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Ricci Curvature Tensor and Non-Riemannian Quantities

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Abstract. There are several notions of Ricci curvature tensor in Finsler geometry and spray geometry. One of them is defined by the Hessian of the well-known Ricci curvature. In this paper we will introduce a new notion of Ricci curvature tensor and discuss its relationship with the Ricci curvature and some non-Riemannian quantities. Using this Ricci curvature tensor, we shall have a better understanding of these non-Riemannian quantities.

1 Introduction

Finsler metrics are just metrics without quadratic structure. Every Finsler metric $F = F(x, y)$ on a manifold M determines a unique spray (a system of geodesics) that is locally expressed in the form $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ on TM. In 1926, L. Berwald introduced two tensors for Finsler metrics and sprays: the Riemannian curvature tensor $R_j{}^i{}_{kl}$ and the Berwald curvature tensor $B_j{}^i{}_{kl}$ ([6]). They are the most basic geometric quantities in Finsler and spray geometries. It is our goal to understand the geometric meanings of these quantities and their relationship.

It is natural to consider the tensor $R_i{}^m{}_{mj}$. In general, $R_i{}^m{}_{mj} \neq R_j{}^m{}_{mi}$. We discover that the anti-symmetric part of $R_i{}^m{}_{mj}$ is a non-Rimannian quantity that is related to the so-called *χ-curvature tensor*, $\chi = \chi_i dx^i$, as follows:

$$R_i{}^m{}_{mj} - R_j{}^m{}_{mi} = \chi_{j\cdot i} - \chi_{i\cdot j}. \tag{1.1}$$

The χ-curvature tensor is originally defined for Finsler metrics ([7]), but it can be easily extended to sprays. More precisely, for a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$, the χ-curvature tensor, $\chi = \chi_i dx^i$, is given by

$$\chi_i = \frac{1}{2}\Big\{ \Pi_{x^m y^i} y^m - \Pi_{x^i} - 2\Pi_{y^i y^m} G^m \Big\},$$

where $\Pi := \frac{\partial G^m}{\partial y^m}$.

Using the above identity (1.1), we obtain the following theorem.

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Theorem 1.1 *For a Finsler metric or a spray on a manifold M, $R_i{}^m{}_{mj} = R_j{}^m{}_{mi}$ if and only if $\chi_i = 0$.*

Clearly, if a spray $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ has the property that $\Pi := \partial G^m/\partial y^m = y^m\partial f/\partial x^m(x)$ for some scalar function $f(x)$ on M, then $\chi = 0$.

The key part of the Riemann curvature tensor is $R^i{}_j := y^j R_j{}^i{}_{kl} y^l$. The well-known Ricci curvature is defined by $\mathrm{Ric} := R^m{}_m$. The Hessian, $\widetilde{\mathrm{Ric}}_{ij} := 1/2\,\mathrm{Ric}_{y^iy^j}$, is a natural candidate for the Ricci curvature tensor ([1]). In this paper we introduce another notion of *Ricci curvature tensor* by

$$\mathrm{Ric}_{ij} := \frac{1}{2}\left\{ R_i{}^m{}_{mj} + R_j{}^m{}_{mi}\right\}.$$

We are going to show that this quantity is a meaningful notion for Finsler metrics and sprays. Let E_{ij} denote the mean Berwald curvature that is given by

$$E_{ij} = \frac{1}{2}\frac{\partial^2\Pi}{\partial y^i\partial y^j},$$

where $\Pi := \frac{\partial G^m}{\partial y^m}$. Akbar-Zadeh also studied the following quantity $H_{ij} := E_{ij|m}y^m$, where "$|$" denotes the horizontal covariant derivative with respect to the Berwald connection ([1]). Recently, several people have made some observations on this quantity ([4, 8]). We shall show the identity

$$\mathrm{Ric}_{ij} - \widetilde{\mathrm{Ric}}_{ij} = H_{ij} = \frac{1}{2}\left\{\chi_{j\cdot i} + \chi_{i\cdot j}\right\}. \tag{1.2}$$

By (1.2), we obtain the following theorem.

