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FORMAL COMPLEXITY OF INVERSE SEMIGROUP RINGS

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A ring $(R, {}^*)$ with involution * is called *formally complex* if $\sum A_i A_i^* = 0$ implies that all A_i are 0. Let $(R, {}^*)$ be a formally complex ring and let S be an inverse semigroup. Let $(R[S], {}^*)$ be the semigroup ring with involution * defined by $\left(\sum a_i s_i\right)^* = \sum a_i^* s_i^{-1}$. We show that $(R[S], {}^*)$ is a formally complex ring. Let $(S, {}^*)$ be a semigroup with proper involution * $(aa^* = ab^* = bb^* \Rightarrow a = b)$ and let $\left(R, {}^{*'}\right)$ be a formally complex ring. We give a sufficient condition for $\left(R[S], {}^{*'}\right)$ to be a formally complex ring and this condition is weaker than * being the inverse involution on S. We illustrate this by an example.

DEFINITIONS: Let $(S, {}^*)$ be a semigroup with involution *; that is * satisfies $(ab)^* = b^*a^*$ and $(a^*)^* = a$. We say that * is *proper* and $(S, {}^*)$ is a *proper *-semigroup* if $(\forall a, b \in S)(aa^* = ab^* = bb^* \Rightarrow a = b)$. In this case we have *-cancellation in the sense that if $aa^*b = aa^*c$ then $a^*b = a^*c$ because then $b^*a.a^*b = b^*a.a^*c = c^*a.a^*c$. Let $(R, {}^*)$ be a ring with involution *; that is * satisfies $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$. Then * is *formally complex* and $(R, {}^*)$ is *a formally complex ring* if for every finite subset $\{a_1, \ldots, a_n\}$ of distinct elements of R, $\sum a_i a_i^* = 0$ implies that all a_i are 0. If for every a in R, $aa^* = 0$ implies that $a = 0$, we say that * is *proper* and $(R, {}^*)$ is a *proper *-ring*. Let $(R, {}^*)$ be a ring with involution and let $\left(S, {}^{*'}\right)$ be a *-semigroup with involution $^{*'}$. We define the natural involution * on the semigroup ring $R[S]$ as follows: $(\sum a_i s_i)^* = \sum a_i^* s_i^{*'}$. In particular, if S is an inverse semigroup and $(R, {}^*)$ is a *-ring, then the natural involution * on $R[S]$ takes the form $(\sum a_i s_i)^* = \sum a_i^* s_i^{-1}$.

It is proved in [**2**] that if S is an inverse semigroup and if $(R, {}^*)$ is a formally complex ring then $(R[S], {}^*)$ is a proper *-ring under the natural involution. We prove that it is formally complex. We need the following lemma proved in [**2**].

LEMMA. *Let S be an inverse semigroup and let $a, b, c \in S$ be such that $aa^{-1} = bc^{-1}$. Then $a^{-1}b = a^{-1}c$.*

THEOREM 1. *Let S be an inverse semigroup and let $(R, {}^*)$ be a formally complex ring. Then $(R[S], {}^*)$, under the natural involution *, is formally complex.*

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PROOF: For $A = \sum a_i s_i \in R[S]$ we set $\operatorname{supp}(A) = \{s_i \mid a_i \neq 0\}$. Let $A_1, \ldots, A_n \in R[S]$ be such that $\sum A_i A_i^* = 0$. Let $X = \bigcup \operatorname{supp}(A_i)$. We have to show that $X = \emptyset$. We prove the equivalent assertion that it is impossible to have $\sum A_i A_i^* = 0$ and $0 < |X| \leqslant n$, where $n \in \mathbb{N}$. If $|X| = 1$ then $A_i = a_i s$ for some fixed $s \in S$ and for $i = 1, \ldots, n$. Then $0 = \sum A_i A_i^* = (\sum a_i a_i^*) s s^{-1}$ which implies that $\sum a_i a_i^* = 0$. Thus all a_i are 0 since R is formally complex, a contradiction. Assume the assertion to be true for $0 < |X| \leqslant n-1$. Let $\sum A_i A_i^* = 0$ such that $|X| = n$. We choose $s \in X$ which is maximal in the sense that $s \leqslant t$ implies that $s = t$ for all $t \in X$. Here $u \leqslant v$ means that $uv^{-1} = uu^{-1}$ which is the Vagner–Preston partial ordering on inverse semigroups. Only one of the following two cases occurs:

(1) $\quad ss^{-1} = s_i s_j^{-1}$ for some $i \neq j$.

