A REPRESENTATION THEOREM FOR DISTRIBUTIVE l-MONOIDS

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ABSTRACT. In this note the Holland representation theorem for l-groups is extended to l-monoids by the following theorem: an l-monoid is distributive if and only if it may be embedded into the l-monoid of order-preserving functions on some totally ordered set. A corollary of this representation theorem is that a monoid is right orderable if and only if it is a subsemigroup of a distributive l-monoid; this result is analogous to the group theory case.

By an l-semigroup we mean a semigroup S equipped with a lattice order so that multiplication distributes over ∧ and ∨ on the left and the right. An l-semigroup is distributive if it is distributive when considered as a lattice. If T is a totally ordered set and S(T) is the set of order-preserving functions on T, then S(T) is a distributive l-monoid which is considered in some detail in [4].

A right l-congruence on an l-semigroup is a right congruence on the semigroup structure and a congruence on the lattice structure. The first theorem provides a criterion for the existence of such congruences. The proof of this theorem relies heavily on an analogous result of T. Merlier [8] for l-congruences. We shall need the following definition: a subset H of an l-semigroup S is a prime l-ideal if it is a prime lattice ideal of S; that is, H is a sublattice of S, if a ≤ b ∈ H then a ∈ H, and if a ∨ b ∈ H, then a or b ∈ H. And we define the relation R_H on S by a R_H b iff (a)H = (b)H where (a)H = {x ∈ S: ax ∈ H}. The relation _H is defined dually with H(a) = {x ∈ S: xa ∈ H}.

**Lemma 1.** If H is a prime l-ideal of S and a, b ∈ S, then
(i) (a)H is a prime l-ideal of S;
(ii) (a ∧ b)H = (a)H ∪ (b)H and (a ∨ b)H = (a)H ∩ (b)H;
(iii) {(a)H: a ∈ S} is totally ordered under set inclusion; and
(iv) _H is a right l-congruence on S for which S/_H is a totally ordered set.

**Proof.** The proofs of (i) and (ii) are straightforward. To show (iii), let a, b ∈ S and assume that (a)H ∉ (b)H. Then there exists x ∈ (a)H \ (b)H. Let y ∈ (b)H. Then by (i) and (ii), x ∧ y ∈ (a)H ∩ (b)H = (a ∨ b)H. Thus x or y ∈ (a)H ∩ (b)H. Since x ∉ (b)H, we must have that y ∉ (a)H. Consequently, (b)H ⊆ (a)H, thus
proving part (iii). That $\mathcal{H}$ is a right congruence follows from ([2], p. 182). And by (ii), $\mathcal{H}$ is a right $l$-congruence. To show that $S/\mathcal{H}$ is totally ordered, let $\bar{a}$ denote the $\mathcal{H}$-class of $S$ containing $a$. Then $\bar{a} \leq \bar{b}$ iff $a \cdot b = a$ iff $(a)H \cup (b)H = (a \cdot b)H = (a)H$ iff $(b)H \subseteq (a)H$. So it follows from (iii) that $S/\mathcal{H}$ is totally ordered.

In an analogous manner we may deduce that when $H$ is a prime $l$-ideal of $S$, $\mathcal{H}$ is a left $l$-congruence on $S$ for which $S/\mathcal{H}$ is totally ordered. Henceforth, we will be assuming that $S$ has a right identity. However, we remark that the following results can be easily dualized.

**Theorem 2.** Let $S$ be an $l$-semigroup with right identity $e$. Then $S$ admits a right $l$-congruence $\rho$ such that $S/\rho$ is totally ordered and nontrivial if and only if $S$ has a proper prime $l$-ideal.

**Proof.** Since $S/\rho$ is totally ordered and nontrivial, $ap \subseteq bp$ for some $a, b \in S$. Set $H = \{x \in S : xp \leq ap\}$. It is easily verified that $H$ is the required proper prime $l$-ideal. Conversely, suppose that $H$ is a prime $l$-ideal such that $a \in H$ but $b \notin H$ for some $a, b \in S$. Then $e \in (a)H$ and $e \notin (b)H$. Thus by Lemma 1, $\mathcal{H}$ is the desired right $l$-congruence.

