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TRANSITIVE VECTOR SPACES OF BOUNDED OPERATORS

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ABSTRACT. The linear subspace **S** of $B(X, Y)$, the space of bounded operators from the Banach space X to the Banach space Y, is said to be transitive if $\mathbf{S}x$ is dense in Y for all $x \neq 0$. We give a number of conditions, involving operators intertwined by **S**, which imply that **S** is not transitive, and conditions which, when $X = Y$, imply that the commutant of **S** is also not transitive.

0. **Introduction.** Suppose that **S** is a linear subspace of $B(X, Y)$, the space of bounded operators from the Banach space X to the Banach space Y. We say that **S** is *transitive* if $\mathbf{S}x$ is dense in Y for all $x \neq 0$. When $X = Y$ and **S** is an algebra, this is of course equivalent to **S** having no proper invariant subspaces. In this paper we give a number of conditions which guarantee that **S** is not transitive, and also conditions under which its commutant is not transitive when $X = Y$. Our results are similar to results proved for algebras in [2], [3], [5], [7]. Notice that when $X = Y$, the commutant of **S** is an algebra, even when **S** is not; so that in this case we will be proving that the commutant has an invariant subspace.

Our major result, Theorem (1.1), gives conditions when the space of operators intertwining two operators K and C is not transitive. In the case $K = C$, Theorem (1.1) reduces to Lomonosov's Theorem [5] on the existence of hyperinvariant subspaces of compact operators. In section 2, we prove, for vector spaces of operators, results similar to those proved in [2] and [3] for algebras intertwining bounded and compact operators. The proofs given for algebras in [2] and [3] need to be modified, and Theorem (2.3) is new even for algebras.

Some of our proofs will use what has come to be called Lomonosov's Lemma [7, Th. 2, p. 222] and its consequences. We start by restating Lomonosov's Lemma for vector spaces of operators.

LOMONOSOV'S LEMMA. *If* **S** *is a transitive subspace of* $B(X, Y)$, *if* K *is a non-zero compact operator from* Y *to* X, *and if* λ *is a non-zero scalar, then there is an operator* S *in* **S** *for which* λ *is an eigenvalue of* SK *and* KS.

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The proof given in [7, pp. 222–223] carries through if $\mathbf{S}$ is a vector space (or even just a convex set) and therefore provides an operator S for which 1 is an eigenvalue of SK. One then just needs to notice that SK and KS always have the same non-zero eigenvalues. Though I originally thought that I would need the above vector-space form of Lomonosov's Lemma, all the proofs below use only the algebra form or one of its consequences.

1. **The space of intertwining operators.** When $K = C$, the following theorem reduces to Lomonosov's Theorem that every compact operator has a hyperinvariant subspace.

THEOREM (1.1). *Suppose that K and C are non-zero bounded operators on the Banach spaces X and Y, respectively, and let $\mathbf{S} = \{S \in B(X, Y): SK = CS\}$. If either K or C is a compact operator and if there is a non-zero bounded operator T from Y to X for which $KT = TC$, then $\mathbf{S}$ is not transitive.*

Proof. We may assume, without loss of generality, that K is one–one. For if x belongs to the null space of K, $\mathbf{S}x$ belongs to the null space of C, and thus is not dense. Similarly we can assume that C has dense range, for if z belongs to the range of K, we have $\mathbf{S}z$ is contained in the range of C. We now consider separately the cases that K is compact and that C is compact.

Case 1. K is compact. Then every TS in $T\mathbf{S}$ commutes with the compact operator K. Let x be a non-zero vector in a hyperinvariant subspace of K. Then $T\mathbf{S}x$ is not dense in X. This completes the proof when T has dense range.

In general let E be the closure of the range of T, and notice that K restricts to a non-zero compact operator $\hat{K}$ on E. Let $\hat{\mathbf{S}}$ be the subspace of $B(E, Y)$ of the restrictions to E of the operators in $\mathbf{S}$. Then for all $\hat{S}$ in $\hat{\mathbf{S}}$ we have $\hat{S}\hat{K} = C\hat{S}$. Also, as an operator from Y to E, T has dense range and satisfies $\hat{K}T = TC$. So, by the dense range case considered above, there is a non-zero x in E for which $\hat{\mathbf{S}}x = \mathbf{S}x$ is not dense in Y.

Case 2. C is compact. Let N be the null space of T. Since $C(N) \subseteq N$ and C has dense range, it induces a non-zero compact operator, $\hat{C}$, on Y/N. Let π be the natural projection from Y onto Y/N, and let $\hat{T}$ be the map induced by T from Y/N to X. For each S in $\mathbf{S}$, $(\pi S)\hat{T}$ commutes with the non-zero compact operator $\hat{C}$. If $y + N$ is a non-zero element of a hyperinvariant subspace of $\hat{C}$, then $\pi\mathbf{S}\hat{T}(y + N) = \pi\mathbf{S}Ty$ is not dense in Y/N. Hence $\mathbf{S}Ty$ is not dense in Y and $Ty \neq 0$. This completes the proof.

2. **Intransitive operator ranges.** We say that a subspace of a Banach space is a *multirange* if it is the span of the range of a bounded multilinear operator from a product of Banach spaces. The most important special case is an *operator range*, that is, the range of a bounded linear operator. A number of invariant-subspace theorems [2], [3], [4], [6] have been proved by considering operator range algebras in the Banach space $B(X)$. Multiranges arise in the

study of joint invariant subspaces [3, Th. (3.2), p. 849]. For instance a two-sided ideal in a Banach algebra is jointly invariant under the algebras of left and of right multiplications (cf. [4]).

