UNIQUE CONTINUATION FOR NON-NEGATIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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Dedicated to Filippo Chiarenza

The aim of this note is to prove the unique continuation property for non-negative solutions of the quasilinear elliptic equation

(*) \( \text{div} A(x, u, \nabla u) = B(x, u, \nabla u). \)

We allow the coefficients to belong to a generalised Kato class.

1. INTRODUCTION

In his paper on Schrödinger semigroups [12] Simon formulated the following conjecture

Let \( \Omega \) be a bounded subset of \( \mathbb{R}^n \) and \( V \) a function defined in \( \Omega \) whose extension with zero values outside \( \Omega \) belongs to the Stummel-Kato class \( S(\mathbb{R}^n) \) (see Definition 2.2). Then the Schrödinger operator \( H = -\Delta + V \) has the unique continuation property,

that is, if \( u \in H^1(\Omega) \) is a solution of equation \( Hu = 0 \) which vanishes of infinite order at one point \( x_0 \in \Omega \) (see Definition 4.2), then \( u \) must be identically zero in \( \Omega \).

A positive answer to Simon’s conjecture was given by Fabes, Garofalo and Lin in [5] for radial potentials \( V \).

At the same time Chanillo and Sawyer in [1] proved the unique continuation property for solutions of the inequality \( |\Delta u| \leq |V||u| \), assuming \( V \) in the Morrey space \( L^{r,n-2r}(\mathbb{R}^n) \) with \( r > (n - 1)/2 \) (see Definition 2.1).

In this note, following an idea of Chiarenza and Garofalo (see [3]), we extend both the above results to the non-negative solutions of a quasilinear elliptic equation of the form

(1.1) \( \text{div} A(x, u, \nabla u) = B(x, u, \nabla u). \)

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Precisely we show that a non-negative solution \( u, u \neq 0 \), of (1.1) cannot have a zero of infinite order, assuming that suitable powers of the coefficients of (1.1) belong to the Morrey space \( L^{r,n-\rho r}(\mathbb{R}^n) \), with \( r \in (1, n/p) \), or to the function space \( \widetilde{M}_p(\mathbb{R}^n) \) (see Theorem 5.1). We denote by \( \widetilde{M}_p(\mathbb{R}^n) \) a generalisation of the Stummel-Kato class (see and Remark 2.5).

We point out that a crucial role in the proof of the Theorem 5.1 is played by Fefferman’s inequality

\[
\int_{\mathbb{R}^n} |u(x)|^p |V(x)| \, dx \leq c \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \quad \forall u \in C_0^\infty(\mathbb{R}^n),
\]

where \( c \) is a positive constant depending on some norm of \( V \). In Section 3 we give a new proof of (1.2) assuming \( V \in \widetilde{M}_p \).

### 2 Some function spaces

We begin this section giving some definitions.

**Definition 2.1:** (Morrey spaces) Let \( q \geq 1, \lambda \in (0, n) \). We say that \( f \in L^q_{\text{loc}}(\mathbb{R}^n) \) belongs to \( L^{q,\lambda}(\mathbb{R}^n) \) if

\[
\sup_{x \in \mathbb{R}^n, \rho > 0} \frac{1}{\rho^\lambda} \int_{B(x, \rho)} |f(y)|^q \, dy \equiv \|f\|^q_{q,\lambda} < +\infty.
\]

Here and in the following, we denote by \( B(x, \rho) \) the ball centred at \( x \) with radius \( \rho \). Whenever \( x \) is not relevant we shall write \( B_\rho \).

**Definition 2.2:** (Stummel-Kato class) Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). For any \( r > 0 \) we set

\[
\eta(r) \equiv \sup_{x \in \mathbb{R}^n} \int_{B(x, r)} \frac{|f(y)|}{|x - y|^{n-2}} \, dy.
\]

We say that \( f \) belongs to \( S(\mathbb{R}^n) \) if

\[
\lim_{r \to 0} \eta(r) = 0.
\]

**Definition 2.3:** Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). For \( p \in (1, n) \) and \( r > 0 \) we set

\[
\phi(r) \equiv \sup_{x \in \mathbb{R}^n} \left( \int_{|z-y| < r} \frac{1}{|x-y|^{n-1}} \left( \int_{|z-z'| < r} \frac{|f(z)|^p}{|z-y|^{n-1}} \, dz \right)^{1/(p-1)} \, dy \right)^{(p-1)}.
\]

We say that \( f \) belongs to the function space \( \widetilde{M}_p(\mathbb{R}^n) \) if

\[
\phi(r) < +\infty, \quad \forall r > 0.
\]
DEFINITION 2.4: We say that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ belongs to the function space $M_p(\mathbb{R}^n)$ if
\[
\lim_{r \to 0} \phi(r) = 0,
\]
where $\phi(r)$ is defined as in Definition 2.3.

