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COMB GRAPHS AND SPECTRAL DECIMATION

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Abstract. We investigate the spectral properties of matrices associated with comb graphs. We show that the adjacency matrices and adjacency matrix Laplacians of the sequences of graphs show a spectral similarity relationship in the sense of work by L. Malozemov and A. Teplyaev (Self-similarity, operators and dynamics, *Math. Phys. Anal. Geometry* **6** (2003), 201–218), and hence these sequences of graphs show a spectral decimation property similar to that of the Laplacians of the Sierpiński gasket graph and other fractal graphs.

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1. Introduction and definitions. Many examples are known of sequences of graphs for which associated matrices of members of the sequence, such as the graph Laplacians, have eigenvalues which are related to each other by iteration of a polynomial or rational function, and have a limiting graph whose spectrum is related to the Julia set of the function.

In the context of fractal graphs this is known as *spectral decimation*, and was first observed for the Sierpiński gasket graph in [**9**] and given a rigorous mathematical treatment in [**6**, **11**]. A generalisation of spectral decimation to a much larger class of self-similar graphs appears in [**8**], in which an abstract concept of *spectral similarity* for operators is developed, leading to a symmetry condition which, if satisfied, ensures that spectral decimation applies to the Laplacian of the graph. In these cases the graph is obtained as a union of a sequence of graphs $(G_n)_{n\in\mathbb{N}}$, and there is a rational function f such that z is an eigenvalue of the Laplacian of G_{n+1} if and only if $f(z)$ is an eigenvalue of the Laplacian of G_n, unless z is a member of a so-called exceptional set $\mathcal{E}$, which does not depend on n. This allows us to find eigenvalues by iteratively solving the equation $f(z) = \lambda$.

The theory of spectral decimation, including spectral self-similarity, is further developed, with some more examples, in [**2**]; for more on spectral decimation on fractals see the book [**12**].

In [**3**], some examples of graphs, constructed as the Schreier graphs of certain groups, are shown to have a similar property, with infinite graphs having spectra related to the Julia set of a simple function.

In this paper we show that the concept of spectral similarity developed in [**8**] for the graph Laplacians of fractal graphs also applies to the adjacency matrices and adjacency matrix Laplacians of the *comb graphs* described in [**1**], and hence show that a property similar to spectral decimation occurs for these graphs. Some relationships involving spectra of operators on comb products were shown in [**10**]; our results provide the link to spectral similarity and the spectral decimation phenomenon for fractals.

1.1. Comb graphs.

We make some definitions, following [**1**].

DEFINITION 1. Given two graphs G_1 and G_2 with a distinguished vertex $o \in V(G_2)$, we define the *comb product* (or *comb graph*) $G_1 \rhd_o G_2$ as a graph formed by taking a copy $G_{2,i}$ of G_2 for every vertex in $V(G_1)$ and attaching $G_{2,i}$ to G_1 by identifying the distinguished vertex o with vertex i of G_1.

If we have a single graph G with a distinguished vertex o, we define the *homogeneous comb product* $G^{\rhd n}$ as the comb product of n copies of G. It is shown in [**1**] that the comb product is associative, so this is well-defined.

1.2. Graphs and matrices.

DEFINITION 2. Given a graph G with vertex set $V(G)$, we can define the following matrices, all $|V(G)| \times |V(G)|$:

The *adjacency matrix* $A(G)$ has $A_{i,j} = 1$ if there is an edge between i and j, and 0 otherwise. In a graph with multiple edges $A_{i,j}$ is the number of edges between i and j. This is used in [**1**].

The usual *graph Laplacian* $\mathcal{L}(G)$ has several alternative definitions, for example in [**5**, **8**], but they are all related by a simple transformation. One definition is that it is the generator matrix of a continuous time random walk on the graph with all vertices having a mean holding time 1, so that, assuming $d_i > 0$ for all i, $\mathcal{L}_{i,i} = -1$ and $\mathcal{L}_{i,j} = A_{i,j}/d_i$ for all pairs of vertices $i \neq j$. (Here d_i is the degree of vertex i in G.) The graph Laplacian is used in the construction of a Laplacian on fractals, see [**7**].

