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GROUP CONGRUENCES ON EVENTUALLY REGULAR SEMIGROUPS

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Abstract

A characterization of group congruences on an eventually regular semigroup S is provided. It is shown that a group congruence is dually right modular in the lattice of congruences on S. Also for any group congruence γ and any congruence ρ on $S, \gamma \vee \rho$ and kernel $\gamma \vee \rho$ are described.

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1. Introduction

D. R. LaTorre [1] gave an alternative characterization for a group congruence on a regular semigroup to that given by R. Feigenbaum [4] in her doctoral dissertation.

Let us recall from [2] that a semigroup S is eventually regular if a power of each element is regular. Throughout this paper S is an eventually regular semigroup and E is the set of idempotents of S. If a is a regular element of S, $V(a)$ denotes the set of inverses of a. For $a \in S$, by "a^n is a-regular" we mean that n is the least positive integer for which a^n is regular. We denote by $\Lambda(S)$ the lattice of congruences of S.

In this paper a characterization for a group congruence on an eventually regular semigroup is provided. In Theorem 3 seveal equivalent expressions for any group congruence on an eventually regular semigroup are given. The join $\gamma \vee \rho$ of a group congruence γ and an arbitrary congruence ρ of an eventually regular semigroup is described in Theorem 5. The next corollary says that every group congruence is a dually right modular element in the lattice of congruences on S, which generalises Corollary 3.2 of [3]. For any congruence ρ on S and any group congruence γ on S, an expression for the kernel of $\gamma \vee \rho$ is obtained: $\operatorname{Ker}(\gamma \vee \rho) = ((\operatorname{Ker}\gamma)\rho)\omega$, which was obtained for regular semigroups in [1]. An expression for $\operatorname{Ker}\gamma \wedge \rho$ is also obtained.

2. Group congruences

A subset H of S is defined to be full if $E \subseteq H$. For any subset H of S the closure $H\omega$ of H is $\{x \in S\colon hx \in H \text{ for some } h \in H\}$; H is said to be closed if $H\omega = H$.

A subset H of S is called self-conjugate if $aHa^{n-1}(a^n)' \subseteq H$ and $a^{n-1}(a^n)'Ha \subseteq H$ for each $a \in S$, where a^n is a-regular, and for each $(a^n)' \in V(a^n)$. This coincides with the definition of self-conjugate in [1] for regular semigroups.

LEMMA 1. *If H is a full self-conjugate subsemigroup of an eventually regular semigroup S, then $H\omega = H$ if and only if, for all $h \in H$ and $x \in S$, $xh \in H$ implies $x \in H$.*

PROOF. Suppose $H\omega = H$ and $h, xh \in H$. Let x^n be x-regular and $(x^n)' \in V(x^n)$. Since H is full we have $x^{n-1}(x^n)'x \in H$. Now $xh, x^{n-1}(x^n)'x \in H$ imply $xhx^{n-1}(x^n)'x \in H$. Since H is self-conjugate, $x^{n-1}(x^n)'(xhx^{n-1}(x^n)'x)x \in H$. Since $x^{n-1}(x^n)'xhx^{n-1}(x^n)'x \in EHE \subseteq H$, we have $x \in H$.

The other implication can be proved similarly.

THEOREM 1. *If H is a full, self-conjugate closed subsemigroup of an eventually regular semigroup S then $\beta_H = \{(a, b) \in S \times S\colon ab^{n-1}(b^n)' \in H \text{ where } b^n$ is b-regular and $(b^n)' \in V(b^n)\}$ is a group congruence on S.*