Some comments are now in order.

REMARK 2.5. We have

(i) $M_p(\mathbb{R}^n) \subset \tilde{M}_p(\mathbb{R}^n);$

(ii) $M_2(\mathbb{R}^n) \equiv S(\mathbb{R}^n).$

(i) is trivial. Concerning (ii), Fubini’s theorem implies
\[
\int_{|z-y|<r} \frac{1}{|x-y|^n|z-y|^{n-1}} \left( \int_{|z-x|<r} \frac{|f(z)|}{|z-y|^{n-1}} \, dz \right) \, dy
= \int_{|z-x|<r} |f(z)| \int_{|z-y|<r} \frac{1}{|x-y|^n|z-y|^{n-1}} \, dy \, dz.
\]
Since
\[
\int_{|z-y|<r} \frac{1}{|x-y|^n|z-y|^{n-1}} \, dy \sim \frac{1}{|z-y|^{n-2}},
\]
we get the conclusion.

Therefore both the function spaces $M_p(\mathbb{R}^n)$ and $\tilde{M}_p(\mathbb{R}^n)$ are generalisations of $S(\mathbb{R}^n).$

3. ON FEFFERMAN’S INEQUALITY

In this section we recall some known results concerning Fefferman’s inequality
\[
(3.1) \quad \int_{\mathbb{R}^n} |u(x)|^p |f(x)| \, dx \leq c \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \quad \forall u \in C_0^\infty(\mathbb{R}^n),
\]
and give a new proof assuming $f \in \tilde{M}_p(\mathbb{R}^n).$

In [7] Fefferman proved (3.1), in the case $p = 2,$ assuming $f \in L^{r,n-2r}(\mathbb{R}^n),$ with $1 < r \leq n/2.$

Later in [10] Schechter showed the same result taking $f$ in the Stummel-Kato class $S(\mathbb{R}^n).$

We stress that it is not possible to compare the assumptions $f \in L^{r,n-2r}(\mathbb{R}^n)$ and $f \in S(\mathbb{R}^n).$

Chiarenza and Frasca [2] generalised Fefferman’s result proving (3.1) under the assumption $V \in L^{r,n-p^*}(\mathbb{R}^n)$ with $r \in (1,n/p)$ and $p \in (1,n).$ Namely they proved the following...
THEOREM 3.1. (See [2, p.407].) Assume $1 < p < n$, $1 < r \leq n/p$, $f \in L^{r,n-pr}(\mathbb{R}^n)$. Then there exists a constant $c$ depending on $n$ and $p$ such that

$$\int_{\mathbb{R}^n} |u^p(x)| |f(x)| \, dx \leq c \|f\|_{r,n-pr} \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

In the following theorem we provide a generalisation of Schecter’s result, proving (3.1) under the assumption $f \in \widetilde{M}_p(\mathbb{R}^n)$, $p \in (1, n)$.

THEOREM 3.2. Assume $f \in \widetilde{M}_p(\mathbb{R}^n)$. Then for any $r > 0$ there exists a positive constant $c(n, p)$ such that

$$\int_{\mathbb{R}^n} |f(x)| |u(x)|^p \, dx \leq c(n, p) \phi(2r) \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx$$

for any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0, r)$.

PROOF: For any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0, r)$, using the well known inequality

$$(3.2) \quad |u(x)| \leq c(n, p) \int_{B(x_0, r)} \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy$$

and Fubini’s theorem, we have

$$(3.3) \quad \int_{\mathbb{R}^n} |f(x)| |u(x)|^p \, dx$$

$$\begin{align*}
&= \int_{B(x_0, r)} |f(x)| |u(x)|^p \, dx \\
&\leq c(n, p) \int_{B(x_0, r)} |f(x)| |u(x)|^{p-1} \left( \int_{B(x_0, r)} \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy \right) \, dx \\
&\leq c(n, p) \int_{B(x_0, r)} |\nabla u(y)| \left( \int_{B(x_0, r)} |f(x)| |u(x)|^{p-1} \frac{1}{|x - y|^{n-1}} \, dx \right) \, dy \\
&\leq c(n, p) \left( \int_{B(x_0, r)} |\nabla u(y)|^p \, dy \right)^{1/p} \\
&\cdot \left[ \int_{B(x_0, r)} \left( \int_{B(x_0, r)} |f(x)| |u(x)|^{p-1} \frac{1}{|x - y|^{n-1}} \, dx \right)^{p/(p-1)} \, dy \right]^{(p-1)/p}.
\end{align*}$$
We also have

