Two structure theorems for homeomorphism groups

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Let \( H(C) \) be the group of homeomorphisms of the Cantor set, \( C \), onto itself. Let \( p : C \rightarrow M \) be a (continuous) map of \( C \) onto a compact metric space \( M \), and let \( G(p, M) \) be \{ \( h \in H(C) \mid \forall x \in C, p(x) = ph(x) \} \). \( G(p, M) \) is a group. The map \( p : C \rightarrow M \) is standard, if for each \( (x, y) \in C \times C \) such that \( p(x) = p(y) \), there is a sequence \( \{x_n\} \subset C \) and a sequence \( \{h_n\} \subset G(p, M) \) such that \( x_n \rightarrow x \) and \( h_n(x_n) \rightarrow y \). Standard maps and their associated groups characterize compact metric spaces in the sense that: Two such spaces, \( M \) and \( N \), are homeomorphic if and only if, given \( p \) standard from \( C \) onto \( M \), there is a standard \( q \) from \( C \) onto \( N \) for which \( G(p, M) = h^{-1}G(q, N)h \), for some \( h \in H(C) \). That is, two compact metric spaces are homeomorphic if and only if they determine, via standard maps, the same classes of conjugate subgroups of \( H(C) \).

The present note exhibits two natural structure theorems relating algebraic and topological properties: First, if \( M = H \cup K \) (\( H, K \neq \emptyset \)), compact metric, and \( p : C \rightarrow M \) are given, then \( G(p, M) \) is isomorphic to a subdirect product of \( G(p, M)/S(p, H \cup K) \) and \( G(p, M)/S(p, K \setminus H) \) where, generally, \( S(p, N) \) is the normal subgroup of homeomorphisms supported on \( p^{-1}(N) \). Second, given \( M \) and \( N \) compact metric and \( p : M \rightarrow N \) continuous and onto, let \( M \neq M = \bigcup \{D_a\}_{a \in A} \neq \emptyset \), where \( \{D_a\}_{a \in A} \) is

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the collection of non-degenerate preimages of points in \( N \).

Then there is a standard \( p : C \to M \) such that \( fp : C \to N \) is standard and there is a homomorphism

\[
H : G(p, M) \to G(fp, N)/S(fp, C1D^A).
\]

Introduction

The results described in the Abstract appeared in [2]. Later, in [3], it was shown that if \( M_1 \) and \( M_2 \) are compact metric spaces, then there are standard maps \( p : C \to M_1 \times M_2 \) and \( p_i : C \to M_i \), \( i = 1, 2 \), such that \( G(p, M_1 \times M_2) = G(p_1, M_1) \cap G(p_2, M_2) \).

In some vague sense "dual" to this is the observation: if \( M = H \cup K \) where \( H \cap K = \emptyset \) and each of \( H \) and \( K \) is closed, then, for \( p : C \to M \), \( G(p, M) \) is the (interior) direct product of the two normal subgroups supported on (the identity on the complement of) the preimages of \( H \) and of \( K \), respectively.

The first of the following theorems concerns the less fortuitous circumstance in which \( H \) and \( K \) intersect.

The second theorem, below, grows out of the observation in [2], that if \( f : M \to N \) is continuous and onto, for \( M \) and \( N \) compact metric, then there is a standard map \( p : C \to M \) such that \( fp : C \to N \) is also standard and \( G(p, M) \subset G(fp, N) \). Theorem 2 is an indication, for many cases, of "how much bigger" \( G(fp, N) \) need be.

First a simple preparatory lemma:

**Lemma.** Given compact metric \( M \), \( N \subseteq M \) and a map \( p : C \to M \), then \( S(p, N) : = \{ h \in G(p, M) | \forall x \in C \setminus p^{-1}(N), h(x) = x \} \) is a normal subgroup of \( G(p, M) \).

**Proof.** Since \( S(p, N) \) is obviously a subgroup, we show only normality: if \( x \notin p^{-1}(N) \), \( h(x) \notin p^{-1}(N) \) for any \( h \in G(p, M) \). Hence for \( f \in S(p, N) \) and \( h \in G(p, M) \), \( h^{-1}fh(x) = x \), for \( x \notin p^{-1}(N) \); and \( h^{-1}fh \in S(p, N) \).
Note that, while $H(C)$ is simple (see [1]), the groups $G(p, M)$ have many different normal subgroups.

**Theorem 1.** Let compact metric $M = H \cup K$ and let $p : C \to M$ be given. Then $G(p, M)$ is isomorphic to a subdirect product of $G(p, M)/S(p, H\setminus K)$ and $G(p, M)/S(p, K\setminus H)$.

Proof. Clearly, $S(p, H\setminus K) \cap S(p, K\setminus H)$ is the identity subgroup. Hence, by a well known theorem about groups, $G(p, M)$ is isomorphic to a subdirect product of $G(p, M)/S(p, H\setminus K)$ and $G(p, M)/S(p, K\setminus H)$. The isomorphism may be chosen so that the elements of $[G(p, M)/S(p, H\setminus K)] \times [G(p, M)/S(p, K\setminus H)]$ in the isomorph of $G(p, M)$ are precisely those of the form $(gS(p, H\setminus K), gS(p, K\setminus H))$, $g \in G(p, M)$.

Generally, "subdirectness" indicates the necessary coupling between restrictions of $G(p, M)$ - homeomorphisms to preimages of non-separated sets.

**Theorem 2.** Let $M$ and $N$ be compact metric spaces and let $f : M \to N$ be continuous and onto. Let $M \neq M - \{D^*_\alpha \neq \emptyset$, where $\{D^*_\alpha \}_{\alpha \in A}$ is the collection of non-degenerate preimages of points in $N$ and $D^*_\alpha$ is their union. Then there is a standard $p : C \to M$ such that $fp : C \to N$ is standard and there is an onto homomorphism $H : G(p, M) \to G(fp, N)/S(fp, N)$. If $CLD^*_\alpha$ is nowhere dense in $M$, $p$ and $H$ can be chosen so that $H$ is an isomorphism.

