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SOME EMBEDDINGS RELATED TO C^*-EMBEDDINGS

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Abstract

A space S is R^*-embedded (G^*-embedded) in a space X if two disjoint regular closed sets (closure disjoint open sets) of S are contained in disjoint regular closed sets (extended to closure disjoint open sets) of X. A space S is R-extendable to a space X if any regular closed set of S can be extended to a regular closed set of X. It is shown that R^*-embedding and G^*-embedding are identical with C^*-embedding for certain fairly general classes of Tychonoff spaces. Under certain conditions it is shown that R-extendability is related to z-embedding. Spaces in which the regular open sets are C and C^*-embedded are also investigated.

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1. Introduction

It is well known that a set S is X^*-embedded in a set X if given two disjoint zero sets Z and H of S, there exists disjoint zero sets $E(Z)$ and $E(H)$ of X such that $E(H) \cap S = H$ and $E(Z) \cap S = Z$. We examine embeddings where Z, H, and $E(Z)$, $E(H)$ are regular closed sets or closure disjoint open sets. We relate these embeddings to C^*-embeddings. A separation axiom, metanormality (every regular closed set is C^*-embedded), plays a significant role in this so properties of this axiom will be investigated. These concepts will be applied to investigating Alexandroff's extension αX and relating it to βX. The question of the C^*-embedding of regular open sets is also studied.

Studies of embeddings involving extending disjoint sets may be found in [2], [3] and [4]. Some like FF-embeddings are closely related to C^*-embeddings. Much of the background and notation is in [13].

DEFINITION 1.1. A set S is FF-embedded [2] (R^*-embedded) [G^* embedded] {GG-embedded [2]} in a space X if given two disjoint closed sets (two disjoint regular closed sets) [two closure disjoint open sets] {two disjoint open sets} of s, they can be extended to two disjoint closed sets {can be extended to two disjoint open sets} of X. A subset A of S is extended to a set $E(A)$ of X if $E(A) \cap S = \varnothing$. A set S is R-extendable to X if for R a regular closed in S, R is extendable to a regular closed set of X. If in the above definiitons of (R^*-embedding) [G^*-embeddings] we replace two by any number of (disjoint regular closed sets) [pairwise closure disjoint open sets} we will use the term (TR^*-embedding) [TG^*-embedding].

2. Basic properties

THEOREM 2.1. *The following are satisfied.*

(a) *A set S is G^*-embedded in X if given two closure disjoint open sets G and H of S, there exists closure disjoint open sets of X, G' and H' such that $G \subset G'$ and $H \subset H'$.*

(b) *A GG-embedded subset is R-extendable. Hence every dense subset of an open set is R-extendable; furthermore regular closed sets are R-extendable.*

(c) *Using the notation $A \to B$ to mean that if a set S is A-embedded in a space X then S is B-embedded in X, $G^* \to R^*$, $TG^* \to TR^*$. For dense subsets of Tychonoff spaces, $G^* \leftrightarrow R^* \to C^*$; for dense subsets of open sets $FF \to G^* \leftrightarrow R^*$.*

(d) *A regular closed set R of X is TR^*-embedded in X.*

(e) *G^*- and R^*-embeddings are transitive; that is, if S is G^*-embedded in X and X is G^*-embedded in Y, then S is G^*-embedded in Y.*

(f) *G^*-embeddings are hereditary, that is if $S \subset T \subset X$ and S is G^*-embedded in X, then S is G^*-embedded in T. If S is dense in T and R^*-embedded in X, S is R^*-embedded in T.*

(g) *If an open set G of X is R^*-embedded (G^*-embedded) in $\overline{G}$, G is R^*-embedded (G^*-embedded) in X. For open sets $R^* \to G^*$.*

(h) *An R^*-embedded (G^*-embedded) dense subset M of an open subset of X is TR^*-embedded (TG^*-embedded).*

(i) *A dense subset of an open set is R^*-embedded if given two disjoint regular closed sets of S they can be extended to disjoint regular closed sets of X (that is, if S is R^*-embedded and R-extendable).*

(j) *In a normal space $FF \to G^* \to R^*$.*

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PROOF. (b) Let R be a regular closed in S. Then $S \sim R$ and R^i have disjoint open extensions $E(S \sim R)$ and $E(R^i)$ in X. Then $E(R^i)^c$, closure in X, is a regular closed extension of R; the R-extendability of open and dense sets follows from the GG-embeddings of these sets [2].