(3.4)
\[
\int_{B(x_0,r)} \left( \int_{B(x_0,r)} |f(x)||u(x)|^{p-1} \frac{1}{|x-y|^{n-1}} \, dx \right)^{p/(p-1)} \, dy
\]
\[
\leq \int_{B(x_0,r)} \left( \int_{B(x_0,r)} \frac{|f(z)|}{|z-y|^{n-1}} \, dx \right)^{1/(p-1)} \int_{B(x_0,r)} \frac{|f(z)||u(z)|^p}{|x-y|^{n-1}} \, dx \, dy
\]
\[
= \int_{B(x_0,r)} |f(x)||u(x)|^p \int_{B(x_0,r)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x_0,r)} \frac{|f(z)|}{|z-y|^{n-1}} \, dz \right)^{1/(p-1)} \, dy \, dx
\]
\[
\leq \phi^{1/(p-1)}(2r) \int_{B(x_0,r)} |f(x)||u(x)|^p \, dx.
\]

By (3.3) and (3.4) we obtain the desired conclusion.

REMARK 3.3. We note that proceeding as in Theorem 3.2 using the representation formula (see, for example [6])

instead of (3.2), it is possible to obtain a Poincaré type inequality. Namely

THEOREM 3.4. Suppose \( u \) is a Lipschitz continuous function on \( \overline{B}_R \), the closure of \( B_R \), and \( f \) is a function defined on \( B_R \) whose extension with zero values outside \( B_R \) belongs to \( M_p(\mathbb{R}^n) \). Then there exists a positive constant \( c \) such that

\[
\int_{B_R} |f(x)||u(x) - u_{B_R}|^p \, dx \leq c \phi(2R) \int_{B_R} |\nabla u(x)|^p \, dx
\]

where \( u_{B_R} \) is the average \((1/|B_R|) \int_{B_R} u(x) \, dx\) where \(|B_R|\) is the Lebesgue measure of \( B_R \).

4. ASSUMPTIONS AND PRELIMINARY RESULTS

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \). The equation we consider is of the form

(4.1) \[ \text{div} \, A(x, u, \nabla u) = B(x, u, \nabla u), \]

where

\[ A(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \]

and

\[ B(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \]
are two continuous functions satisfying the following conditions

\[
\begin{aligned}
&\quad |A(x, u, \xi)| \leq a|\xi|^{p-1} + b(x)|u|^{p-1} \\
&\quad |B(x, u, \xi)| \leq c(x)|\xi|^{p-1} + d(x)|u|^{p-1} \\
&\quad \xi A(x, u, \xi) \geq |\xi|^p - d(x)|u|^p
\end{aligned}
\]  

for almost all, \( x \in \Omega, \forall u \in \mathbb{R}, \forall \xi \in \mathbb{R}^n \). We assume that \( p \) is a fixed number in \((1, n)\), \( a \) is a positive constant and \( b, c \) and \( d \) are measurable functions in \( \Omega \) whose extension with zero values outside \( \Omega \) are such that

\[
\frac{b^p}{(p-1)} c^p, d \in M_p(\mathbb{R}^n),
\]
or

\[
\frac{b^p}{(p-1)} c^p, d \in L_r^{r, n-p} (\mathbb{R}^n) \quad r \in (1, n/p).
\]

**Definition 4.1:** We say that a function \( u \in H^1_{loc}(\Omega) \) is a local weak solution of (4.1) in \( \Omega \) if

\[
\int_{\Omega} \left\{ A(x, u(x), \nabla u(x)) \nabla \phi(x) + B(x, u(x), \nabla u(x)) \phi(x) \right\} dx = 0
\]

for every \( \phi \in C_0^\infty(\Omega) \).

We remark that Definition 4.1 is meaningful by Theorem 3.1 or Theorem 3.2. To state our result we need one more definition.