Another definition is the *combinatorial* or *adjacency matrix Laplacian* $L(G)$, as defined in [**8**], which is related to the generator matrix of a continuous time random walk where the mean holding time of vertex i depends on the vertex degree, so that, assuming $d_i > 0$ for all i, $L_{i,i} = d_i$ and $L_{i,j} = -A_{i,j}$ for all pairs of vertices $i \neq j$.

In the case of a regular graph, such as the Schreier graphs in [**3**], all the above matrices are simple transformations of each other and their eigenvalues will differ by only a linear transformation. However, as discussed in [**4**], for non-regular graphs (including comb graphs) the matrices and hence the behaviour of the eigenvalues may be significantly different.

2. Relationships between eigenvalues.

We start by calculating the relationship between eigenvalues of matrices B and D satisfying a particular relationship, which will apply to the adjacency matrices of comb graphs.

For a square matrix C with rows and columns indexed by V_C, let $\hat{C}$ be the matrix with rows and columns indexed by $V_C \setminus \{o\}$ obtained by removing the row and column corresponding to o from C. We label the characteristic polynomial of a square matrix C as $\chi_C(z)$.

THEOREM 1. *Let us have an $r_1 \times r_1$ matrix B, an $r_2 \times r_2$ matrix C and an $r_1 r_2 \times r_1 r_2$ matrix D. We index the rows and columns of B and C by V_B and V_C respectively, and those of D by $V_B \times V_C$, and assume that we have a distinguished element $o \in V_C$.*

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Let B, C and D be such that

$$D_{(i,j),(l,k)} = \delta_{il} C_{jk} + B_{il} P_{jk},$$

where P is an $r_2 \times r_2$ matrix with rows and columns indexed by V_C such that $P_{jk} = \delta_{jo}\delta_{ko}$, where δ is the usual Kronecker delta.

*Let $\mathcal{H}$ be $\mathbb{C}^{r_1 r_2}$ as acted on by D, $\mathcal{H}_0$ be $\mathbb{C}^{r_1}$ as acted on by B, U be the inclusion operator $\mathcal{H}_0 \to \mathcal{H}$ with $U(e_i) = e_{(i,o)}$ and U^* the orthogonal projector $\mathcal{H} \to \mathcal{H}_0$. Then the matrix D is spectrally similar to B, as defined in Definition 2.1 of [**8**], with functions $\phi_0(z) = 1$ and*

$$\phi_1(z) = -\frac{\chi_C(z)}{\chi_{\hat{C}}(z)}.$$

That is,

$$U^*(D - zI_{r_1 r_2})^{-1} U = (B - \phi_1(z) I_{r_1})^{-1}$$

for all $z \in \mathbb{C}$ for which both sides are well defined, and where we write I_r for the identity matrix on $\mathbb{C}^r$.

Proof. We use Lemma 3.3 of [**8**]. To use this, we set up some notation. Let $\mathcal{H}_1$ be the orthogonal complement to $\mathcal{H}_0$ in $\mathcal{H}$, i.e. the space spanned by the unit vectors $e_{(i,j)}, j \neq o$. Then we order $V_B \times V_C$ with elements of the form (i, o) first, ordered by i, followed by elements of the form (i, j) with $j \neq o$, ordered first by i and then by j.

To use Lemma 3.3 of [**8**] we need to represent D in block form with respect to $\mathcal{H}_0$ and $\mathcal{H}_1$, as

$$\begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix},$$

and show that

$$S - zI_{r_1} - \bar{X}(Q - zI_{r_1(r_2-1)})^{-1} X = B - \phi_1(z) I_{r_1}, \tag{1}$$

where $I_{r_1(r_2-1)}$ is the identity on $\mathcal{H}_1$.