(c) $G^* \to R^*$. Let A and B be disjoint regular closed sets of S; then A^i and B^i, interiors with respect to S, are closure disjoint open sets of S with closure disjoint open extension $E(A^i)$ and $E(B^i)$ in X. The closures of these sets with respect to X are regular closed. The proof of $TG^* \to TR^*$ is similar. For dense subsets of open sets $FF \to G^*$. Let S be dense in G an open set of X. Let U and V be closure disjoint open subset of S. The sets U^{ci} and V^{ci} with respect to G are disjoint open extensions of U_c and V_c (closures of U and V in S), by the FF-embeddings of S in X, the closures in X of U^{ci} and V^{ci} are disjoint. For dense subsets of X, $R^* \to G^*$. The proof is identical with the one above noting that closures of U and V in S and in X are regular closed. For dense sets $R^* \to C^*$. Let S be dense in X and let Z and H be disjoint zero sets of S. Let $G(Z)$ and $G(H)$ be closure disjoint open sets of S containing Z and H respectively. The sets $G(Z)^c$ and $G(H)^c$, closures with respect to S, have disjoint closures in X; so Z and H have disjoint closures in X. Hence S is C^*-embedded in X.

(d) Regular closed subsets of a regular closed set of a space are regular closed.

(f) The first part is immediate. If Q and R are regular closed in S, they are contained in disjoint regular closed sets of x which contain Q^c and R^c, closures with respect to T which are regular closed sets of T.

(g) The R^*-embedding case follows from (d). From the R^*-embedding case and the last part of (c) the G^*-embedding case follows.

(i) This follows from (b).

We omit the proofs of (a), (e), (h) and (j).

REMARK 2.2. (d) is not true in general for TG^*-embeddings and (f) is not in general true for R^*-embeddings (Theorem 6.10).

3. C^*-embeddings of regular open sets

COROLLARY 3.1. *The following are equivalent for a Tychonoff space X.*

(a) *Every regular open set is C^*-embedded in its closure.*

(b) *Every regular open set is R^*-embedded in its closure.*

(c) *Every regular open set is R^*-embedded in X.*

(d) *Every regular open set is G^*-embedded in its closure.*

(e) *Every regular open set is G^*-embedded in X.*

PROOF. (e) ↔ (d) → (b) ↔ (c) → (a) follow from Theorem 2.1, (c) and (g). (a) → (e). Let U and V be closure disjoint open sets of a regular open set G. The sets U^{ci} and V^{ci}, closures and interiors with respect to G, are zero sets of $U^{ci} \cup V^{ci}$ that can be extended to disjoint zero sets of $\overline{U} \cup \overline{V}$ (closures in X). Since $U^{ci} \cup V^{ci}$ is regular open in G and hence in X, then $\overline{U} \cap \overline{V} = \varnothing$ and G is G^*-embedded in X.

Related to Corollary 3.1 is the question as to whether a space is extremally disconnected (ED) if the regular open sets are C^*-embedded. On the positive side we have the following result.

THEOREM 3.2. *An Oz-space (space where open subsets are z-embedded* [6]) *is extremally disconnected (ED) iff every regular open set is C-embedded. A space such that the union of two regular open sets is C^*-embedded is ED.*

PROOF. In an ED space every regular open set is closed and thus C-embedded. Conversely in an Oz-space every regular open set is a cozero set and hence by the C-embedding is a zero-set; so the space is ED. We omit the proof of the last part.

Except to replace C-embedding by well embedding (S is well-embedded in X if S is completely separated from any disjoint zero set of X contained in $X \sim S$ [17]) we can not substantially improve Theorem 3.2. For we have below an example of a Tychonoff space such that every regular open set is C-embedded that is not ED and an Oz-space such that every regular open set is C^*-embedded but is not ED.

EXAMPLE 3.3. Let f be the quotient map and KN be the quotient image of βN obtained by identifying two points of $\beta N \sim N$. Designate the resulting point as $[y]$. We will show that every regular open set of KN is C-embedded. For R regular closed $R \sim R^i = \varnothing$ or $[y]$ since in $\beta N\, f^{-1}(R)$ is open. If $y \notin R$, $R = R^i$ so R^i is C-embedded in X. This also happens if $y \in R^i$. Suppose $y \in R$ and $R \neq R^i$, there exists N_1, N_2 such that $N = N_1 \cup N_2$ and $N_1 \cap N_2 = \varnothing$ and $y \in N_1^c \cap N_2^c$. Thus $KN = \beta N_2 \cup \beta N_2$ and $\beta N_1 \cap \beta N_2 = [y]$. Then neither one of $R^i \cap \beta N_1$ and $R^i \cap \beta N_2$ contains y and y is a limit point of only one of them; for if y were a limit point of both then R would be in KN and $R = R^i$. Then R^i is C-embedded in R since one of $R^i \cap \beta N_1$ and $R^i \cap \beta N_2$ is clopen and R^i is pseudocompact.