In this section, we extend results from [2] and [3] on operator range and multirange algebras of $B(X)$ to multirange subspaces of $B(X)$ and $B(X, Y)$. Theorem (2.3), below, is new even for operator range algebras.

We repeat from [3, p. 847] (cf. [2, p. 57]) the main fact that we will need about multiranges. Recall [1, p. 35] that a Riesz operator is an operator with the same spectral properties as a compact operator.

LEMMA (2.1). *Suppose that M is a multirange in the Banach space X and that K is a Riesz operator on X. If $K(M) \supseteq M$, then there is a spectral projection P of K with finite-dimensional null space for which $P(M) = \{0\}$.*

The next result generalizes [3, Th. (3.3), p. 850].

THEOREM (2.2). *Suppose that $\mathbf{S}$ is an infinite-dimensional multirange in $B(X, Y)$, that K is a bounded operator on X, and that C is a Riesz operator on Y. If $\mathbf{S}K \subseteq C\mathbf{S}$ and if there is a non-zero multirange M for which $K(M) \supseteq M$, then $\mathbf{S}$ is not transitive. When $X = Y$, the commutant of $\mathbf{S}$ is also not transitive.*

Proof. The multirange $\mathbf{S}M \subseteq \mathbf{S}K(M) \subseteq C(\mathbf{S}M)$. Since C is a Riesz operator, it follows from Lemma (2.1) that $\mathbf{S}M$ is finite-dimensional. Hence if x is a non-zero vector in M, then $\mathbf{S}x$ is finite-dimensional and certainly not dense. Also there is a non-zero operator S_0 in the kernel of the map $S \to Sx$ from $\mathbf{S}$ to Y. Hence S_0 has a non-trivial null space, which, when $X = Y$, is an invariant subspace for the commutant of $\mathbf{S}$. This completes the proof.

Some cases when there is an operator range M with $K(M) \supseteq M$ are given in [3, p. 850]. Probably the most important case (cf. [6, Th. 3, p. 116], [2, Th. 7, p. 61]) is when K has a non-zero eigenvalue. In this case we can take M to be the associated eigenspace. The next result is the dual of this special case.

THEOREM (2.3). *Suppose that $\mathbf{S}$ is an infinite-dimensional multirange in $B(X, Y)$, that K is a Riesz operator on X, and that C is a bounded operator on Y. If $C\mathbf{S} \subseteq \mathbf{S}K$ and if C^* has a non-zero eigenvalue, then $\mathbf{S}$ is not transitive. When $X = Y$, the commutant of $\mathbf{S}$ is also not transitive.*

Proof. Let f be an eigenvector for the non-zero eigenvalue λ of C^*. Then $\mathbf{S}^*f = \lambda^{-1}\mathbf{S}^*C^*f \subseteq (K^*/\lambda)\mathbf{S}^*f$. Then, by Lemma (2.1) applied to K^*/λ, there is a spectral projection P of K with finite-dimensional null space for which $P^*(\mathbf{S}^*f) = \{0\}$. Then if x belongs to the range of P, we have $f(Sx) = \{0\}$, so that $\mathbf{S}x$ is not dense in Y. Also, as in the proof of Theorem (2.2), there is an S_0 in $\mathbf{S}$ for which $S_0^*f = 0$. This S_0 does not have dense range, so, when $X = Y$, the closure of the range of S_0 is an invariant subspace of the commutant of $\mathbf{S}$. This completes the proof.

The full strength of the assumption that C or K is a Riesz operator is used in Theorems (2.2) and (2.3) only to prove the intransitivity of the commutant. Essentially the same proofs show that $\mathbf{S}$ is intransitive when the operators involved have a decomposition at 0 in the sense of [3, p. 845] (cf. [2, p. 56]). One just needs to use the full strength of [3, Th. (2.2), p. 847] instead of the special case given in Lemma (2.1), above.

The next corollary is a generalization of [2, Ths. 4 and 6, pp. 59–61] and [3, Th. (3.1), p. 848].

COROLLARY (2.4). *Suppose that $\mathbf{S}$ is an infinite-dimensional multirange in $B(X)$ and that K and C are compact non-zero operators on X. If either $\mathbf{S}K \subseteq C\mathbf{S}$ or $C\mathbf{S} \subseteq \mathbf{S}K$, then the commutant of $\mathbf{S}$ has an invariant subspace.*

Proof. Suppose that the commutant of $\mathbf{S}$ is transitive. It then follows from Lomonosov's Lemma that there are operators A and B in the commutant of $\mathbf{S}$ for which AK and CB have non-zero eigenvalues. Since CB is compact, $(CB)^*$ also has a non-zero eigenvalue. If $\mathbf{S}K \subseteq C\mathbf{S}$, then $\mathbf{S}(AK) \subseteq (AC)\mathbf{S}$ and it follows from Theorem (2.2) that the commutant of $\mathbf{S}$ has an invariant subspace.

If $C\mathbf{S} \subseteq \mathbf{S}K$, then $(CB)\mathbf{S} \subseteq \mathbf{S}(CB)$, and it follows from Theorem (2.3) that the commutant of $\mathbf{S}$ has an invariant subspace.

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