PROOF. Reflexivity follows from $E \subseteq H$. To show symmetry let $(a, b) \in \beta_H$; this implies $ab^{n-1}(b^n)' \in H$, where b^n is b-regular and $(b^n)' \in V(b^n)$. Let a^m be a-regular and $(a^m)' \in V(a^m)$. Since H is self-conjugate,

$$(ba^{m-1}(a^m)'ab^{n-1}(b^n)')(ab^{n-1}(b^n)') \in H.$$

Since H is closed, $ba^{m-1}(a^m)' \in H$, so β_H is symmetric. If $ab^{n-1}(b^n)'$, $bc^{l-1}(c^l)' \in H$ where b^n is b-regular and c^l is c-regular and $(b^n)' \in V(b^n)$, $(c^l)' \in V(c^l)$, we have $ab^{n-1}(b^n)'bc^{l-1}(c^l)' \in H$. Let a^m be a-regular and $(a^m)' \in V(a^m)$. As H is self-conjugate $a^{m-1}(a^m)'(ab^{n-1}(b^n)'bc^{l-1}(c^l)')a \in H$ from which it follows that $c^{l-1}(c^l)'a \in H$. Again $c(c^{l-1}(c^l)'a)c^{l-1}(c^l)'$ belongs to H, giving $ac^{l-1}(c^l)' \in H$, which proves transitivity of β_H.

Hence β_H is an equivalence relation.

To see compatibility of β_H, suppose $ab^{n-1}(b^n)' \in H$, where b^n is b-regular and $(b^n)' \in V(b^n)$ and $c \in S$. If a^m is a-regular and $(a^m)' \in V(a^m)$ we also have $ba^{m-1}(a^m)' \in H$. Let $(bc)^l$ be bc-regular and $((bc)^l)' \in V(bc)$ and let c^k be c-regular and $(c^k)' \in V(c^k)$. Now $c(bc)^{l-1}((bc)^l)'b \in E \subseteq H$. Making use of the self-conjugacy property we have $ac(bc)^{l-1}((bc)^l)'ba^{m-1}(a^m)' \in H$, so $ac(bc)^{l-1}((bc)^l)' \in H$, which gives that β_H is right compatible. We can similarly prove left compatibility. Hence β_H is a congruence.

Since H is a full subsemigroup it is easy to observe that E is contained in a single β_H-class. For any $e \in E$ and $a \in S$ we have $aea^{n-1}(a^n)'$, $eaa^{n-1}(a^n)' \in H$, where a^n is a-regular, so (ae, a), $(ea, a) \in \beta_H$. Hence $\operatorname{Ker}\beta_H$ is the identity element of $S/_{\beta_H}$. For any $a \in S$, if a^n is a-regular, we have $(a\beta_H)(a^{n-1}(a^n)'\beta_H) = (a^{n-1}(a^n)'\beta_H)(a\beta_H) = e\beta_H$. Hence β_H is a group congruence on S.

REMARK. It can be observed that $\operatorname{Ker}\beta_H = H$ and hence H is the identity element of $S/_{\beta_H}$.

THEOREM 2. *The kernel of any group congruence γ on an eventually regular semigroup S is a full, self-conjugate closed subsemigroup.*

PROOF. Clearly kernel γ is a full subsemigroup. Let $a \in S$, $b \in \operatorname{Ker}\gamma$, a^n be a-regular and $(a^n)' \in V(a^n)$. Since $b \in \operatorname{Ker}\gamma$, we have for some $e \in E$ that $(b, e) \in \gamma$, so $(aba^{n-1}(a^n)', aea^{n-1}(a^n)') \in \gamma$. But $aea^{n-1}(a^n)'\gamma = a\gamma e\gamma a^{n-1}(a^n)'\gamma = a\gamma a^{n-1}(a^n)'\gamma = a^n(a^n)'\gamma$, so $aba^{n-1}(a^n)' \in \operatorname{Ker}\gamma$. Similarly we can show that $a^{n-1}(a^n)'ba \in \operatorname{Ker}\gamma$, which proves that $\operatorname{Ker}\gamma$ is self-conjugate. If h, $hx \in \operatorname{Ker}\gamma = e\gamma$ for any $e \in E$ then $e\gamma = h\gamma = hx\gamma = e\gamma x\gamma = x\gamma$ so that $x \in \operatorname{Ker}\gamma$ and hence $\operatorname{Ker}\gamma$ is closed. The theorem follows.