We first consider S, $\mathcal{H}_0 \to \mathcal{H}_0$. We have

$$S_{il} = D_{(i,o),(l,o)} = \delta_{il} C_{oo} + B_{il} P_{oo},$$

but $P_{oo} = 1$, so $S_{il} = \delta_{il} C_{oo} + B_{il}$ and hence

$$S = B + (C_{oo}) I_{r_1}.$$

We now consider X, $\mathcal{H}_0 \to \mathcal{H}_1$ and $\bar{X}$, $\mathcal{H}_1 \to \mathcal{H}_0$. Again using the formula for entries of D,

$$X_{(i,j),l} = D_{(i,j),(l,o)} = \begin{cases} C_{jo} & i = l \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{X}_{i,(l,k)} = D_{(i,0),(l,k)} = \begin{cases} C_{ok} & i = l \\ 0 & \text{otherwise.} \end{cases}$$

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Finally

$$Q_{(i,j),(l,k)} = \begin{cases} C_{jk} & i = l \\ 0 & \text{otherwise,} \end{cases}$$

and so, given our ordering of $V_B \times V_C$, Q is a block matrix with blocks on the diagonal each identical to $\hat{C}$ and zeros elsewhere. Hence

$$((Q - zI_{r_1(r_2-1)})^{-1})_{(i,j),(l,k)} = \begin{cases} ((\hat{C} - zI_{r_2-1})^{-1})_{jk} & i = l \\ 0 & \text{otherwise,} \end{cases}$$

and so

$$(\bar{X}(Q - zI_{r_1(r_2-1)})^{-1}X)_{il} = \begin{cases} \sum_{j,k \in V_C \setminus \{o\}} C_{oj}((\hat{C} - zI_{r_2-1})^{-1})_{jk} C_{ko} & i = l \\ 0 & \text{otherwise,} \end{cases}$$

So the left hand side of (1) becomes

$$B + (C_{oo})I_{r_1} - zI_{r_1} - \left(\sum_{j,k \in V_C \setminus \{o\}} C_{oj}((\hat{C} - zI_{r_2-1})^{-1})_{jk} C_{ko} \right) I_{r_1}$$

and so Lemma 3.3 of [**8**] shows that D is spectrally similar to B with $\phi_0(z) = 1$ and

$$\phi_1(z) = z - C_{oo} + \left(\sum_{j,k \in V_C \setminus \{o\}} C_{oj}((\hat{C} - zI_{r_2-1})^{-1})_{jk} C_{ko} \right).$$

To show that $\phi_1(z) = -\frac{\chi_C(z)}{\chi_{\hat{C}}(z)}$ we define a matrix $K(z)$ with rows and columns indexed by V_C and

$$(K(z))_{jk} = \begin{cases} 1 & j = k = o \\ 0 & j = o \text{ or } k = o \text{ but not both} \\ ((\hat{C} - zI_{r_2-1})^{-1})_{jk} & \text{otherwise} \end{cases}$$

Then

$$\det(K(z)) = (\chi_{\hat{C}}(z))^{-1}$$

and so

$$\det((C - zI_{r_2})K(z)) = \frac{\chi_C(z)}{\chi_{\hat{C}}(z)}.$$

Using the definition of $K(z)$, if $j \neq o$ and $k \neq o$ then

$$\begin{aligned} ((C - zI_{r_2})K(z))_{jk} &= \sum_{q \in V_C \setminus \{o\}} (\hat{C} - zI_{r_2-1})_{jq}((\hat{C} - zI_{r_2-1})^{-1})_{qk} \\ &= \delta_{jk}. \end{aligned}$$

If $j \neq o$ then

$$\begin{aligned} ((C - zI_{r_2})K(z))_{jo} &= \sum_{q \in V_C} C_{jq}(K(z))_{qo} \\ &= C_{jo}. \end{aligned}$$

If $k \neq o$ then

$$\begin{aligned}((C - zI_{r_2})K(z))_{ok} &= \sum_{q \in V_C} C_{oq}(K(z))_{qk} \\ &= \sum_{q \in V_C \setminus \{o\}} C_{oq}((\hat{C} - zI_{r_2-1})^{-1})_{qk}.\end{aligned}$$

Finally

$$\begin{aligned}((C - zI_{r_2})K(z))_{oo} &= \sum_{q \in V_C} (C - zI_{r_2})_{oq}(K(z))_{qo} \\ &= (C - zI_{r_2})_{oo} \\ &= C_{oo} - z.\end{aligned}$$

Putting the above together and considering which terms in the determinant are non-zero,

$$\begin{aligned}\det((C - zI_{r_2})K(z)) &= C_{oo} - z - \left(\sum_{k \in V_C \setminus \{o\}} C_{ko} \sum_{j \in V_C \setminus \{o\}} C_{oj}((\hat{C} - zI_{r_2-1})^{-1})_{jk} \right) \\ &= C_{oo} - z - \left(\sum_{j,k \in V_C \setminus \{o\}} C_{oj}((\hat{C} - zI_{r_2-1})^{-1})_{jk} C_{ko} \right) \\ &= -\phi_1(z).\end{aligned}$$