In this case $s_i \neq s \neq s_j$ since $ss^{-1} = ss_i^{-1}$ implies $s \leqslant s_i$ and so $s = s_i$.

(2) $\quad ss^{-1} = s_i s_j^{-1}$ implies that $i = j$.

In Case (1) let $ss^{-1} = s_i s_j^{-1}$ for some $i, j, i \neq j$. This implies that $s^{-1}s_i = s^{-1}s_j$ by the lemma. But now $|X'| = \left|\bigcup \operatorname{supp} s^{-1}A_i\right| < n$, while $\sum \left(s^{-1}A_i\right)\left(s^{-1}A_i\right)^* = s^{-1}(\sum A_i A_i^*)s = 0$. By induction $X' = \emptyset$. Since s occurs in some A_i we have $s^{-1}A_i = 0$. But then $s^{-1}s = 0$ if $\operatorname{supp}(A_i) = \{s\}$, or there is $t \in X$ distinct from s such that $s^{-1}s = s^{-1}t$. This is impossible.

In Case (2) the coefficient of ss^{-1} in the expansion of $\sum A_i A_i^* = 0$ is a sum of terms of the form cc^* and this coefficient is zero. By the formal complexity of $(R, {}^*)$ all c's involved in this coefficient are zero and so $1 \leqslant |X| < n$. This is impossible. ∎

Let $(S, {}^*)$ be a semigroup with involution *. Let $A \subseteq S$. An element $t \in A$ is called *maximal* if

$$(\forall u, v \in A)((tt^* = tu^* \Rightarrow t = u) \,\&\, ((tt^* = uv^* \& u \neq t \neq v) \Rightarrow t^*u = t^*v)).$$

If every finite non-empty subset of S has a maximal element we say that $(S, {}^*)$ satisfies the *maximal property* and it is an *MP-semigroup*. We prove the following.

PROPOSITION. *Let $(S, {}^*)$ be a proper *-MP-semigroup. Let $(R, {}^*)$ be a formally complex ring. Then $(R[S], {}^*)$ with the natural involution * is formally complex.*

PROOF: The proof is similar to that of the previous theorem. We prove the equivalent assertion that it is impossible to have $\sum A_i A_i^* = 0$ and $1 \leqslant |X| \leqslant n$, where n is any natural number. Here $X = \bigcup \operatorname{supp}(A_i)$. If $|X| = 1$ then $A_i = a_i s$ for some fixed s. Then $0 = \sum A_i A_i^* = (\sum a_i a_i^*) ss^*$ which implies that all a_i are 0 by the formal complexity of $(R, {}^*)$. This is impossible. Assume the assertion to be true for $1 \leqslant |X| \leqslant n-1$. Let $\sum_1^m A_i A_i^* = 0$ and let $|X| = n$. We choose $s \in X$ which is maximal. We have only two cases to consider:

(1) $\quad ss^* = s_i s_j^*$ for some $i \neq j$.

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In this case $s_i \neq s \neq s_j$ by maximality of s.

(2) $ss^* = s_i s_j^*$ implies that $i = j$.

In case (1) let $ss^* = s_i s_j^*$ for some $i \neq j$. This implies that $s^* s_i = s^* s_j$ by maximality of s. But now $|X'| = |\bigcup \operatorname{supp}(s^* A_i)| < n$ while $\sum (s^* A_i)(s^* A_i)^* = s^*(\sum A_i A_i^*) s = 0$. This is impossible unless $X' = \emptyset$. Let $X' = \emptyset$. Then $s^* A_i = 0$, $i = 1, \ldots, n$. Let B_1, B_2, B_3 be elements in $\{A_1, \ldots, A_m\}$ such that $s \in B_1$, $s_i \in B_2$, $s_j \in B_3$. There exist positive integers n_1, n_2, n_3 such that if $B = n_1 B_1 + n_2 B_2 + n_3 B_3$ then $s, s_i, s_j \in \operatorname{supp}(B)$. Since $s^* B_i = 0$ for $i = 1,2,3$ we have $s^* B = 0$. Thus $s^* s = s^* s_i = s^* s_j = s_i^* s = s_j^* s$, $ss^* = s_i s_j^* = s_j^* s_i$. This implies that $ss^* s = s_i s_j^* s = s_i s^* s$. By *-cancellation $ss^* = s_i s^* = ss_i^*$. But then $s = s_i$ by maximality of s and this is impossible. $\square$

