Table 3 is a continuation of Table 1 determined directly from the derivatives, and the values confirm the results obtained above.

<table>
<thead>
<tr>
<th>( n )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f^{(n)}(0) )</td>
<td>0</td>
<td>-5040</td>
<td>0</td>
<td>362880</td>
<td>0</td>
<td>-39916800</td>
<td>0</td>
</tr>
<tr>
<td>( f^{(n)}(1) )</td>
<td>45</td>
<td>-315</td>
<td>1260</td>
<td>0</td>
<td>-56700</td>
<td>623700</td>
<td>-3742200</td>
</tr>
</tbody>
</table>

**TABLE 3**

Even the most sophisticated mathematical software will find it difficult to generate the above formulae. Nevertheless, such packages can be useful in suggesting such patterns, which can then be followed up with some solid mathematics.

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**84.22 On a conjecture of Paul Thompson**

We shall show that, for \( 0 < \sigma < 1 \) and real \( \tau \neq 0 \),

\[
\frac{\arg \left( \sum_{n=1}^{N} n^{-\sigma - it} \right) - \arg \left( \sum_{n=1}^{N-1} n^{-\sigma - it} \right)}{\arg \left( \sum_{n=1}^{N} (1 - \frac{1}{n}) n^{-\sigma - it} \right) - \arg \left( \sum_{n=1}^{N-1} (1 - \frac{1}{n}) n^{-\sigma - it} \right)} \to 1 \text{ as } N \to \infty, \tag{1}
\]

and that convergence is most rapid when \( \zeta(\sigma + it) = 0 \). Here the expression \( \arg a - \arg b \) is taken to be the principal argument of \( a/b \), so that its value lies between \(-\pi\) and \(\pi\). With \( s = \sigma + it \), the function \( \zeta(s) \) is the famous Riemann zeta-function, and ‘most rapid’ will be made precise in the following. Paul Thompson [1] had conjectured that, with \( \sigma = \frac{1}{2} \), the limit formula (1) holds if, and only if, \( \zeta(\sigma + it) = 0 \).

**Some preliminary estimates**

It will be convenient to use order notation for asymptotic analysis which we recall in the following. We write \( f(N) = O(g(N)) \), where \( g(N) > 0 \), to mean that \(|f(N)/g(N)|\) is bounded; in other words there is a number \( K \) such that \(|f(N)| \leq Kg(N)\) for all \( N > 1 \), and we usually apply this for \( N \to \infty \). Similarly, we also write \( f(z) = O(g(z)) \) as \( z \to \infty \), where \( z \) is complex and \( g(z) > 0 \), to mean that \(|f(z)/g(z)|\) is bounded in the punctured neighbourhood of \( \infty \); in other words, there are positive numbers \( K \) and \( r \) such that \(|f(z)| \leq Kg(z)\) for all \( z \) satisfying \( 0 < |z - \infty| < r \).

Let us now write \( C(N) = C(N; \sigma, \tau) \) for the expression of which the limit is stated in (1). We note that both the numerator and the denominator for \( C(N) \) are arguments of complex numbers in the form of a ratio

\[
\frac{\sum_{n=1}^{N} f(n; \sigma, \tau)}{\sum_{n=1}^{N-1} f(n; \sigma, \tau)} = 1 + \frac{f(N; \sigma, \tau)}{\sum_{n=1}^{N-1} f(n; \sigma, \tau)}, \tag{2}
\]

with an appropriate function \( f \). Thus, for the asymptotic expansion, we shall make use of

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\[ \frac{1}{1 + z} = 1 - z + O(|z|^2) \quad \text{as } z \to 0, \quad (3) \]

\[ \arg (1 + z) = \psi(z) + O(|z|^2) \quad \text{as } z \to 0. \quad (4) \]

When applied to \( C(N) \) we shall also require
\[ \left( 1 - \frac{1}{n} \right)^\sigma = 1 - \frac{\sigma}{n} + O(n^{-2}) \quad \text{as } n \to \infty, \quad (5) \]
\[ \sum_{n=1}^{N-1} n^{-s} = \frac{N^{1-s}}{1-s} + \zeta(s) - \frac{N^{-s}}{2} + O(N^{-\sigma-1}) \quad \text{as } N \to \infty. \quad (6) \]

The reader can easily establish the elementary results (3), (4) and (5). Formula (6) is valid for \( \sigma > -1 \) and \( s \neq 1 \). Its derivation appears in [2], wherein much else on \( \zeta(s) \) can be found.