Hence

$$\phi_1(z) = -\frac{\chi_C(z)}{\chi_{\hat{C}}(z)},$$

as claimed. □

Corollary 2. *Under the conditions of Theorem 1, μ is an eigenvalue of D if and only if $f(\mu)$ is an eigenvalue of B, with the same multiplicity, unless μ is an eigenvalue of $\hat{C}$.*

Here $\hat{C}$ is, again, the matrix, with rows and columns indexed by $V_C \setminus \{o\}$, obtained by removing the row and column corresponding to o from C, and $f(z) = -\frac{\chi_C(z)}{\chi_{\hat{C}}(z)}$.

Proof. This follows from Theorem 1 by applying Theorem 3.6 of [**8**]. □

Following [**8**], the set $\mathcal{E}$, consisting of the eigenvalues of $\hat{C}$, where spectral similarity gives no information, is called the *exceptional set*.

The following theorem is essentially an alternative method of proving Corollary 2 without giving the precise result on spectral similarity, but gives additional information about the exceptional set.

Theorem 3. *We assume the same conditions as for Theorem 1. Then if x is an eigenvector of B with eigenvalue λ and y is an eigenvector of $C + \lambda P$ with eigenvalue μ, then we can construct an eigenvector of D with eigenvalue μ.*

Proof. We let $x'_{(i,j)} = x_i y_j$.

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Let $j \neq o$. Then

$$\begin{aligned}(Dx')_{(i,j)} &= \sum_{k \in V_C} C_{j,k} x_i y_k \\ &= x_i \sum_{k \in V_C} (C + \lambda P)_{j,k} y_k \qquad \text{(because } P_{j,k} = 0) \\ &= \mu x_i y_j\end{aligned}$$

Now

$$\begin{aligned}(Dx')_{(i,o)} &= \sum_{k \in V_C} C_{o,k} x_i y_k + \sum_{l \in V_B} B_{i,l} x_l y_o \\ &= \sum_{k \in V_C} C_{o,k} x_i y_k + y_o \sum_{l \in V_B} B_{i,l} x_l \\ &= \sum_{k \in V_C} C_{o,k} x_i y_k + y_o \lambda x_i \qquad \text{(because } Bx = \lambda x) \\ &= \sum_{k \in V_C} (C + \lambda P)_{o,k} x_i y_k \qquad \text{(by the definition of } P) \\ &= \mu x_i y_o\end{aligned}$$

Putting these together,

$$Dx' = \mu x'. \qquad \square$$

We now consider sequences of matrices in which consecutive pairs satisfy the conditions of Theorem 1.

THEOREM 4. *Assume we have an $r \times r$ matrix M with rows and columns indexed by V_M, and let P be the $r \times r$ matrix with rows and columns indexed by V_M and such that $P_{jk} = \delta_{jo}\delta_{ko}$. Then we define a sequence of matrices $(M^{(n)})_{n \in \mathbb{N}}$, with rows and columns indexed by V_M^n, by*
- $M^{(1)} = M$
- $M^{(n)}_{(i,j),(l,k)} = \delta_{il} M_{jk} + M^{(n-1)}_{il} P_{jk}$.

Then, obtaining $\hat{M}$ from M in the same way as we obtained $\hat{C}$ from C above, $M^{(n+1)}$ is spectrally similar to $M^{(n)}$, with function $f(z) = -\frac{\chi_M(z)}{\chi_{\hat{M}}(z)}$. Also μ is an eigenvalue of $M^{(n+1)}$ if it is an eigenvalue of $M + \lambda P$, for λ an eigenvalue of $M^{(n)}$.

Proof. This comes immediately from Theorems 1 and 3, with $B = M^{(n)}$, $C = M$ and $D = M^{(n+1)}$. $\square$

We note that the property of the sequence $(M^{(n)})_{n \in \mathbb{N}}$ required for Theorem 4 is the property used in [**1**] to show that the adjacency matrices of comb graphs can be decomposed into a sum of monotone independent random variables.