**Definition 4.2.** Assume \( w \in L^1_{loc}(\Omega), w \geq 0 \) almost everywhere in \( \Omega \). We say that \( w \) has a zero of infinite order at \( x_0 \in \Omega \) if

\[
\lim_{\rho \to 0} \frac{\int_{B(x_0, \rho)} w(x) \, dx}{|B(x_0, \rho)|^k} = 0 \quad \forall k > 0.
\]

The following two lemmas are known.

**Lemma 4.3.** (See [9].) Assume \( w \in L^1_{loc}(\Omega), w \geq 0 \) almost everywhere in \( \Omega \), \( w \neq 0 \). If

\[
\exists C > 0 : \int_{B(x_0, 2\rho)} w(x) \, dx \leq C \int_{B(x_0, \rho)} w(x) \, dx \quad \forall \rho > 0,
\]

then \( w(x) \) has no zero of infinite order in \( \Omega \).

**Lemma 4.4.** (See [4] and [8].) Let \( B_r \subset \mathbb{R}^n \), \( u \in H^1(B_r) \) be and assume that for all \( B_r \subset B_{\rho} \) there exists a constant \( K \) such that

\[
\left( \int_{B_r} |\nabla u(x)|^p \, dx \right)^{1/p} \leq K r^{(n-p)/p}.
\]

Then there exist two positive constants \( \delta \) and \( C \), depending on \( K, p, n \), such that

\[
\left( \int_{B_{\rho}} e^{\delta u(x)} \, dx \right) \left( \int_{B_{\rho}} e^{-\delta u(x)} \, dx \right) \leq C |B_{\rho}|^2.
\]
5. UNIQUE CONTINUATION

In this section we state and prove our result, namely

**Theorem 5.1.** Let \( u \in H^1(\Omega) \), \( u \geq 0 \), \( u \not\equiv 0 \), be a solution of (4.1) satisfying (4.2) and (4.3) or (4.2) and (4.3)'

Then \( u \) has no zero of infinite order in \( \Omega \).

**Proof:** Let \( x_0 \in \Omega \), let \( B(x_0, R) \) be a ball such that \( B(x_0, 2R) \) is contained in \( \Omega \). Consider any \( B_h \) contained in \( B(x_0, R) \). Let \( \eta \) be a non negative smooth function with support in \( B_{2h} \). Using \( \phi = \eta^p u^{1-p} \) as test function in (4.4) we get (see [11])

\[
(5.1) \quad \int_{\Omega} \left| \nabla \log u(x) \right|^p \eta^p(x) \, dx \leq C_1(p, a) \left\{ \int_{\Omega} \left| \nabla \eta(x) \right|^p \, dx + \int_{\Omega} V(x) \eta^p(x) \, dx \right\},
\]

where \( V \) is defined by

\[
V = C_3(p, a, \text{diam } \Omega) + C_3(p, a, \text{diam } \Omega) + C_3(p, a, \text{diam } \Omega).
\]

By Theorem 3.1 or Theorem 3.2, we have

\[
\int_{\Omega} V(x) \eta^p(x) \, dx \leq C_2(\text{spt } \eta) \int_{\Omega} \left| \nabla \eta(x) \right|^p \, dx.
\]

Inserting this inequality in (5.1), we obtain

\[
(5.2) \quad \int_{\Omega} \eta^p(x) \left| \nabla \log u(x) \right|^p \, dx \leq C_3(p, a, \text{diam } \Omega) \int_{\Omega} \left| \nabla \eta(x) \right|^p \, dx.
\]

Choosing \( \eta \) so that \( \eta = 1 \) in \( B_h \) and \( |\nabla \eta| \leq 3/h \), by (5.2) we have

\[
(5.3) \quad \int_{B_h} \left| \nabla \log u(x) \right|^p \, dx \leq C_4(p, a, \text{diam } \Omega) h^{n-p}.
\]

Therefore, by Lemma 4.4, we have

\[
\int_{B_h} u^\delta(x) \, dx \int_{B_h} u^{-\delta}(x) \, dx \leq C |B_h|^2,
\]

that is, \( u^\delta \) belongs to the Muckenhoupt class \( A_2 \) for some \( \delta > 0 \) (see [3] and [6]). Now it is well known that \( A_2 \) implies the doubling property for \( u_\delta \), that is, the assumption of Lemma 4.3. So the conclusion follows for \( u^\delta \) and hence also for \( u \). \( \square \)
REFERENCES


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