It is clear that Example 3.3 is not Oz unlike the next example in which the regular open sets are C^*-embedded but not C-embedded.

EXAMPLE 3.4. Let X be the quotient map of $\Pi = N \cup D$ [13, page 197] obtained by identifying two points of D. By the same argument as in Theorem 1 we can show that regular open sets of X are C^*-embedded in their closure. We can show that X is Oz by using the fact that every point of X is a zero set and the fact that Π is Oz. This establishes that regular closed sets are C^*-embedded since Oz-spaces are metanormal [15] and [19].

The following example shows that a space may satisfy the conditions of Corollary 3.1 but not every regular open set is C^*-embedded in the space.

EXAMPLE 3.5. Let M be a discrete set of cardinality c. We partition M into an uncountable number of denumerable infinite families $\{P_a\}$. Let $P'_a = \mathrm{cl}_{\beta M} P_3 \sim M$. The c sets $\{P_a\}$ are pairwise disjoint subsets of $\beta M \sim M$. Construct D by assigning one point d_a for each set P'_a and let $D = \{d_a\}$. Compare with construction of Π [13]. Using the quotient topology, let $X = N \cup D \cup M$ where $N \cup D = \Pi$ and $D \cup M \subset \beta M$ and $(N \cup D) \cap (D \cup M) = D$. The regular open set M is C^*-embedded in $D \cup M$ but not in X. Otherwise D would be C^*-embedded in Π since $D \cup M$ is normal, since the family $\{d_a \cup P_a\}$ is discrete. We will show that every regular open set is C^*-embedded in its closure. Let R be regular closed. Then $R = R_M \cup R_N$ where $R_N = (R \cap N)^c$ and $R_M = (R \cap M)^c$ and R_M and R_N are regular closed. Consequently $R^i = (R_N \cap R_M) \cup (R \cap N) \cup (R \cap M)$. For let $y \in R_N \cap R_M$, then $y \in (R \cap N)^c \cap (R \cap M)^c$ and $(R \cap N)^c$ is open in $N \cup D$ and $(R \cap M)^c$ is open in $D \cup M$. There exists open G_y of $(R \cap N)^c$ such that $G_y \cap D = \{y\}$ and G open H_y of $(R \cap M)^c$ such that $H_y \cap D = \{y\}$, $G_y \subset (R \cap N)^c$, $H_y \subset (R \cap M)^c$. The set $H_y \cup G_y$ is open in X and contained in R. If f is continuous on R^i, f_n (f restricted to N) can be extended continuously on R_M. Thus f can be extended continuously to R since f is uniquely defined on $R_M \cap R_N$.

4. Relation to C^*-embeddings and metanormal spaces

DEFINITION 4.1. A space X is metanormal (MN) if given A and B regular closed, $A \cap B = \varnothing$, there exists disjoint open sets G and H such that $A \subset G$, $B \subset H$.

These spaces were studied systematically in [19] and [21] and called κ-normal and mildly normal respectively.

THEOREM 4.2 [19] and [21]. *The following are equivalent for a space X.*

(a) *X is MN.*

(b) *Given A and B, disjoint regular closed sets, they are completely separated in X.*

(c) *Every regular closed set of X is C^*-embedded.*

Using the fact [2] that a set S is z-embedded in a space X if any two disjoint cozero sets may be extended to disjoint cozero sets of X, we may establish the following equivalence for MN.

(d) *Every regular closed set is z-embedded.*

THEOREM 4.3. *The following hold.*

(a) *If $S \subset X$, S is MN, then $C^* \to G^* \to R^*$.*

(b) *If X is MN, then for a subset S, $R^* \to G^* \to C^*$.*

(c) *If X is MN and S is R^*-embedded or G^*-embedded, then S is MN.*

PROOF. (a) $C^* \to G^*$. Let G and H be closure disjoint open sets of S. Let Z and Q be disjoint zero sets of S such that $G \subset Z$ and $H \subset Q$ guaranteed by Theorem A. Let $E(Z)$ and $E(Q)$ be disjoint zero set extensions of Z and Q respectively which are in turn contained in closure disjoint open sets G' and H'. Since $G \subset G'$, $H \subset H'$, and by Theorem 1, S is G^*-embedded and hence R^*-embedded in X.