REMARK. It can be easily seen that for any group congruence γ on S, $\beta_{\operatorname{Ker}\gamma} = \gamma$, and the mapping $H \mapsto \beta_H$ is an inclusion preserving one-to-one correspondence between the set of all full, self-conjugate closed subsemigroups of S and the set of group congruences on S.

THEOREM 3. *If γ is a group congruence on an eventually regular semigroup S and $\operatorname{Ker}\gamma = H$, then the following are equivalent:*

(1) $a\gamma b$;

(2) $ba^{m-1}(a^m)' \in H$ *where a^m is a-regular and $(a^m)' \in V(a^m)$;*

(3) $a^{m-1}(a^m)'b \in H$ *where a^m is a-regular and $(a^m)' \in V(a^m)$;*

(4) $b^{n-1}(b^n)'a \in H$ *where b^n is b-regular and $(b^n)' \in V(b^n)$;*

(5) $axb^{n-1}(b^n)' \in H$ *for some $x \in H$ and b^n is b-regular and $(b^n)' \in V(b^n)$;*

(6) $bxa^{m-1}(a^m)' \in H$ *for some $x \in H$ and a^m is a-regular and $(a^m)' \in V(a^m)$;*

(7) $a^{m-1}(a^m)'xb \in H$ *for some $x \in H$ and a^m is a-regular and $(a^m)' \in V(a^m)$;*

(8) $b^{n-1}(b^n)'xa \in H$ *for some $x \in H$ and b^n is b-regular and $(b^n)' \in V(b^n)$;*

(9) $xa = by$ *for some* $x, y \in H$;
(10) $ax = yb$ *for some* $x, y \in H$;
(11) $HaH \cap HbH \neq \varnothing$.

PROOF. That (1) implies (2) follows from the fact that γ is symmetric. Assume (2), namely that $ba^{m-1}(a^m)' \in H$. Since H is self-conjugate,

$$b^{n-1}(b^n)'ba^{m-1}(a^m)'b \in H.$$

As H is closed we get $a^{m-1}(a^m)'b \in H$. Hence (2) implies (3). Since H is full, self-conjugate we have $a^{m-1}(a^m)'b^n(b^n)'a \in H$. Now $a^{m-1}(a^m)'b \in H$ implies $b^{n-1}(b^n)'a \in H$. So (3) implies (4). If $b^{n-1}(b^n)'a \in H$, we get $bb^{n-1}(b^n)'ab^{n-1}(b^n)' \in H$, so $ab^{n-1}(b^n)' \in H$, which proves (1), (2), (3), (4) are equivalent.

(5)⇒(6). Assume $axb^{n-1}(b^n)' \in H$. Since $x \in H$ we have $xb^{n-1}(b^n)'bx \in H$ so $axb^{n-1}(b^n)'bxa^{m-1}(a^m)' \in H$. Since H is closed, $bxa^{m-1}(a^m)' \in H$.

(6)⇒(7). If $bxa^{m-1}(a^m)' \in H$ for some $x \in H$, then $bxa^{m-1}(a^m)'x \in H$ and also $b^{n-1}(b^n)'bxa^{m-1}(a^m)'xb \in H$. Since H is closed, $a^{m-1}(a^m)'xb \in H$.

(7)⇒(8). Assume $a^{m-1}(a^m)'xb \in H$, where $x \in H$. Since

$$b^{n-1}(b^n)'xaa^{m-1}(a^m)'xb \in H$$

and H is closed we get $b^{n-1}(b^n)'xa \in H$.

(8)⇒(9). If $b^{n-1}(b^n)'xa \in H$ for some x in H, then $b^{n-1}(b^n)'xa = y$, where $y \in H$ and hence $b^n(b^n)'xa = by$. Put $b^n(b^n)'x = x_1$. Then $x_1a = by$ for some $x_1, y \in H$.