Theorem 1.2 *For a spray on a manifold M, the following are equivalent:*

(i) $\mathrm{Ric}_{ij} = \widetilde{\mathrm{Ric}}_{ij}$;
(ii) $H_{ij} = 0$;
(iii) $\chi_i = f_{ij}(x)y^j$ *with* $f_{ij} + f_{ji} = 0$.

In particular, $\mathrm{Ric}_{ij} = 0$ *if and only* $\mathrm{Ric} = 0$ *and* $H_{ij} = 0$.

If a spray G is induced by a Ricci-flat Berwald metric, then $\mathrm{Ric}_{ij} = 0$. Sprays with $\mathrm{Ric}_{ij} = 0$ and $H_{ij} = 0$ deserve further study.

For a Finsler metric $F = F(x,y)$ on an n-dimensional manifold, one may compare Ric_{ij} with the fundamental tensor $g_{ij} = \frac{1}{2}[F^2]_{y^iy^j}$ and compare Ric with F^2. We find that Ric_{ij} being isotropic is stronger than that Ric being isotropic. More precisely, we have the following theorem.

Theorem 1.3 *For a Finsler metric F on an n-dimensional manifold,* $\mathrm{Ric}_{ij} = (n-1)\kappa(x)g_{ij}$ *if and only if* $\mathrm{Ric} = (n-1)\kappa(x)F^2$ *and* $\chi_i = f_{ij}(x)y^j$ *with* $f_{ij} + f_{ji} = 0$.

It is known that if a Randers metric $F = \alpha + \beta$ is of isotropic Ricci curvature, $\mathrm{Ric} = (n-1)\kappa(x)F^2$, then the S-curvature is constant (3.3), hence $\chi = 0$ by (3.1). By Theorem 1.3, we obtain the following corollary.

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Corollary 1.4 *For any Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, $\mathrm{Ric}_{ij} = (n-1)\kappa(x)g_{ij}$ if and only if $\mathrm{Ric} = (n-1)\kappa(x)F$. In this case, $\chi_i = 0$ and the S-curvature is constant.*

Sprays and Finsler metrics with $\chi_i = f_{ij}(x)y^j$ and $f_{ij} + f_{ji} = 0$ deserve further study.

2 Preliminaries

We shall begin with sprays since our geometric quantities in this paper are defined for sprays. A spray on a manifold M is a vector field on the tangent bundle TM, which is locally expressed in any standard local coordinates (x^i, y^i) in TM as follows:

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are positively homogeneous in y, $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, $\forall \lambda > 0$. Roughly speaking, a spray is just a system of parametrized curves (*geodesics*) uniquely determined by their initial directions:

$$\frac{d^2x^i}{dt^2} + G^i\Big(x, \frac{dx}{dt}\Big) = 0.$$

Every Finsler metric $g = g(x, y)$ on a manifold induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ by

$$G^i = \frac{g^{il}}{4}\Big\{ \frac{\partial}{\partial x^j}(g_{kl}) + \frac{\partial}{\partial x^k}(g_{jl}) - \frac{\partial}{\partial x^l}(g_{jk}) \Big\} y^j y^k,$$

where $g_{ij} := \frac{1}{2} g_{y^i y^j}$.

Let $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ be a spray on a manifold M. Put

$$N^i_j := \frac{\partial G^i}{\partial y^j}, \quad \Gamma^i_{jk} := \frac{\partial^2 G^i}{\partial y^j \partial y^k}.$$

Let $\omega^i := dx^i$, $\omega^{n+i} := dy^i + N^i_j dx^j$, and $\omega_j^{\ i} := \Gamma^i_{jk} dx^k$. We have $d\omega^i = \omega^j \wedge \omega_j^{\ i}$. The local curvature form is defined by

$$\Omega_j^{\ i} := d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i}.$$

Expressing

$$\Omega_j^{\ i} = \frac{1}{2} R_j^{\ i}{}_{kl}\, \omega^k \wedge \omega^l - B_j^{\ i}{}_{kl}\, \omega^k \wedge \omega^{n+l},$$

we obtain two tensors $R_j^{\ i}{}_{kl}$ and $B_j^{\ i}{}_{kl}$. We call them the *Riemann curvature tensor* and the *Berwald curvature tensor*, respectively ([6]).

