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A REPRESENTATION THEOREM FOR DISTRIBUTIVE l-MONOIDS

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ABSTRACT. In this note the Holland representation theorem for l-groups is extended to l-monoids by the following theorem: an l-monoid is distributive if and only if it may be embedded into the l-monoid of order-preserving functions on some totally ordered set. A corollary of this representation theorem is that a monoid is right orderable if and only if it is a subsemigroup of a distributive l-monoid; this result is analogous to the group theory case.

By an *l-semigroup* we mean a semigroup S equipped with a lattice order so that multiplication distributes over $\wedge$ and $\vee$ on the left and the right. An l-semigroup is *distributive* if it is distributive when considered as a lattice. If T is a totally ordered set and $S(T)$ is the set of order-preserving functions on T, then $S(T)$ is a distributive l-monoid which is considered in some detail in [4].

A *right l-congruence* on an l-semigroup is a right congruence on the semigroup structure and a congruence on the lattice structure. The first theorem provides a criterion for the existence of such congruences. The proof of this theorem relies heavily on an analogous result of T. Merlier [8] for l-congruences. We shall need the following definition: a subset H of an l-semigroup S is a *prime l-ideal* if it is a prime lattice ideal of S; that is, H is a sublattice of S, if $a \leq b \in H$ then $a \in H$, and if $a \wedge b \in H$, then a or $b \in H$. And we define the relation $\mathcal{R}_H$ on S by $a\,\mathcal{R}_H b$ iff $(a)H = (b)H$ where $(a)H = \{x \in S\colon ax \in H\}$. The relation ${}_H\mathcal{R}$ is defined dually with $H(a) = \{x \in S\colon xa \in H\}$.

LEMMA 1. *If H is a prime l-ideal of S and $a, b \in S$, then*
(i) *$(a)H$ is a prime l-ideal of S;*
(ii) *$(a \wedge b)H = (a)H \cup (b)H$ and $(a \vee b)H = (a)H \cap (b)H$;*
(iii) *$\{(a)H\colon a \in S\}$ is totally ordered under set inclusion; and*
(iv) *$\mathcal{R}_H$ is a right l-congruence on S for which $S/\mathcal{R}_H$ is a totally ordered set.*

Proof. The proofs of (i) and (ii) are straightforward. To show (iii), let $a, b \in S$ and assume that $(a)H \nsubseteq (b)H$. Then there exists $x \in (a)H \backslash (b)H$. Let $y \in (b)H$. Then by (i) and (ii), $x \wedge y \in (a)H \cap (b)H = (a \vee b)H$. Thus x or $y \in (a)H \cap (b)H$. Since $x \notin (b)H$, we must have that $y \in (a)H$. Consequently, $(b)H \subseteq (a)H$, thus

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proving part (iii). That $\mathcal{R}_H$ is a right congruence follows from ([2], p. 182). And by (ii), $\mathcal{R}_H$ is a right l-congruence. To show that $S/\mathcal{R}_H$ is totally ordered, let $\bar{a}$ denote the $\mathcal{R}_H$-class of S containing a. Then $\bar{a} \leq \bar{B}$ iff $\overline{a \wedge b} = \bar{a}$ iff $(a)H \cup (b)H = (a \wedge b)H = (a)H$ iff $(b)H \subseteq (a)H$. So it follows from (iii) that $S/\mathcal{R}_H$ is totally ordered.

In an analogous manner we may deduce that when H is a prime l-ideal of S, ${}_H\mathcal{R}$ is a left l-congruence on S for which $S/{}_H\mathcal{R}$ is totally ordered. Henceforth, we will be assuming that S has a right identity. However, we remark that the following results can be easily dualized.

THEOREM 2. *Let S be an l-semigroup with right identity e. Then S admits a right l-congruence ρ such that S/ρ is totally ordered and nontrivial if and only if S has a proper prime l-ideal.*

Proof. Since S/ρ is totally ordered and nontrivial, $a\rho \subset b\rho$ for some a, $b \in S$. Set $H = \{x \in S : x\rho \leq a\rho\}$. It is easily verified that H is the required proper prime l-ideal. Conversely, suppose that H is a prime l-ideal such that $a \in H$ but $b \notin H$ for some a, $b \in S$. Then $e \in (a)H$ and $e \notin (b)H$. Thus by Lemma 1. $\mathcal{R}_H$ is the desired right l-congruence.