(9)⇒(10). $xa = by$ for some $x, y \in H$ implies $a^m(a^m)'xab^{n-1}(b^n)'b = a^m(a^m)'byb^{n-1}(b^n)'b$, so $a(a^{m-1}(a^m)'xab^{n-1}(b^n)'b) = (a^m(a^m)'byb^{n-1}(b^n)')b$, which says $ax_1 = y_1b$ for some $x_1, y_1 \in H$.

(10⇒(11). If $ax = yb$ for some $x, y \in H$ then $xaxy = xyby$ so $HaH \cap HbH \neq \varnothing$.

(11)⇒(5). $HaH \cap HbH \neq \varnothing$ implies $h_1ah_2 = t_1bt_2$ for some $h_1, h_2, t_1, t_2 \in H$. Now $h_1ah_2 = t_1at_2$ implies $a^m(a^m)'h_1ah_2b^{n-1}(b^n)'b = a^m(a^m)'t_1bt_2b^{n-1}(b^n)'b$, so $a(a^{m-1}(a^m)'h_1ah_2b^{n-1}(b^n)'b) = (a^m(a^m)'t_1bt_2b^{n-1}(b^n)')b$. Hence $ax = yb$ for some $x, y \in H$, which implies $axb^{n-1}(b^n)' = yb^n(b^n)' \in H$. Hence (5) to (11) are equivalent.

(1)⇒(9). If $ab^{n-1}(b^n)' = y \in H$ then $ab^{n-1}(b^n)'b = yb$, so $ax = yb$ for some $x, y \in H$.

(5)⇒(4). Now $axb^{n-1}(b^n)' \in H$ implies $a^{m-1}(a^m)'axb^{n-1}(b^n)'a \in H$, so $b^{n-1}(b^n)'a \in H$, which completes the proof.

THEOREM 4. *If H is a full, eventually regular subsemigroup of an eventually regular semigroup S, then for any $a \in H$, a^n is a-regular in H and $x \in V(a^n)$ imply $x \in H$.*

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PROOF. Let a^n be a-regular and take any inverse $(a^n)'$ of a^n in H. If $x \in V(a^n)$ then $x = xa^nx = (xa^n)(a^n)'(a^nx) \in H$.

In [1], for any group congruence γ and any congruence ρ on a regular semigroup S, it is shown that $\gamma \vee \rho$ is equal to $\gamma \circ \rho \circ \gamma$. In the following theorem we prove the corresponding result for eventually regular semigroups.

THEOREM 5. *If γ is a group congruence on S and ρ is any congruence on S, then $\gamma \vee \rho = \gamma \circ \rho \circ \gamma$.*

PROOF. It suffices to prove $\rho \circ \gamma \circ \rho \subseteq \gamma \circ \rho \circ \gamma$.

Let $(x, y) \in \rho \circ \gamma \circ \rho$. Then for some $a, b \in S$ we have $(x, a) \in \rho$. $(a, b) \in \gamma$, $(b, y) \in \rho$. let a^m be a-regular and let b^n be b-regular. From (x, a), $(b, y) \in \rho$ it follows that $(b^n(b^n)'x, b^n(b^n)'a)$, $(b^n(b^n)'a, yb^{n-1}(b^n)'a) \in \rho$. Now $(a, b) \in \gamma$ implies $((a^m)'a^{m-1}, b^{n-1}(b^n)') \in \gamma$, since γ is a group congruence. Also we have $(x, b^n(b^n)'x) \in \gamma, (b^n(b^n)'x, yb^{n-1}(b^n)'a) \in \rho$, and $(yb^{n-1}(b^n)'a, y(a^m)'a^m)$, $(y(a^m)'a^m, y) \in \gamma$, so $(x, y) \in \gamma \circ \rho \circ \gamma$, and the theorem is proved.