The key part of the Riemann curvature tensor is $R^i{}_k := y^j R_j^{\ i}{}_{kl} y^l$. We have the following important identity:

$$R_j^{\ i}{}_{kl} = \frac{1}{3}\Big\{ R^i{}_{k\cdot l\cdot j} - R^i{}_{l\cdot k\cdot j} \Big\}. \tag{2.1}$$

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By (2.1), we can see that the two curvature tensors $R_j{}^i{}_{kl}$ and $R^i{}_k$ essentially contain the same geometric data. Without confusion, we still call $R_j{}^i{}_{kl}$ the *Riemann curvature tensor*.

The Ricci curvature is defined by

$$\mathrm{Ric} := R^m{}_m = y^i R_i{}^m{}_{mj} y^j. \tag{2.2}$$

This is a scalar function on TM. A spray is said to be *Ricci-flat* if $\mathrm{Ric} = 0$. A Finsler metric g is called an *Einstein metric* if the Ricci curvature is isotropic: $\mathrm{Ric} = (n-1)\kappa(x)g$.

A spray is called a *Berwald spray* if $B_j{}^i{}_{kl} = 0$. For a Berwald spray, $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ at every point. This is the key property that ensures the exponential map $\exp_x\colon T_xM \to M$ is C^∞ at the origin $0 \in T_xM$.

The mean Berwald curvature tensor is defined by $E_{ij} := \frac{1}{2}B_i{}^m{}_{mj}$. A spray is said to *weakly Berwald* if $E_{ij} = 0$. Akbar-Zadeh [1] introduced and studied the so-called H-curvature tensor, $H = H_{ij}dx^i \otimes dx^j$: $H_{ij} := E_{ij|m}y^m$.

3 Non-Riemannian Quantities

Let $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ be a spray on a manifold M. Take a volume form $dV = \sigma(x)dx^1\cdots dx^n$ on M. Put $\Pi := \partial G^m/\partial y^m$. The S-curvature of (G, dV) is given by

$$S := \Pi - y^m\frac{\partial}{\partial y^m}(\ln\sigma).$$

It is a globally defined scalar function on TM ([6]). For different volume forms, the S-curvature differs by a closed 1-form on M.

Definition 3.1 ([3,7]) With the S-curvature, we define $\chi = \chi_i dx^i$ by

$$\chi_i := \frac{1}{2}\Big\{ S_{\cdot i|m}y^m - S_{|i}\Big\}. \tag{3.1}$$

For simplicity, we call χ the *χ-curvature tensor* .

In local coordinates,

$$\chi_i = \frac{1}{2}\Big\{ \Pi_{x^m y^i}y^m - \Pi_{x^i} - 2\Pi_{y^i y^m}G^m\Big\}. \tag{3.2}$$

Thus χ is independent of dV. The χ-curvature tensor was discussed in [7].

Note that for a Finsler metric $F = F(x, y)$ on an n-manifold M, if the S-curvature is almost constant, *i.e.*,

$$S = (n+1)cF + \theta, \tag{3.3}$$

where $c = constant$ and θ is a closed 1-form on M, then $\chi = 0$ by (3.1).

The mean Berwald curvature can be expressed by

$$E_{ij} = \frac{1}{2}S_{\cdot i\cdot j} = \frac{1}{2}\frac{\partial^2\Pi}{\partial y^i\partial y^j},$$

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where $\Pi = \partial G^m/\partial y^m$. Put $H_{ij} := E_{ij|m}y^m$. We call H_{ij} the *H-curvature tensor*. This quantity is introduced by Akbar-Zadeh [1] in order to study Finsler metrics of scalar flag curvature.

We have the following lemma.