Proof. Let $\{T^1_i\}_{i=1}^{n(1)}$, $\{T^2_i\}_{i=1}^{n(2)}$, ..., be a sequence of finite closed covers of $M$ with the properties:

1) mesh of $\{T^k_i\}_{i=1}^{n(k)}$ and of $\{f(T^k_i)\}_{i=1}^{n(k)} < 1/k$;

2) $T^k_i \cap T^k_j \neq \emptyset$ and hence $f(T^k_i) \cap f(T^k_j) \neq \emptyset$ is the union of two or more elements of $\{T^{k+1}_i\}_{i=1}^{n(k+1)}$ and $\{f(T^{k+1}_i)\}_{i=1}^{n(k+1)}$. 


3) for each \( k \), \( \text{Cl}(M-C^{(k)}) \) is the union of elements of
\[
\left\{ T_i^k \right\}_{i=1}^{n(k)}
\]
each of which is the closure of an open set in \( M \).

Let the \( T_i^k \)'s be listed so that the closures of the open sets which cover \( \text{Cl}(M-C^{(k)}) \) occur first: \( T_i^k \subset \text{Cl}(M-C^{(k)}) \), \( 1 \leq i \leq N(k) < n(k) \).

Now divide the interval \([0, 1]\) into \( 2m(1) \) equal subintervals, labeling every second one of these, end points included, as
\( E_1^1, E_2^1, \ldots, E_{n(1)}^1 \). Given the interval \( E_i^k(1), E_i^k(2), \ldots, E_i^k(k) \), where
\( i(1), \ldots, i(k) \) is such that \( T_i^k(1) \supset T_i^k(2) \supset \ldots \supset T_i^k(n(k)) \), divide it into \( 2m(i(k)) \) equal subintervals where \( m(i(k)) \) is the number of elements of \( \left\{ T_i^{k+1} \right\}_{i=1}^{n(k+1)} \) contained in \( T_i^k(k) \). Denote every second one of these by
\[
E_{i(1)}^{k+1}, E_{i(2)}^{k+1}, \ldots, E_{i(k)}^{k+1}
\]
where the \( j(n) \)'s are the subscripts of the elements of the \( (k+1) \)-st cover which are contained in \( T_i^k(k) \).

Set \( C = \bigcap_{k=1}^{\infty} \left( \bigcup E_i^k(i(1), \ldots, i(k)) \right) \) for all sequences \( i(1), \ldots, i(k) \)
for which \( T_i^1(1) \supset T_i^2(2) \supset \ldots \supset T_i^k(k) \). \( C \) is a Cantor set. Let
\( F_n(x) = T_i^n(i(n)) \) for \( x \in C \cap E_i^n(i(1), \ldots, i(n)) \), where \( i(1), \ldots, i(n) \) is such that \( T_i^1(1) \supset \ldots \supset T_i^n(i(n)) \). Let \( p(x) = \bigcap_{n=1}^{\infty} F_n(x) \), a (continuous) map of \( C \) onto \( M \). Observe, as in [2], that \( fp : C \to N \) is also standard by condition 2) above.

(The routine constructions of \( C \) and \( p \) have been repeated to
permit an identification of the parts of \( C \) with which we deal next.)

Letting
\[
C_1 = \bigcup_{k=1}^{\infty} \bigcup_{1 \leq i(k) \leq N(k)} \left( C \cap E^k_{\epsilon(1), \ldots, \epsilon(k)} \right) = \bigcup_{1 \leq i(1) \leq N(1)} \left( C \cap E^1_{\epsilon(1)} \right),
\]
and \( C_2 = C \setminus C_1 \), observe that \( h(C_1) = C_1 \) by the third condition above, on the covers of \( M \) for \( h \in G(p, M) \). Otherwise, points in the \( fp \)-preimage of \( M - \text{ClD}_\alpha^4 \) would be carried onto points of its complement.

Since \( C_1 \) is both open and closed, each homeomorphism in \( G(p, M) \) may be expressed, uniquely except for order, as the product of homeomorphisms supported on (the identity on the complement of) each of \( C_1 \) and \( C_2 \). Thus, \( G(p, M) \) is the interior direct product of its normal subgroups \( G_1 \) and \( G_2 \), the homeomorphisms supported on \( C_1 \) and \( C_2 \), respectively.

Likewise, \( G(fp, N) \) is the interior direct product of its normal subgroups of homeomorphisms supported on each of \( C_1 \) and \( C_2 \), respectively. The first of these is \( G_1 \) again. The second is \( S(fp, \text{ClD}_\alpha^4) \), since none of the points in \( C_1 \) which are in the preimage of a point of \( \text{ClD}_\alpha^4 \) can be permuted without moving points in the preimages of points outside \( \text{ClD}_\alpha^4 \).

The obvious homomorphism \( H \) is suggested by the diagram
\[
G(p, M) \to \left[ G(p, M)/G_2 \right] \cong G_1 \to G(fp, N)/S(fp, \text{ClD}_\alpha^4).
\]

If \( \text{ClD}_\alpha^4 \) is nowhere dense in \( M \), one can, for each \( k \), let \( N(k) = n(k) \), so that each of \( G_2 \) and \( S(fp, \text{ClD}_\alpha^4) \) is the identity subgroup, and \( H \) becomes an isomorphism. If \( \text{ClD}_\alpha^4 \) is not nowhere dense, obvious examples show the above construction must fail to yield an isomorphism.

References


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