In [3] the modularity relation M on a lattice was given by aMb if $(x \vee a) \wedge b = x \vee (a \wedge b)$ for all $x \leq b$; and M^* denotes its dual. An element $d \in L$ is right modular if aMd for all $a \in L$. Proposition 2.3(ii) in [3] says that in a semigroup S, for $\gamma \in \Lambda(S)$, if $\gamma \vee \rho = \gamma \circ \rho \circ \gamma$ for all $\rho \in \Lambda(S)$ then γ is a dually right modular element.

COROLLARY. *On an eventually regular semigroup S, each group congruence is a dually right modular element of $\Lambda(S)$.*

THEOREM 6. *For any congruence ρ and any group congruence γ on an eventually regular semigroup S, $a(\gamma \vee \rho)b$ if and only if $xa\rho by$ for some $x, y \in \operatorname{Ker}\gamma$.*

PROOF. As Theorems 4 and 6 in [1] have been shown to apply to S, the proof is the same as that of Theorem 7 in [1].

The following theorem corresponds to Theorem 8 in [1] for regular semigroups, which describes $\operatorname{Ker}(\gamma \vee \rho)$, for any group congruence γ and any congruence ρ.

THEOREM 7. *For any congruence ρ and any group congruence γ on an eventually regular semigroup S,* $\operatorname{Ker}(\gamma \vee \rho) = ((\operatorname{Ker}\gamma)\rho)\omega$.

PROOF. Take $x \in \operatorname{Ker}(\gamma \vee \rho)$. Then there exists $e \in E$ with $(x, e) \in \gamma \vee \rho$. From the previous theorem we get $(xp, qe) \in \rho$ for some $p, q \in \operatorname{Ker}\gamma$, since $qe \in \operatorname{Ker}\gamma$ and $xp \in (\operatorname{Ker}\gamma)\rho$. Since $p \in \operatorname{Ker}\gamma \subseteq (\operatorname{Ker}\gamma)\rho$ we get $x \in ((\operatorname{Ker}\gamma)\rho)\omega$. Conversely if $x \in ((\operatorname{Ker}\gamma)\rho)\omega$, then $hx \in ((\operatorname{Ker}\gamma)\rho)$ for some $h \in (\operatorname{Ker}\gamma)\rho$ from which it follows that $(hx, y) \in \rho$ and $(h, y_1) \in \rho$, for some $y, y_1 \in \operatorname{Ker}\gamma$. Now

$y, y_1 \in \operatorname{Ker}\gamma$ implies $(y, e), (y_1, e) \in \gamma$ for some $e \in E$, so $hx, h \in \operatorname{Ker}\gamma \vee \rho$. Thus $x \in \operatorname{Ker}(\gamma \vee \rho)$, since $\operatorname{Ker}(\gamma \vee \rho)$ is closed.

COROLLARY. $\gamma \vee \rho = S \times S$ *if and only if* $((\operatorname{Ker}\gamma)\rho)\omega = S$.

The next result corresponds to Lemma 4 of [1] for regular semigroups, and the proof is similar.

LEMMA 2. *For any congruence ρ and any group congruence γ on an eventually regular semigroup S,* $\operatorname{Ker}(\gamma \cap \rho) = \operatorname{Ker}\gamma \cap \operatorname{Ker}\rho$.

Let ρ be an idempotent-separating congruence on S and let γ be a group congruence on S. Corollary 4 of [1] states that if S is regular then $\operatorname{Ker}\gamma \cap \operatorname{Ker}\rho = E$ implies $\gamma \cap \rho = \iota$. This implication is not true in general for eventually regular semigroups, as the following example shows.

Let us take the three element null semigorup $S = \{a, b, 0\}$ with zero element 0 (that is, $xy = 0$ for all $x, y \in S$) and put $\gamma = S \times S$, and let ρ be the congruence with partition $\{\{a, b\}, \{0\}\}$. Then ρ (and γ) are idempotent-separating, $\rho \cap \gamma = \rho$, and $\operatorname{Ker}\rho \cap \operatorname{Ker}\gamma = \{0\} = E$.

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