We now use our theorems to obtain two results concerning $(G^{\rhd n})_{n \in \mathbb{N}}$, the sequence of homogeneous comb products based on an initial graph G.

COROLLARY 5. *Letting $A^{(n)}$ be the adjacency matrix of $G^{\rhd n}$, μ is an eigenvalue of $A^{(n+1)}$ if and only if μ is an eigenvalue of $A(G) + \lambda P$ with λ an eigenvalue of $A^{(n)}$, and $A^{(n+1)}$ is spectrally similar to $A^{(n)}$.*

Proof. It is shown in [**1**] that this sequence satisfies the conditions of Theorem 4, with $M = A(G)$, the adjacency matrix of the initial graph. $\square$

Corollary 6. *Letting $L^{(n)}$ be the adjacency matrix Laplacian of $G^{\triangleright n}$, μ is an eigenvalue of $L^{(n+1)}$ if and only if μ is an eigenvalue of $L(G) + \lambda P$ with λ an eigenvalue of $L^{(n)}$, and $L^{(n+1)}$ is spectrally similar to $L^{(n)}$.*

Proof. To establish this, we need to show that the sequence of adjacency matrix Laplacians satisfies the conditions of Theorem 4. To do this we proceed inductively, and calculate

$$\delta_{il} M_{jk} + M^{(n-1)}_{il} P_{jk}. \tag{2}$$

There are three cases:

- If $(i, j) = (l, k)$ this is

$$\delta_{ii} M_{jj} + M^{(n-1)}_{ii} P_{jj} = M_{jj} + M^{(n-1)}_{ii} P_{jj}$$

  which is M_{jj} if $j \neq o$ and $M_{jj} + M^{(n-1)}_{ii}$ if $j = o$, which by the definition of the comb product is the degree of (i, j) in $G^{\triangleright n}$ and hence equal to $M^{(n)}_{(i,j),(l,k)}$.
- If $i \neq l$ then 2 becomes $M^{(n-1)}_{il} P_{jk}$, which is -1 if and only if $j = k = o$ and $i \sim l$ in $G^{\triangleright(n-1)}$, i.e. if and only if $(i, j) \sim (l, k)$ in $G^{\triangleright n}$.
- If $i = l$ and $j \neq k$ then (2) equals M_{jk}, which is -1 if and only if $j \sim k$ in the initial graph G.

Hence in all cases (2) is equal to $M^{(n)}_{(i,j),(l,k)}$, as required. $\square$

Because the condition that μ is an eigenvalue of $A(G) + \lambda P$ (or $L(G) + \lambda P$) can be related to the rational function f from Corollary 2, Corollaries 5 and 6 show that a variant of spectral decimation exists for the adjacency matrices and adjacency matrix Laplacians of homogeneous comb products, regardless of the initial graph G.

Putting our results together, we obtain the following description of the spectra of the adjacency matrices and adjacency matrix Laplacians of homogeneous comb products:

Theorem 7. *Let $\lambda^{(1)}_1, \ldots, \lambda^{(1)}_N$ (where N is the number of vertices of G) be the eigenvalues of the adjacency matrix of G, and, for $n \geq 1$, let $\lambda^{(n+1)}_1, \ldots, \lambda^{(n+1)}_{N^{n+1}}$ be the N^{n+1} values (up to multiplicity) which are eigenvalues of $A(G) + \lambda^{(n)}_j$ for some $j \in \{1, \ldots, N^n\}$.*

Then the spectrum of the adjacency matrix $A(G^{\triangleright n})$ consists of the values $\lambda^{(n)}_1, \ldots, \lambda^{(n)}_{N^n}$.

Similarly, let $\nu^{(1)}_1, \ldots, \nu^{(1)}_N$ be the eigenvalues of the adjacency matrix Laplacian of G, and let $\nu^{(n+1)}_1, \ldots, \nu^{(n+1)}_{N^{n+1}}$ be the N^{n+1} values (up to multiplicity) which are eigenvalues of $L(G) + \nu^{(n)}_j$ for some $j \in \{1, \ldots, N^n\}$.

Then the spectrum of the adjacency matrix Laplacian $L(G^{\triangleright n})$ consists of the values $\nu^{(n)}_1, \ldots, \nu^{(n)}_{N^n}$.

