq X$, at most 1000 say. Now this is done for all $x \in X_2 g_j$, and it turns out that after the second run only a very few points have not been proven to belong to a particular H-orbit, of course in particular including those which actually belong to $X_3 \dot\cup X_4 \subseteq X$.

We could repeat this further by testing even more points, but instead we note that we have already found good lower bounds for the matrix entries $p_{h,2,j} \in \mathbb{N}_0$. Now we have the weighted rows sums $\sum_{j=1}^{10} k_j p_{h,2,j} = k_2 k_h$, and the integrality conditions $k_j p_{h,2,j} = k_h p_{j,2,h}$, which in particular imply that $p_{h,2,j} = 0$ if and only if $p_{j,2,h} = 0$. It turns out that these conditions are sufficient to find all the matrix entries $p_{h,2,j} \in \mathbb{N}_0$, for $h, j \in \{1, \ldots, 10\}$ such that $[h, j] \notin \{[3,3], [3,4], [4,3], [4,4]\}$.

Table 3: Intersection matrix P_2 of $\mathbb{B}$ on $2^{1+22}.\mathrm{Co}_2$.

i	k_i	1	2	3	4	5	6	7	8	9	10
1	1	.	1	.	.	.	.	.	.	.	.
2	93 150	93150	925	63	15	1	.	.	.	.	.
3	7 286 400	.	4928	**63**	**120**	42	.	1	.	.	.
4	262 310 400	.	42240	**4320**	**1815**	420	.	30	15	.	.
5	4 196 966 400	.	45056	24192	6720	1807	891	272	120	.	27
6	9 646 899 200	.	.	.	.	2048	891	512	.	100	36
7	470 060 236 800	.	.	64512	53760	30464	24948	10287	5040	3850	3060
8	537 211 699 200	.	.	.	30720	15360	.	5760	3495	4125	4320
9	4 000 762 036 224	.	.	.	.	.	41472	32768	30720	31175	32256
10	6 685 301 145 600	.	.	.	.	43008	24948	43520	53760	53900	53451

The result is given in Table 3, where the as yet unknown entries are indicated in bold face.

Actually, there are only a very few possibilities for the unknown entries left, which can be checked using the following additional necessary condition. Since all the constituents of 1_H^G are $\mathbb{Q}$-valued, the field $\mathbb{Q}$ is a splitting field of the associated endomorphism ring E, and hence in particular the characteristic polynomial of P_2 splits into linear factors over $\mathbb{Q}$. The latter condition turns out to be fulfilled by precisely one of the possibilities left, thus completing P_2.

4.7.
To conclude, we determine the row eigenspaces of $P_2^{\mathrm{tr}} \in \mathbb{Q}^{10\times 10}$, and find eight 1-dimensional and a single 2-dimensional one. Computing the degrees of the Fitting correspondents associated with the 1-dimensional eigenspaces, by the formula given in 2.2, we conclude that we have found $\mathrm{Irr}(E) \setminus \{\lambda_2, \lambda_4\}$, using the notation in Table 4, where the degrees of the Fitting correspondents and a basis $\{\varphi_1, \varphi_2\} \subseteq \mathbb{Q}^{10}$ of the 2-dimensional eigenspace of P_2^{tr} are given as well.

Finally, to determine the as yet unknown characters λ_2 and λ_4 we proceed as follows. For $j \in \{2, 4\}$ we have $\lambda_j = \varphi_1 + x_j \varphi_2$, for some $x_j \in \mathbb{Z}$. The formula for the degrees of Fitting correspondents, applied to $\varphi_1 + X \cdot \varphi_2 \in \mathbb{Q}[X]^{10}$, leads to the quadratic

Table 4: Character table of $\mathbb{B}$ on $2^{1+22}.Co_2$.

λ	χ_λ	1	2	3	4	5
1	1	1	93150	7286400	262310400	4196966400
2	**96 255**	1	−2025	772200	−5702400	42768000
3	9 458 750	1	10287	215424	3777840	25974432
4	**347 643 114**	1	−2025	99000	356400	−5702400
5	4 275 362 520	1	495	48960	−334800	1631520
6	9 287 037 474	1	3375	28800	356400	1015200
7	536 105 794 455	1	1095	1560	7200	−113280
8	635 966 233 056	1	−425	9400	−3600	−57600
9	4 375 623 425 250	1	135	−360	−12960	17280
10	6 145 833 622 500	1	−153	−936	8640	1152
φ_1		1	−2025	107129	283239	−5117112
φ_2		0	0	11	−99	792

λ	6	7	8	9	10
1	9646899200	470060236800	537211699200	4000762036224	6685301145600
2	290816000	−2714342400	5474304000	8833204224	−11921817600
3	35514368	607533696	100362240	−42467328	−730920960
4	8806400	0	45619200	−191102976	141926400
5	2769920	−9636480	−12441600	−2359296	20321280
6	−870400	−6652800	4147200	−14155776	16128000
7	81920	107520	−921600	2555904	−1720320
8	−115200	358400	−76800	1409024	−1523200
9	−40960	138240	414720	−884736	368640
10	32768	−129024	−207360	294912	0
φ_1	12211712	−32776128	111171456	−82132992	−3745280
φ_2	4608	−44352	88704	147456	−197120

equation

$$\frac{11707448673375}{\chi_{\lambda_j}(1)} = \frac{9563}{294400} \cdot X^2 + \frac{6905057}{147200} \cdot X + \frac{14897519123}{294400}$$

having $x_j \in \mathbb{Z}$ as one of its solutions. Since the degrees of the Fitting correspondents are $\chi_{\lambda_2}(1) = 96255$ and $\chi_{\lambda_4}(1) = 347643114$, this yields

$$x_2 \in \{\frac{-591998657}{9563}, 60461\} \quad \text{and} \quad x_4 \in \{\frac{-6743057}{9563}, -739\}.$$

Hence we have $\lambda_2 = \varphi_1 + 60461 \cdot \varphi_2$ and $\lambda_4 = \varphi_1 - 739 \cdot \varphi_2$, and we are done.

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