(b) $R^* \to G^*$. Let G and H be closure disjoint open sets of S then G^c and H^c (closures in S) are contained in disjoint regular closed sets A and B of X. There exists closure disjoint open sets of X, A' and B' such that $A \subset A'$ and $B \subset B'$ by Theorem 4.2. Then $G \subset A'$ and $H \subset B'$. An application of Theorem 2.1 completes the proof. $G^* \to C^*$. Let Z and Q be disjoint zero sets of S. They are contained in closure disjoint open sets G and H of S. Let $E(G)$ and $E(H)$ be closure disjoint open extensions of G and H respectively in X. By Theorem 4.2, G and H will be contained in disjoint zero sets of X establishing the C^*-embedding of S in X.

(c) Let A and B be disjoint regular closed sets of S. By the R^*-embedding of S in X, there exists disjoint regular closed sets of X, A' and B' such that $A \subset A'$ and $B \subset B'$. By the MN property of X there exists disjoint open sets of X, G and H such that $A' \subset G$, $B' \subset H$. Then $G \cap S$ and $H \cap S$ are disjoint open sets of S such that $A \subset G \cap S$ and $G \subset H \cap S$. The proof is completed by noting that $G^* \to R^*$.

COROLLARY 4.4. *If S is Tychonoff and MN and KS is a T_2 compactification of S then $KS = \beta S$ if S is R^*-embedded $=$ G^*-embedded in KS.*

COROLLARY 4.5. *For X Tychonoff, X is R^*-embedded $=$ G^*-embedded in βX if and only if X is MN.*

We note with Theorems 2.1 and 4.3 the following equivalences may be added to Theorem 4.2.

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(e) *Every R^*-embedded subset is C^*-embedded.*
(f) *Every regular closed set is G^*-embedded.*
(g) *Every R^*-embedded subset is G^*-embedded.*
(h) *The R^*-embedded and the G^*-embedded subsets are identical.*

COROLLARY 4.6. *In a hereditary MN-space the C^*-embedded, G^*-embedded and R^*-embedded subsets are identical.*

EXAMPLE 4.7. The example Π mentioned in Example 3.4 is hereditary *ED* so Π is hereditary *MN* but Π is not normal [13, page 197].

Lane [15] and Shchepin [19] have shown that every *Oz*-space is *MN* and Noble [18] has shown that the product of real lines is *Oz*.

EXAMPLE 4.8. A product of real lines is not *ED* or necessarily normal but is *MN*.

EXAMPLE 4.9. The set of all countable ordinals is not *Oz* [6] but is normal and hence is *MN*.

COROLLARY 4.10. *A Tychonoff space X is R^*-embedded (G^*-embedded) in every Tychonoff space in which X is embedded if and only if X is almost compact and MN.*

Later in Example 5.4, we will show that an almost compact space is not necessarily *MN*.

We can show that every dense subset of a Tychonoff space is *MN* if every open set is *MN*. This condition neither implies that the space is hereditary *MN* nor that every C^*-embedding or even C-embedding is a G^*-embedding in the space. For we may embed any Tychonoff space S, in the product of real lines in which every open set is *Oz* [6] and hence *MN* as a C^*-embedded subset. Since S may not be *MN*, S say may not be R^*-embedded or G^*-embedded in X.

5. Alexandroff's extension αX

DEFINITION 5.1 [5]. An open filter $\mathscr{G}$ is regular (completely regular) if for $G \in \mathscr{G}$ there exists $H \in \mathscr{G}$ there exists $H \in \mathscr{G}$ such that $\overline{H} \subset G$ (and a real function f on X such that $f(H) = 0$ and $f(X \sim G) = 1$).

Alexandroff [1] identified a fixed regular open ultrafilter (completely regular open ultrafilter) with the adherent point of the filter. The space $\alpha X(\alpha' X)$ consisted of $X \cup \{\mathscr{F}_a\colon a \in A\}$ where each $\mathscr{F}_a$ is a free regular (completely

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regular) open ultrafilter of X. A base for the topology on $\alpha X(\alpha' X)$ consists of the sets of the form $G' = G \cup \{\mathscr{F}_a\colon\ G \in \mathscr{F}_a\}$, that is, the strict extension [5]. Alexandroff showed that $\alpha' X = \beta X$. Votavova [23] proved the following theorem connecting αX and βX.