Lemma 3.2

$$H_{ij} = \frac{1}{2}\Big\{\chi_{i\cdot j} + \chi_{j\cdot i}\Big\}. \tag{3.4}$$

Proof In local coordinates

$$H_{ij} = y^m\frac{\partial E_{ij}}{\partial x^m} - E_{im}\frac{\partial G^m}{\partial y^j} - E_{mj}\frac{\partial G^m}{\partial y^i} - 2G^m\frac{\partial E_{ij}}{\partial y^m}.$$

Differentiating χ, we get

$$\begin{aligned}\chi_{i\cdot j} &= y^m\frac{\partial E_{ij}}{\partial x^m} - 2E_{im}\frac{\partial G^m}{\partial y^j} - 2G^m\frac{\partial E_{ij}}{\partial y^m} + \frac{1}{2}\Big\{\frac{\partial^2\Pi}{\partial x^j\partial y^i} - \frac{\partial^2\Pi}{\partial x^i\partial y^j}\Big\}\\ &= H_{ij} + E_{jm}\frac{\partial G^m}{\partial y^i} - E_{im}\frac{\partial G^m}{\partial y^j} + \frac{1}{2}\Big\{\frac{\partial^2\Pi}{\partial x^j\partial y^i} - \frac{\partial^2\Pi}{\partial x^i\partial y^j}\Big\}.\end{aligned}$$

This gives (3.4). ■

By Lemma 3.2, $\chi_i = 0$ implies that $H_{ij} = 0$. The converse might not be true.

Differentiating (3.4) with respect to y^k yields

$$H_{ij\cdot k} = \frac{1}{2}\Big\{\chi_{i\cdot j\cdot k} + \chi_{j\cdot i\cdot k}\Big\}. \tag{3.5}$$

Then

$$H_{jk\cdot i} = \frac{1}{2}\Big\{\chi_{j\cdot k\cdot i} + \chi_{k\cdot j\cdot i}\Big\}, \tag{3.6}$$

$$H_{ki\cdot j} = \frac{1}{2}\Big\{\chi_{k\cdot i\cdot j} + \chi_{i\cdot k\cdot j}\Big\}. \tag{3.7}$$

Then (3.5)+(3.7)−(3.6) yields

$$\chi_{i\cdot j\cdot k} = \frac{1}{2}(\chi_{i\cdot j\cdot k} + \chi_{i\cdot k\cdot j}) = H_{ij\cdot k} + H_{ki\cdot j} - H_{jk\cdot i}. \tag{3.8}$$

By (3.8), we obtain the following lemma.

Lemma 3.3 *Let $\theta = \theta_i(x)y^i$ be a 1-form and let $\Phi = \Phi(x,y)$ be homogenous in y of degree one. Then*

$$H_{ij} = \frac{1}{2}\theta\Phi_{\cdot i\cdot j} \tag{3.9}$$

if and only if

$$\chi_i = \frac{1}{2}(\theta\Phi_{\cdot i} - \theta_i\Phi) + f_{ij}y^j, \tag{3.10}$$

where $f_{ij} = f_{ij}(x)$ are scalar functions satisfying $f_{ij} + f_{ji} = 0$.

Proof If (3.9) holds, then $H_{ij\cdot k} = \frac{1}{2}(\theta_k \Phi_{\cdot i\cdot j} + \theta \Phi_{\cdot i\cdot j\cdot k})$. Plugging this into (3.8) we have

$$\chi_{i\cdot j\cdot k} = \frac{1}{2}(\theta_k \Phi_{\cdot i\cdot j} + \theta_j \Phi_{\cdot k\cdot i} - \theta_i \Phi_{\cdot j\cdot k} + \theta \Phi_{\cdot i\cdot j\cdot k}),$$

which is equivalent to

$$(2\chi_{i\cdot j} - \theta \Phi_{\cdot i\cdot j} - \theta_j \Phi_{\cdot i} + \theta_i \Phi_{\cdot j})_{\cdot k} = 0.$$

