A PRESENTATION OF THE GROUPS PSL(2, p) WITH THREE DEFINING RELATIONS

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H. Behr and J. Mennicke (**1**) have proven that the group PSL(2, p) can be presented by the following system of generators and relations:

$$S^p = T^2 = (ST)^3 = (S^2TS^{\frac{1}{2}(p+1)}T)^3 = 1 \qquad (p > 2). \tag{1}$$

From this presentation, it follows that the three relations

$$S^p = (ST)^3, \quad T^2 = 1, \quad (S^2TS^{\frac{1}{2}(p^2+1)}T)^3 = 1 \tag{2}$$

for the same generators S and T suffice if $p > 3$, $p \neq 17$. If $p = 3$, it is well known that the relations $S^3 = 1$, $T^2 = 1$, and $(ST)^3 = 1$ define PSL(2, 3). For $p = 2$, the relations $S^3 = 1$, $T^2 = 1$, and $(ST)^2 = 1$ define PSL(2, 2). For $p = 17$, the three relations

$$S^{17} = (ST)^3, \quad T^2 = 1, \quad (S^2TS^9T)^3 = 1 \tag{3}$$

will suffice.

Indeed, the group G, with generators S, T and defining relations (2), contains the subgroup $\langle S^p \rangle$ in its centre. This is because S^p commutes both with S and with ST; hence, S^p commutes with every element of G. The factor group $F = G/\langle S^p \rangle$ has generators $\bar{S} = S/\langle S^p \rangle$, $\bar{T} = T/\langle S^p \rangle$ with the defining relations (1). Hence, $F \cong$ PSL(2, p), a simple non-abelian group. Because of the choice of the relation

$$(S^2TS^{\frac{1}{2}(p^2+1)}T)^3 = 1$$

in place of the original

$$(S^2TS^{\frac{1}{2}(p+1)}T)^3 = 1,$$

it follows that the factor commutator group of G is 1.* Hence, by the theory of the Schur multiplier, either

$$G \cong \mathrm{PSL}(2, p) \quad \text{or} \quad G \cong \mathrm{SL}(2, p)$$

(cf. **2**). The latter possibility is excluded since $T^2 = 1$. Hence,

$$G \cong \mathrm{PSL}(2, p).$$

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*For $p = 17$, the original choice was good enough.

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