THEOREM 5.2. *A necessary and sufficient condition that $\alpha X = \beta X$ is that for every sequence of sets $\{A_n\colon\ A_n^c \subset A_{n-1}^i, A_n^i \neq 0,\ n = 1, 2, \dots\}$ there exists a continuous function to the reals f and k such that $f(\sim A_1) \subset [0]$ and $f(A_k) \subset [1]$. The superscripts i and c mean interior and closure respectively.*

It is immediate that if X is MN, X satisfies the condition of Votavova.

THEOREM 5.3. (a) *If X is MN, $\alpha X = \beta X$.* (b) *If X is R^*-embedded in αX, αX is regular.* (c) *If* (b) *holds and $\alpha X = \beta X$, X is MN.*

PROOF. (b) Let $x \in G^*$ a member of the base of αX. There exist $H \in \mathscr{G}x$ such that $\overline{H} \subset G$. Then $x \in \overline{H}^* \subset G^*$. The latter relation is due to the R^*-embedding of X in αX since G^* is regular open. (c) follows from Theorem 4.3.

EXAMPLE 5.4 (JACK PORTER). This is an example of a space satisfying the condition of Theorem 5.2 which is not MN. Let T_1, T_2 and T_3 be copies of the Tychonoff Plank and let $X = T_1 \cup T_2 \cup T_3$ which N_1 and N_2 and W_3 identified using the quotient topology. The one free regular ultrafilter is completely regular. Since T_3 and T_1 are regular closed sets of X not contained in disjoint open sets of X, X is not MN.

The above example also shows that an almost compact space may not be MN and also is an example of C^*-embedded subset that is not R^*- or G^*-embedded (in βX), by Theorems 4.3, and 5.2.

6. Metatransitivity of embeddings

DEFINITION 6.1. Let S be dense in X (in T) and let $S \subset T \subset X$. If S is A-embedded in X implies T is A embedded in X then A-embeddings are said to be metatransitive (paratransitive). If S is A-embedded in X implies S is A-embedded in T then A-embeddings are said to be hereditary.

It is well known that C- and C^*-embeddings are paratransitive; but FF-embeddings and z-embeddings are not metatransitive.

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EXAMPLE 6.2. Since N is Lindelöf, N is z-embedded in any compactification KN [7]. If KN is the compactification of Example 3.3, then $KN - [y]$ is not z-embedded in KN.

EXAMPLE 6.3. If M is a non-normal dense subset of βN, then M is not FF-embedded in βN, whereas N being normal, is FF-embedded in βN.

THEOREM 6.4. *R^*-embeddings are paratransitive.*

PROOF. Let S be R^*-embedded in X and be dense in $T \subset X$. Let A and B be disjoint regular closed sets of T, then $A \cap S$ and $B \cap S$ are regular closed sets of S for if $A = G^c$, G open in T, A must be the closure of $G \cap S$ in T. Otherwise there exists $F = (G \cap S)^c$, $F \subset A$, $F \neq A$. So $F \subset G$ and thus $G \sim F \neq \emptyset$ contrary to S being dense in T. So $A \cap S$ is the closure of $G \cap S$ in S; let A' and B' be disjoint regular closed sets of X suth that $A \cap S \subset A'$ and $B \cap S \subset B'$. Then $A \subset A'$, $B \subset B'$, since $A \subset (A \cap S)^c$ in X and $B \subset (B \cap S)^c$ in X. The R^*-embedding of T in X follows.

COROLLARY 6.5. *G^*-embeddings are metratransitive.*

PROOF. A dense embedding is an R^*-embedding if it is a G^*-embeddings.
We will later show that G^*-embeddings are not paratransitive.

COROLLARY 6.6. *Let $S \subset T \subset \beta S$. If S is MN then T is MN.*

PROOF. By Theorem 4.3, S is G^*-embedded in βS and by Corollary 6.5, T is G^*-embedded in βS; by Theorem 4.3 T in MN.

COROLLARY 6.7. *If $S \subset T \subset \beta S$ and S is FF-embedded in βS, then T is G^*-embedded in βS.*

PROOF. S is normal [2].

COROLLARY 6.8. *The Tychonoff Plank T and the square plank, $W^* \times W^* - (w^*, w^*)$ are MN.*

PROOF. The set $W \times N$ is normal and C^*-embedded in T and $W \times W$ is normal and C^*-embedded in $W^* \times W^*$. An application of Corollary 6.6 completes the proof.

We note that Zaitsev [19] has shown that the Niemytski plane is MN.

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REMARK 6.9. A theorem of the type of Corollary 6.6 is satisfied for many properties, such as pseudocompactness, extremal disconnectedness, basic disconnectness, being an F-space and connectness [13]. If we replace βS by υS, the property is satisfied by being a P-space, [13], Oz [6] and weakly δ-normally separated [16].