THEOREM 3. *Let S be an l-semigroup with right identity e. Then S is distributive if and only if S admits a collection $\{H_\lambda : \lambda \in \Lambda\}$ of prime l-ideals such that $\bigcap\{\mathcal{R}_{H_\lambda} : \lambda \in \Lambda\}$ is the identity.*

Proof. ($\Leftarrow$) In this case S as a lattice may be represented as a subdirect product of totally ordered sets and so is clearly distributive.

($\Rightarrow$) Given a, $b \in S$, Iseki [7] has shown that the lattice S admits a prime l-ideal H such that $a \in H$ and $b \notin H$. Since S has a right identity, $a \not\equiv b(\mathcal{R}_H)$. And thus the intersection of all such right l-congruences is clearly the identity.

Holland's representation theorem [6] states that each l-group may be l-embedded into the l-group of order-preserving permutations of some totally ordered set. Since an l-group is always distributive ([1], p. 19), the generalization of this to l-semigroups is the following theorem. The proof merely mimics the one for groups.

THEOREM 4. *An l-semigroup with right identity is distributive if and only if it may be l-embedded into the l-monoid of order-preserving functions on some totally ordered set.*

Proof. It is obvious that any l-subsemigroup of $S(T)$ for some totally ordered set T is distributive. Conversely, suppose that S is a distributive l-semigroup with right identity. Then S admits a collection $\{\rho_\lambda : \lambda \in \Lambda\}$ of right l-congruences such that each S/ρ_λ is totally ordered, and $\bigcap \rho_\lambda$ is the identity. Choose a total order for Λ, and then order the disjoint union $\bigcup(S/\rho_\lambda)$

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lexicographically. That is, $[a]_\lambda \le [b]_\mu$ if and only if $\lambda < \mu$, or $\lambda = \mu$ and $[a]_\lambda \le [b]_\mu$. This is the totally ordered set T. Then define $\phi : S \to S(T)$ by setting $[a]_\lambda(s\phi) = [as]_\lambda$. This is well-defined since the ρ_λ's are right congruences. It is one-to-one since if $a \ne b$, there exists ρ_λ such that $a \notin b(\rho_\lambda)$ and so $[l]_\lambda(a\phi) \ne [l]_\lambda(b\phi)$. Finally, it is easily checked that ϕ is an l-homomorphism.

In a partially ordered semigroup an element s is right positive if $xs \ge x$ for all x; left positive and right and left negative are defined analogously ([5], p. 216). But in $S(T)$, left and right coincide and in fact an element is positive (negative) if it is greater (less) than the identity e. Furthermore, direct calculation reveals that each element $\alpha \in S(T)$ may be written as $\alpha = (\alpha \vee e)(\alpha \wedge e)$, a product of a negative and positive element, just as in l-groups. Our representation theorem enables us to now conclude that this decomposition property also holds for arbitrary distributive l-monoids.

A monoid S is *right-orderable* if it admits a total order which is preserved by multiplication on the right. From Holland's representation theorem it follows easily that the class of right-orderable groups coincides with the class of subgroups of l-groups ([3] and [6]). The same proof and Theorem 4 allows us to conclude that the class of right-orderable monoids coincides with the class of submonoids of distributive l-monoids.

Can Theorem 4 be stated for distributive l-semigroups lacking a one-sided identity? If so, such a semigroup would then be embeddable in an l-semigroup with identity, namely $S(T)$. Thus this question reduces to the following: which distributive l-semigroups can be embedded into some distributive l-monoid?

References

1. A. Bigard, K. Keimel and S. Wolfenstein, *Groupes et Anneaux Réticulés*, Springer-Verlag, Berlin, 1977.

2. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Volume II, AMS, Providence, 1967.

3. P. Conrad, *Right-ordered groups*, Michigan Math. J. **6** (1959), 267–275.

4. C. C. Edwards and M. Anderson, *Lattice properties of the symmetric weakly inverse semigroup on a totally ordered set*, J. Austral. Math. Soc. **31** (1981), 395–404.

5. L. Fuchs, *Teilweise Geordnete Algebraische Strukturen*, Vanderhoeck and Ruprecht, Göttingen, 1966.

6. W. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math J. **10** (1963), 399–408.

7. K. Iséki, *A characterization of distributive lattices*, Nederl. Akad. Wetensch. Proc. Ser A **54** (1951), 388–389.

8. T. Merlier, *Sur les demi-groupes reticules et les o-demi-groupes*, Semigroup Forum **2** (1971), 64–70.

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