Then there exist scalar functions $f_{ij} = f_{ij}(x)$ such that

$$2\chi_{i\cdot j} - \theta \Phi_{\cdot i\cdot j} - \theta_j \Phi_{\cdot i} + \theta_i \Phi_{\cdot j} = 2 f_{ij}.$$

Thus,

$$\chi_{i\cdot j} = \frac{1}{2}(\theta \Phi_{\cdot i\cdot j} + \theta_j \Phi_{\cdot i} - \theta_i \Phi_{\cdot j}) + f_{ij}.$$

Contracting this equation with y^j, we get (3.10). By (3.9), it is easy to see $f_{ij} + f_{ji} = 0$. The sufficiency is obvious. ■

When $\Phi = \Phi(x, y)$ in Lemma 3.3 is a positive definite Finsler metric and χ is the χ-curvature tensor of Φ, the condition in (3.9) means that $H = H_{ij} dx^i \otimes dx^j$ almost vanishes.

As a special case of Lemma 3.3, we immediately obtain the following corollary.

Corollary 3.4 *$H_{ij} = 0$ if and only if $\chi_i = f_{ij} y^j$, where $f_{ij} = f_{ij}(x)$ are scalar functions satisfying $f_{ij} + f_{ji} = 0$.*

A spray G on a manifold M is said to be *S-closed* if in any standard local coordinates in TM, $\Pi = \frac{\partial G^m}{\partial y^m}$ is a gradient of a local scalar function on M, *i.e.*,

$$\Pi = y^m \frac{\partial f}{\partial y^m}$$

for some local function $f = f(x)$. A Finsler metric is said to be *S-closed* if its spray is S-closed. It is easy to see that every Riemannian metric is S-closed.

We have the following proposition.

Proposition 3.5 *If a spray is S-closed, then $\chi = 0$.*

Proof By assumption, in local coordinates, there is a local function $f = f(x)$ such that

$$\Pi = y^m \frac{\partial f}{\partial x^m}.$$

Plugging it into (3.2) yields that $\chi_i = 0$.

The converse is obviously not true. See Example 3.6. ■

Example 3.6 Let $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ be a spray on an open subset $U \subset R^n$ with $G^i = P y^i$, where $P_{y^i x^m} y^m - P_{x^i} = 0$. One can verify that $\chi_i = 0$. Assume that P is not a 1-form; then G is not S-closed.

4 Ricci Curvature Tensors

Akbar-Zadeh [1] defines a notion of Ricci curvature tensor by the Hessian

$$\widetilde{\mathrm{Ric}}_{ij} := \frac{1}{2}\,\mathrm{Ric}_{y^i y^j}\,.$$

From (2.2), it is also natural to study the tensor $R_i{}^m{}_{mj}$. By the Bianchi identity and (2.1) we have $R_i{}^m{}_{mj} - R_j{}^m{}_{mi} = R_m{}^m{}_{ij}$. Thus, $R_i{}^m{}_{mj}$ is not symmetric in i, j, in general. $R_i{}^m{}_{mj}$ might not be a good candidate for the Ricci curvature tensor. Therefore we define a new notion of *Ricci curvature tensor* by

$$\mathrm{Ric}_{ij} := \frac{1}{2}\Big\{ R_i{}^m{}_{mj} + R_j{}^m{}_{mi}\Big\},$$

We have

$$\mathrm{Ric} = \mathrm{Ric}_{ij}\, y^i y^j = \widetilde{\mathrm{Ric}}_{ij} y^i y^j.$$

The difference $\mathrm{Ric}_{ij} - \widetilde{\mathrm{Ric}}_{ij}$ is an interesting non-Riemannian quantity.

By (2.1), we have

$$\mathrm{Ric}_{ij} = \frac{1}{6}\Big\{ 2R^m{}_{m\cdot i\cdot j} - R^m{}_{i\cdot m\cdot j} - R^m{}_{j\cdot m\cdot i}\Big\} = \widetilde{\mathrm{Ric}}_{ij} - \frac{1}{6}\Big\{ R^m{}_{m\cdot i\cdot j} + R^m{}_{i\cdot m\cdot j} + R^m{}_{j\cdot m\cdot i}\Big\}. \tag{4.1}$$

By (4.1), we obtain the following lemma.

