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A PREEMPTIVE PRIORITY QUEUE WITH A GENERAL BULK SERVICE RULE

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In this paper a single server preemptive priority queueing system, consisting of two types of units, with unlimited Poisson inputs and exponential service time distributions, is studied. The higher priority units are served in batches according to a general bulk service rule and they have preemptive priority over lower priority units. Steady state queue length distributions, stability condition and the mean queue lengths are obtained.

Introduction

Queues with more than one type of input and with different priority rules have been solved under various assumptions. Heathcote [2] has studied a preemptive priority queueing problem with two priorities. Sivasamy [4] has analysed a non-preemptive priority queue by introducing general bulk service rule in the lower priority queue. In this paper, we apply similar ideas as in Heathcote [2] and Sivasamy [4] to study a preemptive priority queueing problem by introducing the general bulk service rule in the top priority queue.

The system $M, M|M^{a,b}, M|1$

Consider a single server facility in which two independent Poisson classes of units, to be called type-1 units (top priority or higher priority units) and type-2 units (lower priority or nonpriority units), arrive at rates λ_1 and λ_2 respectively and form separate queues. Let

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the service times of the type-1 and type-2 units be independently exponentially distributed random variables with means μ_1^{-1} and μ_2^{-1} respectively. The type-1 units have preemptive priority over the type-2 units and it operates as follows:

As soon as a type-1 unit arrives, the server breaks his serving to a type-2 unit if any, and remains idle. Then the idle server opens service for type-1 units when the queue length reaches size 'a' units. If, at a service epoch, the server finds 'q' type-1 units in the system where $1 \leqslant q \leqslant a-1$, no matter how many type-2 units are present, he waits until there accumulates 'a' type-1 units, whereupon he removes the batch of 'a' units for service; if he finds a or more but at most b, he takes them all in the batch and if he finds more than 'b' units, he picks a batch of b units at random for service, while other units wait in the queue. If none of the type-1 units is present at an epoch, the server starts servicing the type-2 units. Under this preemptive priority discipline, a type-2 unit may be preempted any number of times. Hence we propose to study $M, M|M^{a,b}, M|1$, preemptive priority queueing system.

This queueing system, in a real life situation, may be observed in the following taxidriver cum repairman problem. The server is a taxi driver who is also attending to the service of automobiles if he is free from taxi driving. While he is at service as a repairman, if there is a demand for his taxi, then he abandons the automobile service but waits till a minimum of 'a' customers accumulate for the taxi service and takes up the taxi driving with this batch of 'a' customers. Then at the subsequent service epochs, the server attends these two different jobs, as indicated in the description of the queueing system under study.

Steady state probabilities and equations

$P_{m,n}$: The steady state probability that there are $m(0 \leqslant m \leqslant a-1)$ type-1 units and $n(n \geqslant 0)$ type-2 units present in the system. Obviously the server will be at rest when $(1 \leqslant m \leqslant a-1, n \geqslant 0)$ and $(m=0, n=0)$ and he will be busy with the type-2 units when $(m = 0, n \geqslant 1)$.

$Q_{m,n}$: The steady state probability that the server is busy with a batch of type-1 units and there are m type-1 units (excluding the type-1 units included in the batch under service) and n type-2 units wait in the respective queues. It is non-zero for all $m,n \geq 0$.

The steady state probability equations of the queueing model under study are then given by:

$$(1) \qquad -(\lambda_1+\lambda_2)P_{0,0} + \mu_2 P_{0,1} + \mu_1 Q_{0,0} = 0$$

$$(2) \qquad -(\lambda_1+\lambda_2+\mu_2)P_{0,n} + \lambda_2 P_{0,n-1} + \mu_2 P_{0,n+1} + \mu_1 Q_{0,n} = 0 \quad (n \geq 1)$$

$$(3) \qquad -(\lambda_1+\lambda_2)P_{m,0} + \lambda_1 P_{m-1,0} + \mu_1 Q_{m,0} = 0 \quad (1 \leq m \leq a-1)$$

$$(4) \qquad -(\lambda_1+\lambda_2)P_{m,n} + \lambda_1 P_{m-1,n} + \lambda_2 P_{m,n-1} + \mu_1 Q_{m,n} = 0 \quad (1 \leq m \leq a-1;\ n \geq 1)$$

