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AUSLANDER GENERATORS AND HOMOLOGICAL CONJECTURES

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Abstract. Let A be an artin algebra with representation dimension not more than 3. Assuming that ${}_AV$ is an Auslander generator and $M \in \mathrm{add}_A V$, we show that both $\mathrm{findim}(\mathrm{End}_A M)$ and $\mathrm{findim}(\mathrm{End}_A M)^{op}$ are finite, and consequently the Gorenstein symmetry conjecture, the Wakamatsu-tilting conjecture and the generalized Nakayama conjecture hold for $\mathrm{End}_A M$.

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1. Introduction and main result. Let A be an artin algebra. The finitistic dimension of A, denoted by $\mathrm{findim}A$, is defined to be the supremum of the projective dimensions of all finitely generated modules of finite projective dimension. The famous finitistic dimension conjecture asserts that $\mathrm{findim}A$ is always finite.

Igusa and Todorov [**3**] presented a good way to test the finitistic dimension conjecture. In particular, they proved that $\mathrm{findim}A$ is finite, provided that the representation dimension of A, denoted by $\mathrm{repdim}A$, is not more than 3. Recall that $\mathrm{repdim}A = \inf\{\mathrm{gd}(\mathrm{End}_A V) \mid V \text{ is a generator–cogenerator}\}$, where gd denotes the global dimension and $\mathrm{End}_A V$ denotes the endomorphism algebra of ${}_AV$. A generator–cogenerator such as $\mathrm{repdim}A = \mathrm{gd}(\mathrm{End}_A V)$ is called an Auslander generator. In general, an artin algebra may have many Auslander generators, see for instance [**2**].

Our main result is stated as follows.

THEOREM 1.1. *Let A be an artin algebra with* $\mathrm{repdim}A \leq 3$. *Assume that ${}_AV$ is an Auslander generator. Then both* $\mathrm{findim}(\mathrm{End}_A M)$ *and* $\mathrm{findim}(\mathrm{End}_A M)^{op}$ *are finite, whenever* $M \in \mathrm{add}_A V$.

Theorem 1.1 generalizes the main result of [**6**]. It is not known if $\mathrm{findim}A^{op}$ is finite, provided that $\mathrm{findim}A$ is finite in general, where A^{op} denotes the opposite algebra of A.

We recall the following well-known conjectures (see, for instance, [**1**, **4**]). Here E is an artin algebra.

Gorenstein symmetry conjecture. $\mathrm{id}({}_EE) < \infty$ if and only if $\mathrm{id}(E_E) < \infty$, where id denotes the injective dimension.

Wakamatsu-tilting conjecture. Let ${}_E\omega$ be a Wakamatsu-tilting module.
(1) If $\mathrm{pd}_E\omega < \infty$, then ω is tilting.
(2) If $\mathrm{id}_E\omega < \infty$, then ω is co-tilting.

Generalized Nakayama conjecture. Each indecomposable injective E-module occurs as a direct summand in the minimal injective resolution of ${}_E E$.

It is well known that the finitistic dimension conjecture hold for E and E^{op} implies that all the above conjectures hold. Hence, we have the following corollary.

COROLLARY 1.2. *Let A be an artin algebra with* $\operatorname{repdim} A \leq 3$. *Assume that ${}_A V$ is an Auslander generator. Then the Gorenstein symmetry conjecture, the Wakamatsu-tilting conjecture and the generalized Nakayama conjecture hold for* $\operatorname{End}_A M$ *whenever $M \in \operatorname{add}_A V$.*

Since representation-finite algebras and torsionless-finite algebras have representation dimension of not more than 3 (see [**5**]), we obtain the following result as special cases.

COROLLARY 1.3. *Let $A = \operatorname{End}_\Lambda M$, where Λ and M satisfy one of the following conditions:*

(1) *Λ is a representation-finite algebra and M is any Λ-module, or*
(2) *Λ is a torsionless-finite algebra and M is torsionless or co-torsionless or a direct sum of torsionless and co-torsionless modules.*

Then both $\operatorname{findim} A$ *and* $\operatorname{findim} A^{op}$ *are finite. In particular, the Gorenstein symmetry conjecture, the Wakamatsu-tilting conjecture and the generalized Nakayama conjecture hold for A.*

2. The proof.

Let A be an artin algebra. We denote $A-\mathrm{mod}$ the category of all finite generated left A-modules. Assume $M \in A-\mathrm{mod}$. We denote $\operatorname{pd}_A M$ the projective dimension of ${}_A M$ and $\Omega^i_A M$ the ith syzygy of M. Throughout the paper, $\mathbf{D}$ denotes the usual duality functor between $A-\mathrm{mod}$ and $A^{op}-\mathrm{mod}$.

The following lemma is well known.

LEMMA 2.1. *Let A be an artin algebra and let V be a generator–cogenerator in* $A-\mathrm{mod}$. *The following are equivalent for a non-negative integer n.*

(1) $\operatorname{gd}(\operatorname{End}_A V) \leq n + 2$.
(2) *For any $X \in A-\mathrm{mod}$, there is an exact sequence $0 \to V_n \to \cdots \to V_1 \to V_0 \to X \to 0$ with each $V_i \in \operatorname{add}_A V$ such that the corresponding sequence induced by the functor* $\operatorname{Hom}_A(V, -)$ *is also exact.*

The following lemma collects some important properties of the Igusa–Todorov functor introduced in [**3**].