REMARK. Let $(R, {}^*)$ be a formally complex ring with 1. Let n be a positive integer. Let e_{ij} be the $n \times n$ matrix with 1 as the ij-entry and 0 elsewhere, $1 \leqslant i, j \leqslant n$. Then $E = \{e_{ij}\} \cup \{0\}$ is an inverse semigroup with $e_{ij}^{-1} = e_{ji}$. Let $M_n(R)$ be the ring of all $n \times n$ matrices with entries in R. Then $M_n(R) = R[E]$. Let * be the involution on $M_n(R)$ defined by $[a_{ij}]^* = [a_{ij}^*]^t$ and let $^{*'}$ be the natural involution on $R[E]$. Then $(M_n(R), {}^*) = \left(R[E], {}^{*'}\right)$, a formally complex ring by the theorem. Of course this can be proved directly. The same thing can be said about finite sums of such matrix rings.

We now give an example of a formally complex ring over a semigroup which is not an inverse semigroup.

EXAMPLE. There is a finite regular *-semigroup $(S, {}^*)$ which is not an inverse semigroup and such that for any formally complex ring $(R, {}^*)$ the semigroup ring $(R[S], {}^*)$ with the natural involution is formally complex.

PROOF: Let S be the set of all 2×2-matrices, with entries from $\mathbb{Z}_3$, of the form $A_n = ne_{11}$, $B_n = ne_{22}$, $C_n = n(e_{11} + e_{12} + e_{21} + e_{22})$, $D_n = ne_{12}$, D_n^t, $E_n = ne_{11} + ne_{12}$, E_n^t, $F_n = ne_{12} + ne_{22}$, F_n^t, 0. Since $n \in \mathbb{Z}_3$ there are 19 elements in S. It is easy to check that S is a semigroup. Let $*$ be the transpose mapping on S. We show that * is a proper involution. Let $X, Y \in S$ be such that $XX^t = XY^t = YY^t$. Then $(X - Y)(X - Y)^t = 0$ in the ring of 2×2-matrices in $\mathbb{Z}_3$. If $X - Y = ae_{11} + be_{12} + ce_{21} + de_{22}$ it follows that $a^2 + b^2 = c^2 + d^2 = 0$. But in $\mathbb{Z}_3$ this implies that $a = b = c = d = 0$. Thus $X = Y$ and $(S, {}^*)$ is a proper *-semigroup.

We show that S is regular. Using the fact that $n^3 = n$ in $\mathbb{Z}_3$ it is easily verified that $A_n^3 = A_n$, $B_n^3 = B_n$, $C_n^3 = C_n$, $E_n^3 = E_n$, $F_n^3 = F_n$ and $D_n E_n^t D_n = D_n$. Thus S is regular.

We show that S is not an inverse semigroup. Let $x = e_{11}$, $y = e_{12}$, $z = e_{11} + e_{21}$. Then it is easily verified that $zyz = z$, $yzy = y$, $zxz = z$, $xzx = x$. Thus z has two inverses and S is not an inverse semigroup.

To show that $(R[S], {}^*)$ is formally complex we notice that in S, $xx^t = xy^t$ implies that $x = y$ and $xx^t = uv^t$ implies that $u = v$. Thus if $\emptyset \neq S' \subseteq S$ then all elements of S' are maximal and so the previous proposition applies. We conclude that $(R[S], {}^*)$ is formally complex. ∎

The semigroup S can be described verbally as the set of all multiples of elementary matrices, all multiples of matrices with one zero row (column) and the other row (column) equal to $(1, 1)$ and all matrices with equal entries.

Finally we prove the following:

THEOREM 2. *Let $(S, {}^*)$ be a proper-*-MP semigroup and let $(R, {}^*)$ be a formally complex ring. Then $R[S]$ is semisimple.*

PROOF: Let A be an element of the radical of $R[S]$. Then AA^* is in the radical and so there is a natural number n such that $(AA^*)^n = 0$. We can assume that $n = 2^k$. Using the properness of the natural involution * on $R[S]$ we conclude that $AA^* = 0$ and so $A = 0$.

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