$$(5) \qquad -(\lambda_1+\lambda_2+\mu_1)Q_{0,0} + \lambda_1 P_{a-1,0} + \mu_1 \sum_{k=a}^{b} Q_{k,0} = 0$$

$$(6) \qquad -(\lambda_1+\lambda_2+\mu_1)Q_{0,n} + \lambda_1 P_{a-1,n} + \lambda_2 Q_{0,n-1} + \mu_1 \sum_{k=a}^{b} Q_{k,n} = 0 \quad (n \geq 1)$$

$$(7) \qquad -(\lambda_1+\lambda_2+\mu_1)Q_{m,0} + \lambda_1 Q_{m-1,0} + \mu_1 Q_{m+b,0} = 0 \quad (m \geq 1)$$

$$(8) \qquad -(\lambda_1+\lambda_2+\mu_1)Q_{m,n} + \lambda_1 Q_{m-1,n} + \lambda_2 Q_{m,n-1} + \mu_1 Q_{m+b,n} = 0 \quad (m,n \geq 1)$$

Note that when $a = 1$, equations (3) and (4) cannot occur.

Let us define the following generating functions,

$$H_m(y) = \sum_{n=0}^{\infty} P_{m,n} y^n , \quad |y| \leq 1 ; \quad 0 \leq m \leq a-1$$

$$F_m(y) = \sum_{n=0}^{\infty} Q_{m,n} y^n , \quad |y| \leq 1 ; \quad m \geq 0 .$$

Writing $\alpha(y) = \lambda_1 + \lambda_2 - \lambda_2 y$ and omitting its argument y , equations (1) to (8) reduce to,

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$$\mu_1 F_{m+b}(y) - (\alpha+\mu_1)F_m(y) + \lambda_1 F_{m-1}(y) = 0 \quad m \geqslant 1 \tag{9}$$

$$\mu_1 \sum_{k=a}^{b} F_k(y) - (\alpha+\mu_1)F_0(y) + \lambda_1 H_{a-1}(y) = 0 \tag{10}$$

$$\mu_1 F_m(y) - \alpha H_m(y) + \lambda_1 H_{m-1}(y) = 0 \quad 1 \leqslant m \leqslant a-1 \tag{11}$$

$$\mu_1 F_0(y) - [\alpha + \mu_2(1-\tfrac{1}{y})] H_0(y) = \mu_2(\tfrac{1}{y}-1)P_{0,0} . \tag{12}$$

The difference equation (9) can be written as,

$$h(E)F_{m-1}(y) = 0, \quad m = 1,2,\ldots \tag{13}$$

so that the characteristic equation becomes,

$$h(z) \equiv \mu_1 z^{b+1} - (\alpha+\mu_1)z + \lambda_1 = 0 \tag{14}$$

Hence by Rouche's theorem, it can be shown that there will be only one zero of $h(z) = 0$ inside the unit circle $|z| = 1$. Denote this root by $w = w(y)$. Since $\sum_{m=0}^{\infty} F_m(y) < \infty$, we must have,

$$F_m(y) = F_0(y)w^m; \quad m \geqslant 1 \tag{15}$$

Solving equation (11) recursively, we get,

$$H_m(y) = (\frac{\lambda_1}{\alpha})^m H_0(y) + \left[\frac{(\lambda_1/\alpha)^m - w^m}{1-w^b}\right] F_0(y) \quad 1 \leqslant m \leqslant a-1 \tag{16}$$

Substituting for $F_k(y)$, $k = a$, $a+1,\ldots b$ and $H_{a-1}(y)$ in (10) we obtain,

$$F_0(y) = \frac{\lambda_1(1-w)(1-w^b)w(\lambda_1/\alpha)^{a-1}}{\lambda_2(1-y)(w^{a+1} - w^{b+1}) + \lambda_1(1-w)\left[1-w(\lambda_1/\alpha)^{a-1}\right]} H_0(y) \tag{17}$$