**Theorem 3.** Let $S$ be an $l$-semigroup with right identity $e$. Then $S$ is distributive if and only if $S$ admits a collection $\{H_\lambda : \lambda \in \Lambda\}$ of prime $l$-ideals such that $\bigcap \{H_\lambda : \lambda \in \Lambda\}$ is the identity.

**Proof.** ($\Rightarrow$) In this case $S$ as a lattice may be represented as a subdirect product of totally ordered sets and so is clearly distributive.

($\Leftarrow$) Given $a, b \in S$, Iseki [7] has shown that the lattice $S$ admits a prime $l$-ideal $H$ such that $a \in H$ and $b \notin H$. Since $S$ has a right identity, $a \neq b(\mathcal{H})$. And thus the intersection of all such right $l$-congruences is clearly the identity.

Holland’s representation theorem [6] states that each $l$-group may be $l$-embedded into the $l$-group of order-preserving permutations of some totally ordered set. Since an $l$-group is always distributive ([1], p. 19), the generalization of this to $l$-semigroups is the following theorem. The proof merely mimics the one for groups.

**Theorem 4.** An $l$-semigroup with right identity is distributive if and only if it may be $l$-embedded into the $l$-monoid of order-preserving functions on some totally ordered set.

**Proof.** It is obvious that any $l$-subsemigroup of $S(T)$ for some totally ordered set $T$ is distributive. Conversely, suppose that $S$ is a distributive $l$-semigroup with right identity. Then $S$ admits a collection $\{\rho_\lambda : \lambda \in \Lambda\}$ of right $l$-congruences such that each $S/\rho_\lambda$ is totally ordered, and $\bigcap \rho_\lambda$ is the identity. Choose a total order for $\Lambda$, and then order the disjoint union $\bigcup (S/\rho_\lambda)$.
lexicographically. That is, \([a]_\lambda \leq [b]_\mu\) if and only if \(\lambda < \mu\), or \(\lambda = \mu\) and \([a]_\lambda \subseteq [b]_\lambda\). This is the totally ordered set \(T\). Then define \(\phi : S \rightarrow S(T)\) by setting \([a]_\lambda (s \phi) = [as]_\lambda\). This is well-defined since the \(\rho_\lambda\)'s are right congruences. It is one-to-one since if \(a \neq b\), there exists \(\rho_\lambda\) such that \(a \notin b(\rho_\lambda)\) and so \([I]_\lambda (a \phi) \neq [I]_\lambda (b \phi)\). Finally, it is easily checked that \(\phi\) is an \(l\)-homomorphism.

In a partially ordered semigroup an element \(s\) is right positive if \(xs \geq x\) for all \(x\); left positive and right and left negative are defined analogously ([5], p. 216). But in \(S(T)\), left and right coincide and in fact an element is positive (negative) if it is greater (less) than the identity \(e\). Furthermore, direct calculation reveals that each element \(\alpha \in S(T)\) may be written as \(\alpha = (\alpha \vee e)(\alpha \wedge e)\), a product of a negative and positive element, just as in \(l\)-groups. Our representation theorem enables us to now conclude that this decomposition property also holds for arbitrary distributive \(l\)-monoids.

A monoid \(S\) is right-orderable if it admits a total order which is preserved by multiplication on the right. From Holland’s representation theorem it follows easily that the class of right-orderable groups coincides with the class of subgroups of \(l\)-groups ([3] and [6]). The same proof and Theorem 4 allows us to conclude that the class of right-orderable monoids coincides with the class of submonoids of distributive \(l\)-monoids.

Can Theorem 4 be stated for distributive \(l\)-semigroups lacking a one-sided identity? If so, such a semigroup would then be embeddable in an \(l\)-semigroup with identity, namely \(S(T)\). Thus this question reduces to the following: which distributive \(l\)-semigroups can be embedded into some distributive \(l\)-monoid?

REFERENCES