THEOREM 6.10. *In general* (a) $G^* \nrightarrow C^*$. (b) $C^* \nrightarrow G^*$. (c) *R^*-embedding are not hereditary*, (d) *G^*-embeddings are not paratransitive.*

PROOF. Let $X = N \cup D \subset \beta N$, $N \cap D = \varnothing$, D countable and discrete. Let $Y = X \cup T$, where T is the Tychonoff Plank and N is also the side of the Tychonoff Plank homeomorphic to the integers and the topology in Y is the quotient topology. (a) The set N is C^*-embedded in X but not in $Y = X \cup T$. For if N were C^*-embedded in Y, X would be C^*-embedded in Y, X would be C^*-embedded in Y and hence in T. However similar to the proof of Corollary 3.1 (a) $\to$ (c), N is G^*-embedded in Y. (b) the space Y is not MN since the regular closed set X is not C^*-embedded in Y, so by Theorem 4.3, Y is not G^*-embedded in βY. (c), the set D is R^*-embedded in Y since X is regular closed in Y (Theorem 2.1): By Theorem 4.3, D is not R^*-embedded in MN-space T since D is not C^*-embedded in T. (d) the set X is not G^*-embedded in Y even though N is G^*-embedded in Y by part (a). For if X were G^*-embedded in Y, D would be G^*-embedded in Y and hence in T since G^*-embeddings are hereditary.

We note that we could have used Example 3.5 to prove Theorem 6.10.

THEOREM 6.11. *In an MN-space X an R^*-embedding is hereditary and G^*-embedding is paratransitive.*

PROOF. From Theorem $A(h)$ the R^*- and G^*-embedded sets are identical and R^*-embeddings are paratransitive (Theorem 6.4) and G^*-embeddings are hereditary (Theorem 2.1).

7. R^*- and G^*-embeddings of every set

DEFINITION 7.1 [10]. A space X is seminormal if given two disjoint closed sets A and B of X, there is a regular closed set R such that $A \subset R \subset \sim B$.

It follows that an MN seminormal space is normal.

THEOREM 7.2. (a) *A space is normal if and only if every closed set of X is G^*-embedded.* (b) *A space X is seminormal if and only if every closed set of X is R^*-embedded.*

PROOF. We prove only (b). Let A and B be regular closed sets of a closed set F of a seminormal space X. There exists disjoint regular closed sets of X, A' and B' such that $A \subset A'$, $B \subset B'$ so F is R^*-embedded in X. Conversely suppose every closed set of X is R^*-embedded and A and B are disjoint closed sets of X. Then A and B are disjoint regular closed sets of $A \cup B$, so A and B are contained in disjoint regular closed sets of X.

THEOREM 7.3 (JACK PORTER). *Let S be Tychonoff and not normal. Then S may be embedded as a closed set of seminormal Tychonoff space that is not normal.*

PROOF. Katětov [14] has shown that any Hausdorff space S may be embedded in a semiregular Hausdorff space X as a closed set. Dickman and Zame [10] showed that X is seminormal and Porter showed that X is Tychonoff if S is Tychonoff.

THEOREM 7.4. *The following are equivalent for a space X.*
(a) *X is extremally disconnected (ED).*
(b) *Every dense open set is R^*-embedded.*
(c) *Every dense open set is G^*-embedded.*

PROOF. (a) → (b), since $ED \to MN$ every dense and open set is MN and C^*-embedded so these sets are G^*-embedded. (b) → (c) from Theorem 2.1. (c) → (a). Let G and H be disjoint open sets of X. Then $U = G \cup H$ is dense and open in X. Since G and H are closed and disjoint in U they have closure disjoint extensions in X. So X is ED.

THEOREM 7.5. *The following are equivalent for a space X.*
(a) *X is normal and hereditarily ED.*
(b) *Every subset of X is G^*-embedded.*
(c) *Every subset of X is R^*-embedded.*

PROOF. (a) → (b), (a) → (c) since very subset is C^*-embedded [13] and MN, every subset is G^*- and R^*-embedded by Theorem 4.3. (b) → (c) by Theorem 2.1. (c) → (a). The space X is ED by Theorem 7.4 and hence MN. Hence every R^*-embedded set is C^*-embedded; so X is hereditary ED and normal.

8. Remarks on MN-spaces

The following theorem is easily proved.