**Estimation of the denominator for \( C(N) \)**

From (6), we find that, as \( N \to \infty \),
\[ \sum_{n=1}^{N-1} n^{-ir} = \frac{N^{1-ir}}{1 - ir} + O(1), \quad \text{and} \quad \sum_{n=1}^{N-1} n^{-1-ir} = O(1), \]
and hence, by (5),
\[ \sum_{n=1}^{N-1} \left( 1 - \frac{1}{n} \right)^\sigma n^{-ir} = \sum_{n=1}^{N-1} n^{-ir} - \sigma \sum_{n=1}^{N-1} n^{-1-ir} + O\left( \sum_{n=1}^{N-1} n^{-2} \right) = \frac{N^{1-ir}}{1 - ir} + O(1), \]
where we have used the fact that \( \sum n^{-2} \) is absolutely convergent. Together with (3) and (5) we deduce that
\[ \frac{\sum_{n=1}^{N} (1 - 1/n)^\sigma n^{-ir}}{\sum_{n=1}^{N-1} (1 - 1/n)^\sigma n^{-ir}} = 1 + \frac{1 - \frac{1}{N}}{1 - ir} \left( 1 + O(N^{-1}) \right) \]
\[ = 1 + \frac{1 - ir}{N} + O(N^{-2}), \]
and hence, by (4),
\[ \arg \left( \frac{\sum_{n=1}^{N} (1 - 1/n)^\sigma n^{-ir}}{\sum_{n=1}^{N-1} (1 - 1/n)^\sigma n^{-ir}} \right) = \frac{\tau}{N} + O(N^{-2}). \quad (7) \]

**Estimation of the numerator for \( C(N) \)**

A similar argument applied to the numerator for \( C(N) \) yields, for \( \sigma > -1 \) and \( s \neq 1 \),
\[ \frac{\sum_{n=1}^{N} n^{-s}}{\sum_{n=1}^{N-1} n^{-s}} = 1 + \frac{1 - s}{N} \left( 1 + (s - 1) \zeta(s) N^{s-1} + O(N^{-1}) \right)^{-1}. \]

We note that
\[ \left| (1 - s) \zeta(s) N^{s-1} + O(N^{-1}) \right|^2 = O(N^{2\sigma-2}) + O(N^{\sigma-2}) + O(N^{-2}), \]
and all three error terms are simply \( O(N^{2\sigma-2}) \), if we restrict our attention to
\[ \sigma > 0. \] Thus, on applying (3), we now have
\[ \sum_{n=1}^{N} N^{-s} = 1 + \frac{1-s}{N} - (1-s)^2 \zeta(s) N^{s-2} + O(N^{-2}) + O(N^{2\sigma - 3}), \]
and hence, by (4),
\[ \arg \left( \frac{\sum_{n=1}^{N} n^{-s}}{\sum_{n=1}^{N} n^{-s}} \right) = -\frac{\tau}{N} - \mathcal{O}((1-s)^2 \zeta(s) N^{s-2}) + O(N^{-2}) + O(N^{2\sigma - 3}). \] (8)

As a little aside, we remark that the argument also gives the following formula for \( \zeta(s) \) in the 'critical strip' \( 0 < \sigma < 1 \),
\[ \zeta(s) = \lim_{N \to \infty} \left( \frac{N^{-2s}}{(1-s)^2} \left( 1 - \frac{\sum_{n=1}^{N} n^{-s}}{\sum_{n=1}^{N} n^{-s}} \right) + \frac{N^{1-s}}{1-s} \right). \]

**Estimation of \( C(N) \)**

From (7) and (8), we see that both the numerator and the denominator for \( C(N) \) have the asymptotic value \(-\tau/N\) as \( N \to \infty \) for \( 0 \leq \sigma < 1 \) (\( \sigma < 1 \) comes from \( \mathrm{Re}(s-2) < -1 \) in (8)), so that \( C(N) \to 1 \) as \( N \to \infty \). In other words, we have established Thompson's limit formula (1). Indeed, from (7) and (8), the expression \( C(N) \) itself has the asymptotic formula
\[ C(N) = 1 + \frac{N}{\tau} \mathcal{O}((1-s)^2 \zeta(s) N^{s-2}) + O(N^{-1}) + O(N^{2\sigma - 2}). \]

Note that, if \( \zeta(s) \neq 0 \), then the second term on the right-hand side here will be present, and it will not be \( O(f(N) N^{\sigma-1}) \) for any \( f(N) \to 0 \) as \( N \to \infty \). On the other hand, if \( \zeta(s) = 0 \) then this second term will be absent. Therefore, for \( 0 < \sigma < 1 \),
\[ C(N) = \begin{cases} 1 + O(N^{\sigma-1}) & \text{always} \\ 1 + O(N^{-1}) + O(N^{2\sigma - 2}) & \text{if, and only if, } \zeta(s) = 0. \end{cases} \]

The reason for imposing the condition \( \sigma \neq 0 \) is that, as it stands, the 'only if' part of the statement might not be valid. However, it is known that \( \zeta(s) \) has no zero on the line \( \sigma = 0 \), so that no harm is done even if we write \( 0 \leq \sigma < 1 \). Readers who are not familiar with the theory of \( \zeta(s) \) may be interested to know that the distribution of the zeros of \( \zeta(s) \) is closely related to the distribution of primes, and the fact that \( \zeta(s) \) has no zero on \( \sigma = 0, 1 \) is used to prove the prime number theorem.

**References**


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