Lemma 4.1 *The difference* $\mathrm{Ric}_{ij} - \widetilde{\mathrm{Ric}}_{ij}$ *is given by*

$$\mathrm{Ric}_{ij} - \widetilde{\mathrm{Ric}}_{ij} = -\frac{1}{6}\Big\{ R^m{}_{m\cdot i\cdot j} + R^m{}_{i\cdot m\cdot j} + R^m{}_{j\cdot m\cdot i}\Big\}. \tag{4.2}$$

The χ-curvature tensor can be expressed in terms of the Riemann curvature.

Lemma 4.2 ([3]) *For any spray,*

$$\chi_i := -\frac{1}{6}\Big\{ 2R^m{}_{i\cdot m} + R^m{}_{m\cdot i}\Big\}. \tag{4.3}$$

By (4.3), we can show the following lemma.

Lemma 4.3 *The difference* $\mathrm{Ric}_{ij} - \widetilde{\mathrm{Ric}}_{ij}$ *is given by*

$$\mathrm{Ric}_{ij} - \widetilde{\mathrm{Ric}}_{ij} = \frac{1}{2}\Big\{ \chi_{j\cdot i} + \chi_{i\cdot j}\Big\}. \tag{4.4}$$

Proof By (2.1), we have

$$\begin{aligned} R_i{}^m{}_{mj} &= -\frac{1}{3}\Big\{ R^m{}_{j\cdot m\cdot i} - R^m{}_{m\cdot j\cdot i}\Big\} \\ &= \frac{1}{2} R^m{}_{m\cdot i\cdot j} - \frac{1}{6}\Big\{ 2R^m{}_{j\cdot m\cdot i} + R^m{}_{m\cdot j\cdot i}\Big\} = \widetilde{\mathrm{Ric}}_{ij} + \chi_{j\cdot i}. \end{aligned} \tag{4.5}$$

This gives (4.4). ■

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Proof of Theorem 1.1 By (4.5), we immediately obtain the following:

$$R_i{}^m{}_{mj} - R_j{}^m{}_{mi} = \chi_{j\cdot i} - \chi_{i\cdot j}. \tag{4.6}$$

Assume that $R_i{}^m{}_{mj} = R_j{}^m{}_{mi}$. By (4.6), we have $\chi_{i\cdot j} - \chi_{j\cdot i} = 0$. Contracting this with y^j and using $\chi_j y^j = 0$, we get $\chi_i = 0$. ■

By Lemmas 3.2 and 4.3, we obtain the following proposition.

Proposition 4.4 *For any spray,*

$$\mathrm{Ric}_{ij} - \widetilde{\mathrm{Ric}}_{ij} = H_{ij}. \tag{4.7}$$

We prove (4.7) via the χ-curvature tensor . It will be an interesting problem to give a proof of (4.7) without using the χ-curvature tensor .

Remark 4.5 By (4.2) and (4.7), we obtain

$$H_{ij} = -\frac{1}{6}\left\{ R^m{}_{m\cdot i\cdot j} + R^m{}_{i\cdot m\cdot j} + R^m{}_{j\cdot m\cdot i} \right\}.$$

The relation between H_{ij} and $R^i{}_k$ for Finsler metrics is given by X. Mo [4]. We can easily see that if G is R-quadratic in y ($R^i{}_k = R_j{}^i{}_{kl}(x) y^j y^l$), then $H_{ij} = 0$.

Proof of Theorem 1.2 The theorem follows from Corollary 3.4 and (4.7). ■

Proof of Theorem 1.3 Assume that $\mathrm{Ric}_{ij} = (n-1)\kappa(x) g_{ij}$. Then $\mathrm{Ric} = \mathrm{Ric}_{ij}\, y^i y^j = (n-1)\kappa(x)F^2$ and $\widetilde{\mathrm{Ric}}_{ij} = (n-1)\kappa(x) g_{ij}$. By (4.4), we get $H_{ij} = 0$. ■

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