THEOREM 8.1. *Every Oz-space is MN* [15]. *An MN F′-space is an F-space. A δ-normal* [16] *MN-space is weakly δ-normally separated. Every regular closed set of a space X is C-embedded if X is MN and wewakly δ-normally separated.*

EXAMPLE 8.2. There exists a δ-normal space that is not MN. Mack [16] has constructed a δ-normal space that is not weakly δ-normally separated. Such a space cannot be MN by Theorem 8.1.

EXAMPLE 8.3. There exists an F'-space S that is not MN. Comfort, Hindman and Negrepontis [9] have constructed an F'-space that is not an F-space. By Theorem 8.1, S is not MN.

The question arises in view of Theorem 4.3 as to how far from normality is MN or how general are spaces in which we can describe C^*-embeddings in terms of regular closed sets or closure disjoint open sets. A normal space S is C^*-embedded in every normal space in which S is embedded as a closed set whereas a Tychonoff space S is C^*-embedded in every Tychonoff space in which S is embedded as a closed set if and only if S is almost compact.

THEOREM 8.4. *A Tychonoff space S is* (a) *C^*-embedded* ((b) *z-embedded*) *in every Oz-space in which S is embedded as a closed set if and only if S is almost compact* (*or Lindelöf*).

PROOF. Every almost compact Tychonoff space S is C^*-embedded in every Tychonoff space in which S is embedded, a result of Hewitt. A Tychonoff space S is z-embedded in every Tychonoff space in which S is embedded if S is almost compact or Lindelöf, a result of Blair and Hager [7]. (a) If S is not almost compact, let KS be a compactification of S such that $KS \neq \beta S$. We embed K in the product of closed intervals, $\Pi\{I_a\colon a \in A,\ A \text{ countable},\ I_a = [0,1]\}$. In turn we embed this product in $Y = \Phi\{J_a\colon a \in A, J_a = 0,2\}$. In turn we embed this product in $Y = \Pi\{J_a\colon a \in A, J_a = [0,2]\}$. Set $X = Y \sim (KS \sim S])$. Since X contains the Σ-product with $x_a = 2$ for each a, X is C^*-embedded in Y [12] based on work of H. H. Corson and A. M. Gleason. So that S is closed in X but not C^*-embedded in X. Otherwise S would be C^*-embedded in Y and hence in KS. The product Y is Oz being the product of separable metric spaces [18]. Since X is dense in Y, X is also Oz [6]. The result follows. (b) If S is not Lindelöf or almost compact, there is a compactification KS in which S is not z-embedded [7].

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Construct Y and X as above since Y is Oz, X is z-embedded in Y [6]. So S cannot be z-embedded in X. Otherwise S would be z-embedded in Y and hence in KS.

DEFINITION 8.5. A property P is a Tychonoff C^*-associate (z-associate) if for a space S having a property P to be C^*-embedded in every Tychonoff space having property P in which S is embedded as a closed set, it is necessary that the S be almost compact (almost compact or Lindelöf).

COROLLARY 8.6. *The properties Oz, MN and weakly δ-normally separated are Tychonoff C^*-associates and z-associates.*

PROOF. $Oz \to MN$ by Theorem 8.1 and $Oz \to$ weakly δ-normally separated; for in an Oz-space, regular closed sets are zero sets [6].

From Theorem 7.3 we can prove the following corollary of Theorem 8.4.

COROLLARY 8.7. *Seminormality is* (a) *a Tychonoff C^*-associated* (b) *a Tychonoff z-associate.*

PROOF. (a) Let S be a Tychonoff space that is not almost compact. By Theorem 8.4 S is embedded in an Oz-space X as a closed and not C^*-embedded set. Then X may be embedded as a closed subset of a Tychonoff seminormal space Y by Theorem 7.3. Then S is closed in Y but not C^*-embedded in Y. The proof of (b) is similar.

By the method of Corollary 8.7, we can show that P is a Tychonoff C^*-associate if every Tychonoff space can be embedded as a closed set of a Tychonoff space having property P.

For purpose of comparison a stronger condition than MN, almost normal, is mentioned here.

DEFINITION 8.8 [20]. A space is AN if given two disjoint closed sets A and B, one regular closed, there exists two disjoint open sets G and H such that $A \subset G$, $B \subset H$. (There is an equivariant formulation with G and H zero sets.)