Proof. This follows from Corollaries 5 and 6. $\square$

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3. Localisation and high multiplicity. The Laplacians associated with self-similar fractal graphs [7] in many cases show high multiplicity of eigenvalues associated with localised eigenfunctions, i.e. eigenfunctions which are non-zero only on some subset of the graph.

We show that this can also happen for comb graphs.

LEMMA 8. *If there is a non-trivial symmetry of the graph G which fixes a distinguished vertex o, then we can find an eigenfunction x of the adjacency matrix such that $x_o = 0$. This result also applies to the graph and adjacency matrix Laplacians.*

Proof. Let σ be such a symmetry. As σ is non-trivial we can find an eigenfunction of $A(G)$, x', which is not fixed by σ. Then $\sigma x'$ is another eigenfunction, with the same eigenvalue, and so $x = x' - \sigma x'$ is an eigenfunction which is 0 on vertices fixed by σ, including o.

This also applies to the graph and adjacency matrix Laplacians. $\square$

This allows us to construct localised eigenfunctions:

COROLLARY 9. *If G satisfies the conditions of Lemma 8, then the adjacency matrix of a homogeneous comb product $G^{\rhd n}$ has an eigenvalue λ with at least $|V(G)|^{(n-1)}$ linearly independent eigenfunctions each localised on at most $|V(G)| - 1$ vertices.*

Again, this result also applies to the graph and adjacency matrix Laplacians.

Proof. This is based on the structure of the graph $G^{\rhd n}$, which consists of $|V(G)|^{(n-1)}$ copies of G attached to $G^{\rhd(n-1)}$ at the distinguished vertex o, vertex (i, j) being naturally identified with vertex j in copy i of G.

So we can take any one of these copies of G, l say, and construct an eigenfunction y with $y_{(l,j)} = x_j$ (where x_j is the eigenfunction constructed for G in Lemma 8, with eigenvalue λ), and $y_{(i,j)} = 0$ if $i \neq l$, and hence non-zero on at most $|V(G)| - 1$ vertices. The structure of the graph and the definition of the matrices ensure that this is indeed an eigenfunction with eigenvalue λ. As there are $|V(G)|^{(n-1)}$ possible values of l, there are $|V(G)|^{(n-1)}$ linearly independent eigenfunctions constructed by this method. $\square$

Note that, for the adjacency matrix and adjacency matrix Laplacian, we can now find other eigenvalues with high multiplicity and localised eigenfunctions by applying Corollaries 5 and 6.

4. Examples.

4.1. Adjacency matrix examples.

EXAMPLE 1. We let the initial graph G be the complete graph on three vertices, with adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then the eigenvalues of $A(G)$ are -1, -1 and 2 and if λ is an eigenvalue of $A(G^{\rhd(n-1)})$ then by Corollary 5 it generates eigenvalues of $A(G^{\rhd n})$ which are eigenvalues

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of

$$A + \lambda P = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

the eigenvalues of which are given by $\mu = -1$ and the solutions of the equation

$$f(\mu) = \frac{\mu^2 - \mu - 2}{\mu - 1} = \lambda.$$

So we can generate eigenvalues of any $A(G^{\triangleright n})$ by starting with the eigenvalues of $A(G)$ and iteratively solving the equation $f(\mu) = \lambda$.

One property of this example is that an eigenvalue $\mu = -1$ is generated by any λ. Hence the multiplicity of -1 as an eigenvalue of $A(G^{\triangleright n})$ is at least the number of eigenvalues of $A(G^{\triangleright (n-1)})$, which is 3^{n-1}. Furthermore this ensures that the roots of $f(\mu) = -1$ have multiplicity 3^{n-2} as eigenvalues of $A(G^{\triangleright n})$, etc. This generates eigenvalues with high multiplicity.

This eigenvalue -1 is an eigenvalue of the matrix $\hat{A}$ formed by removing the row and column corresponding to o from A, and hence a member of the exceptional set for the spectral similarity. This association between the eigenvalues with high multiplicity and the exceptional set also occurs in the fractal graphs case in [**8**].