LEMMA 2.2. *For any artin algebra A, there is a functor Ψ which is defined on the objects of* $A-\mathrm{mod}$ *and takes non-negative integers as values, such that*

(1) $\Psi(M) = \operatorname{pd}_A M$, *provided that* $\operatorname{pd}_A M < \infty$.
(2) $\Psi(X) \leq \Psi(Y)$ *whenever* $\operatorname{add}_A X \subseteq \operatorname{add}_A Y$. *The equation holds in case* $\operatorname{add}_A X = \operatorname{add}_A Y$.
(3) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in* $A-\mathrm{mod}$ *with* $\operatorname{pd}_A Z < \infty$, *then* $\operatorname{pd}_A Z \leq \Psi(X \oplus Y) + 1$.

Let A be an artin algebra and $M \in A-\mathrm{mod}$ with $E = \operatorname{End}_A M$. Then M is also a right E-module. It is well known that $(M \otimes_E -, \operatorname{Hom}_A(M, -))$ is a pair of adjoint functors and that, for any E-module Y, there is a canonical homomorphism σ_Y :

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$Y \to \mathrm{Hom}_A(M, M \otimes_E Y)$ defined by $n \to [t \to t \otimes n]$. It is easy to see that σ_Y is an isomorphism, provided that Y is a projective E-module.

The following lemma is essential.

LEMMA 2.3. *Let $M \in A$−mod and $E = \mathrm{End}_A M$. Then, for any $X \in E$−mod, $\Omega_E^2 X \simeq \mathrm{Hom}_A(M, Y)$ for some $Y \in A$−mod.*

Proof. Consider the exact sequence

$$0 \to \Omega_E^2 X \to E_1 \to E_0 \to X \to 0$$

with $E_0, E_1 \in E$−mod projective. Applying the functor $M \otimes_E -$, we obtain an induced exact sequence

$$0 \to Y \to M \otimes_E E_1 \to M \otimes_E E_0 \to M \otimes_E X \to 0,$$

for some $Y \in A$−mod. Now applying the functor $\mathrm{Hom}_A(M, -)$, we further have an induced exact sequence

$$0 \to \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, M \otimes_E E_1) \to \mathrm{Hom}_A(M, M \otimes_E E_0).$$

Moreover, there is the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to & \Omega_E^2 X & \to & E_1 & \to & E_0 \\
 & \downarrow \phi & & \downarrow \sigma_{E_1} & & \downarrow \sigma_{E_0} \\
0 \to & \mathrm{Hom}_A(M, Y) & \to & \mathrm{Hom}_A(M, M \otimes_E E_1) & \to & \mathrm{Hom}_A(M, M \otimes_E E_0).
\end{array}$$

Since $E = \mathrm{End}_A M$ and $E_0, E_1 \in \mathrm{add}_E E$, the canonical homomorphisms σ_{E_0} and σ_{E_1} are isomorphisms. It follows that $\Omega_E^2 X \simeq \mathrm{Hom}_A(M, Y)$. □

Proof of Theorem 1.1. Let $E := \mathrm{End}_A M$. Suppose that $X \in E$−mod and $\mathrm{pd}_E X < \infty$. Then $\mathrm{pd}_E(\Omega_E^2 X) < \infty$. Moreover, $\Omega_E^2 X \simeq \mathrm{Hom}_A(M, Y)$ for some $Y \in A$−mod, by Lemma 2.3. Since ${}_A V$ is a generator–cogenerator such that $\mathrm{gd}(\mathrm{End}_A V) \leq 3$, by Lemma 2.1 we obtain an exact sequence

$$0 \to V_1 \to V_0 \to Y \to 0 \, (\dagger)$$

with $V_0, V_1 \in \mathrm{add}_A V$ such that the corresponding sequence induced by the functor $\mathrm{Hom}_A(V, -)$ is also exact. Note that $M \in \mathrm{add}_A V$, so the sequence (†) also stays exact under the functor $\mathrm{Hom}_A(M, -)$. Thus, we have the following exact sequence in E−mod:

$$0 \to \mathrm{Hom}_A(M, V_1) \to \mathrm{Hom}_A(M, V_0) \to \mathrm{Hom}_A(M, Y) \to 0.$$

Now by Lemma 2.2, we have that

$$\begin{aligned}
\mathrm{pd}_E X &\leq \mathrm{pd}_E(\Omega_E^2 X) + 2 \\
&= \mathrm{pd}_E(\mathrm{Hom}_A(M, Y)) + 2 \\
&\leq \Psi(\mathrm{Hom}_A(M, V_0) \oplus \mathrm{Hom}_A(M, V_1)) + 1 + 2 \\
&\leq \Psi(\mathrm{Hom}_A(M, V)) + 1 + 2 < \infty.
\end{aligned}$$

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It follows that findim$E$ is finite.

Now consider algebras A^{op} and $E^{op}(= \mathrm{End}_A M)^{op})$. Since ${}_A V$ is a generator–cogenrerator in A−mod, ${}_{A^{op}}\mathbf{D}V$ is also a generator–cogenrerator in A^{op}−mod. Moreover, if $\mathrm{gd}(\mathrm{End}_A V) \leq 3$, then $\mathrm{gd}(\mathrm{End}_{A^{op}}\mathbf{D}V) \leq 3$, since $\mathrm{End}_{A^{op}}\mathbf{D}V \simeq (\mathrm{End}_A V)^{op}$. Finally, if $M \in \mathrm{add}_A V$, then $\mathbf{D}M \in \mathrm{add}_{A^{op}}\mathbf{D}V$ and $(\mathrm{End}_A M)^{op} \simeq \mathrm{End}_{A^{op}}\mathbf{D}M$. Thus, the previous argument shows that $\mathrm{findim}(\mathrm{End}_A M)^{op}(= \mathrm{findim}(\mathrm{End}_{A^{op}}\mathbf{D}M))$ is also finite. □

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