Every extremally disconnected space is AN. We have

$$\begin{array}{lcl} ED & \to & AN \\ \downarrow & & \downarrow \\ Oz & \to & MN + \text{weakly } \delta\text{-normally separated.} \end{array}$$

Unlike MN and seminormal, AN is not a Tychonoff C^*-associate. We can easily prove using the equivalent formulation of Definition 6.1:

THEOREM 8.9. *Let S be an AN-space which is Lindelöf or almost compact. Then S is C^*-embedded in every AN-space in which S is embedded as a closed set.*

EXAMPLE 8.10. The Tychonoff plank is not AN as it has a closed countable subet that is not C^*-embedded contrary to Theorem 8.9.

EXAMPLE 8.11. The square plank is MN (Example 6.2) and satisfies Z, being countable compact [24], is not AN. For the closed sets W_1 and W_2 are contained in disjoint open sets G_1 and G_2 respectively but are not contained in disjoint zero sets [13]. So that $\overline{G}_1$ and W_2 are not completely separated.

REMARK 8.12. Some results in regard to absolute C^*-embedding of properties betwen Tychonoff and normal may be found in [4]. The absolute C^*-embedding properties of ED space were investigated by Dow [11].

9. R-extendability

The concept of R-extendability was defined in Definition 1.1 and in Theorem 2.1 it was shown that a GG-embedded subset was R-extendable. Analogous to Theorem 4.3 we prove

THEOREM 9.1.(a) *A z-embedded Oz-space X is R-extendable.* (b) *An R-extendable subset S of an Oz-space X is z-embedded and* (c) *Oz.* (d) *A space X is Oz if every R-extendable set of X is z-embedded.*

PROOF. (a) From [2] a z-embedded Oz-space is GG-embedded and hence R-extendable. (b) Let Z be a zero set of S. Then $S - Z = \bigcup H_n$ where each H_n is a zero set of S. There exists regular closed neighborhoods of Z, $\{R_n\}$, such that $R_n \cap H_n = \varnothing$. Then R_n has a regular closed extension $E(R_n)$ which by the Oz-property of X is a zero set. Then $\bigcap E(R_n)$ is a zero set extensions of Z in X. (c) If R is a regular closed set of S, $E(R)$ is a zero set by the Oz-property of X. So $R = S \cap E(R)$ is a zero set of S. (d) From [2] and Theorem 2.1 every open set is R-extendable and hence z-embedded. So X is Oz.

COROLLARY 9.2. *An Oz-space is weakly perfectly normal* [6] *if every subset is R-extendable.*

COROLLARY 9.3. *A Tychonoff space S is R-extendable in every Tychonoff space in which S is embedded if S is almost compact or Lindelöf, and Oz.*

EXAMPLE 9.4. The set $\beta N \sim N$ is G^*- and R^*-embedded in βN but is not R-extendable as $\beta N \sim N$ is not Oz and βN is Oz.

EXAMPLE 9.5. The set N is z-embedded and Oz. So N is R-extendable to T, the Tychonoff Plank; but N *is not* R^*-*embedded in* T.

From Theorem 2.1 it follows that every subset of a completely normal space is R-extendable since every subset is GG-embedded. However the result can be improved.

THEOREM 9.6. *The following are equivalent for a space X.*

(a) *X is hereditary seminormal.*

(b) *Every closed subset of X is R-extendable.*

(c) *Every subset of X is R-extendable.*

COROLLARY 9.7. *If every closed set is R-extendable then every closed set is R^*-embeddable.*

Analogous to Theorem 9.1 we have the following results.

THEOREM 9.8. (a) *Let $S \subset X$. An R^*-embedded ED-space S is R-extendable to X.* (b) *An R-extendable subset of an ED-space X is R^*-embedded in X and* (c) *ED.*

COROLLARY 9.9. *The following are equivalent for a space X.*

(a) *X is ED and every subset is R-extendable.*

(b) *Every subset of X is R^*-embedded.*

(c) *X is hereditarily ED and normal.*

10. Conclusion

In relating R^*- and G^*-embedding to C^*-embedding, the MN property plays a vital role. Singal and Singal [21] noted that the MN property was preserved by a mapping that is both open and closed. We note using the following further equivalence for MN in Theorem 4.2 we can prove that the MN property is preserved under cozero set preserving mappings.

(i) Two closure disjoint open subsets of X are completely separated. Example 3.5 shows that MN is not preserved under closed maps. In regard to products there are some results in [19] including that the product of two metanormal spaces may not be MN. This was pointed out to the author by E. Van Douwen. Analogous to a theorem of Blair [6] on Oz-spaces one can prove the following.

THEOREM 10.1. *Assume that each finite subproduct in* $X = \prod AX_a$ *satisfies the countable chain condition. If every countable subproduct of* X *is* MN *then* X *is* MN.

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