Example 2. For a more complex example showing eigenvalues with high multiplicity, we consider an initial graph G with adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Then the eigenvalues of $A(G)$ are 0 and $\pm\frac{\sqrt{10\pm 2\sqrt{17}}}{2}$ and if λ is an eigenvalue of $A(G^{\triangleright (n-1)})$ then by Corollary 5 it generates eigenvalues of $A(G^{\triangleright n})$ which are eigenvalues of

$$A + \lambda P = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

the eigenvalues of which are given by $\mu = 0$ and the solutions of the equation

$$f(\mu) = \frac{\mu^5 - 5\mu^3 + 2\mu}{\mu^4 - 4\mu^2} = \lambda.$$

So we can generate eigenvalues of any $A(G^{\triangleright n})$ by starting with the eigenvalues of $A(G)$ and iteratively solving the equation $f(\mu) = \lambda$.

The behaviour is now similar to that of the previous example. The eigenvalue $\mu = 0$ is generated by any λ and is a member of the exceptional set for the spectral similarity, generating eigenvalues with high multiplicity.

4.2. The adjacency matrix Laplacian.

EXAMPLE 3. We consider the adjacency matrix Laplacian of the graph in the second example above. The initial matrix $L(G)$ is

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

and we are interested in the eigenvalues of

$$L(G) + \lambda P = \begin{pmatrix} 1+\lambda & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

which are given by the equation

$$\mu^5 - 10\mu^4 + 34\mu^3 - 46\mu^2 + 20\mu = \lambda(\mu^4 - 9\mu^3 + 26\mu^2 - 26\mu + 4).$$

Here 2 is an eigenvalue of $L(G) + \lambda P$ for any λ, and the remaining eigenvalues are given by the solutions of $g(\mu) = \lambda$, where

$$g(\mu) = \frac{\mu^5 - 10\mu^4 + 34\mu^3 - 46\mu^2 + 20\mu}{\mu^4 - 9\mu^3 + 26\mu^2 - 26\mu + 4}.$$

Again there are eigenvalues of high multiplicity, and the eigenvalue 2 is a member of the exceptional set for the spectral similarity.

4.3. Examples without high multiplicity.

EXAMPLE 4. A simple example starts with the graph G being the complete graph on two vertices, with adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This example does not satisfy the symmetry condition of Lemma 8.

In this case eigenvalues of $A(G) + \lambda P$ are given by solutions to

$$\lambda\mu = \mu^2 - 1.$$

There are no values which are solutions to this for all λ, no values which are eigenvalues of both A and $\hat{A}$, and there are no eigenvalues of $A(G^{\triangleright n})$ with high multiplicity: the eigenvalues of $A(G^{\triangleright n})$ can be generated by starting with the eigenvalues of $A(G)$, -1 and 1, and iteratively solving the equation $h(\mu) = \lambda$, where

$$h(\mu) = \frac{\mu^2 - 1}{\mu},$$

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giving 2^n distinct eigenvalues of $A(G^{\triangleright n})$. The only value in the exceptional set is $\mu = 0$, for which $h(\mu)$ is undefined; it does not occur as an eigenvalue.

EXAMPLE 5. For a more complex example not satisfying the symmetry condition of Lemma 8, we consider a graph G with

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

In this case eigenvalues of $A(G) + \lambda P$ are given by solutions to

$$\lambda(\mu^4 - 4\mu^2 - 2\mu + 1) = \mu^5 + 5\mu^3 + 2\mu^2 - 3\mu.$$

Again, there are no values which are solutions to this for all λ, no values which are eigenvalues of both A and $\hat{A}$, and there are no eigenvalues of $A(G^{\triangleright n})$ with high multiplicity: the eigenvalues of $A(G^{\triangleright n})$ can be generated by starting with the 5 distinct eigenvalues of $A(G)$ and iteratively solving the equation $h(\mu) = \lambda$, where

$$h(\mu) = \frac{\mu^5 + 5\mu^3 + 2\mu^2 - 3\mu}{\mu^4 - 4\mu^2 - 2\mu + 1},$$

giving 5^n distinct eigenvalues of $A(G^{\triangleright n})$. The values in the exceptional set (roots of $\mu^4 - 4\mu^2 - 2\mu + 1 = 0$, for which $h(\mu)$ is undefined) do not occur as eigenvalues.

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