Using (17) in (12), we see that,

$$H_0(y) = \left[\lambda_2(1-y)(w^{a+1} - w^{b+1}) + \lambda_1(1-w)\left\{1-w(\lambda_1/\alpha)^{a-1}\right\}\right] \left[\frac{\mu_2(1-y)}{A(y)}\right] P_{0,0} . \tag{18}$$

where,

(19) $$A(y) = \lambda_1(1-y)[\mu_2-\mu_2 w-\lambda_2 y][1-w(\lambda_1/\alpha)^{a-1}] + \lambda_1\mu_1(1-w^b)wy\{(\lambda_1/\alpha)^{a-1}-1\} - \lambda_2(1-y)[\alpha y-\mu_2(1-y)](w^{a+1}-w^{b+1}) .$$

Now the joint probability generating function of the number of type-1 and type-2 units, is given by,

(20) $$\phi(x,y) = \left[\frac{B(y)}{A(y)}\right]\frac{\mu_2(1-y)}{(1-xw)(\alpha-\lambda_1 x)} P_{0,0}$$

where

(21) $$B(y) = \alpha\left[\lambda_2(1-y)(w^{a+1}-w^{b+1}) + \lambda_1(1-w)\right](1-xw)\left[1-(\lambda_1 x/\alpha)^a\right] + \lambda_1(1-w)[\alpha-\lambda_1 x](\lambda_1/\alpha)^{a-1}(x^a w^{a+1}-w^{b+1})$$

The solution is now complete except for the unknown $P_{0,0}$. Let $w(y) \to \theta$ as $y \to 1$. Thus from (14) we see that,

(22) $$\frac{\lambda_1}{\mu_1} = \frac{\theta(1-\theta^b)}{1-\theta} .$$

Since $\phi(1,1) = 1$, we have

(23) $$P_{0,0} = \frac{(1-\theta)(1-\rho_2-\theta) - (a-1)(1-\theta)\rho_2 - \rho_2(\theta^{a+1}-\theta^{b+1})}{a(1-\theta) + (\theta^{a+1}-\theta^{b+1})}$$

which is the probability that the idle server finds no unit of either type in the system. Hence the steady state solution exists only if,

(24) $$1 - \theta > (1-\theta)(\theta+a\rho_2) + \rho_2(\theta^{a+1}-\theta^{b+1}) .$$

The probability that the server finds no unit of either type in the system or he is busy with a type-2 unit, is given by

(25) $$H_0(1) = \frac{(1-\theta)^2}{a(1-\theta) + \theta^{a+1} - \theta^{b+1}} .$$

The probability that the system contains m units of type-1, no matter how many type-2 units are present, when the server is idle, is given by,

$$H_m(1) = \frac{1-\theta^{m+1}}{1-\theta} H_0(1) \quad 1 \leqslant m \leqslant a-1 . \tag{26}$$

The probability that there are m units of type-1, when the server is busy with a batch of type-1 units is

$$F_m(1) = \frac{1-\theta^b}{1-\theta} \theta^{m+1} H_0(1) \quad m \geqslant 0 . \tag{27}$$

Now the mean number of type-1 units in the system may be obtained by differentiating the joint probability generating function $\phi(x,y)$ with respect to x and setting $x = y = 1$ or from the partial generating functions $F_m(y)$ and $H_m(y)$. By either method we obtain finally

$$\begin{aligned} L_{q1} &= \sum_{m=0}^{a-1} m\, H_m(1) + \sum_{m=0}^{\infty} m\, F_m(1) \\ &= \frac{1}{1-\theta}\left[\frac{\lambda_1\theta}{\mu_1(1-\theta)} + \frac{a(a-1)}{2} + \frac{a\theta^{a+1}(1-\theta) - \theta^2(1-\theta^a)}{(1-\theta)^2}\right] H_0(1) \end{aligned} \tag{28}$$

If we compare this result with the mean queue length of $M\,|M^{a,b}|\,1$ queueing system discussed in Medhi [3] or in Gohain and Borthakur [1], it is clear that the mean number (28), is not changed by the imposition of the preemptive priority discipline.

Similarly the mean number of type-2 units in the system is given by

$$L_2 = \sum_{m=0}^{a-1} H_m'(1) + \sum_{m=0}^{\infty} F_m'(1) .$$

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