AN ORDER-PRESERVING REPRESENTATION THEOREM FOR COMPLEX BANACH ALGEBRAS AND SOME EXAMPLES

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Let $A$ be a complex Banach algebra with unit $e$ of norm one. We show that $A$ can be represented on a compact Hausdorff space $\Omega$ which arises entirely out of the algebraic and norm structures of $A$. This space induces an order structure on $A$ that is preserved by the representation. In the commutative case, $\Omega$ is the spectrum of $A$, and we have a generalization of Gelfand's representation theorem for commutative complex Banach algebras with unit. Various aspects of this representation are illustrated by considering algebras of $n \times n$ complex matrices.

This paper is a revised and expanded version of the unpublished work [8] by the first author.

The following notations are used throughout the paper: cl denotes closure, co denotes convex span of, $A'$ is the dual space of $A$, $w^*$-topology $= \sigma(A', A)$-topology, and maximal left ideal is abbreviated to m.l.i.

1. A wedge constructed from the algebraic and norm structures of $A$. Let $J$ be a m.l.i. of $A$. Since $J$ is closed and $e$ is at unit distance from $J$, there exists (by the Hahn–Banach theorem) an element $f$ in $A'$ such that $f(J) = (0)$ and $f(e) = 1 = ||f||$. Hence the following sets are nonempty:

- $B = \{f \in A' : f(e) = ||f|| = 1\}$,
- $M = \{f \in B : f(J) = (0) \text{ for some m.l.i. } J \text{ of } A\}$,
- $\Omega = w^*\text{-cl } M$,
- $S = w^*\text{-cl co } M$,
- $C' = \bigcup \{\lambda S : \lambda \geq 0\}$,
- $K' = C' - iC'$,
- $K = \{x \in A : \text{Re } f(x) \geq 0 \text{ for all } f \in K'\}$.

The sets $B$, $S$, and $\Omega$ are compact and Hausdorff in the $w^*$-topology. We have $\Omega \subseteq S \subseteq B$ and $B$ and $S$ are convex. Our aim is to represent $A$ as a subspace of $C(\Omega)$, the space of all complex-valued continuous functions on $\Omega$ with sup norm, and to consider various aspects of this representation (Examples: In general, $M$ is not $w^*$-closed and not in one-to-one correspondence with the maximal left ideals).

We observe here that, if we let $H(A) = \{x \in A : f(x) \text{ is real for all } f \text{ in } B\}$ and $K(A) = \{x \in A : f(x) \geq 0 \text{ for all } f \text{ in } B\}$, then $K(A) + iK(A) \subseteq K$. In fact, $K(A) + iK(A) \subseteq K_\mathbb{R} \subseteq K$.

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where $K_B = \{ x \in A : \text{Re} f(x) \geq 0, \text{Im} f(x) \geq 0, \text{for all } f \in B \}$. We show below (§ 2) that $A = K - K$ so that, if $K(A) + iK(A) = K$, then $A = H(A) + iH(A)$, and hence [6] $A$ is a $C^*$-algebra. Conversely, if $A$ is a $C^*$-algebra, then $A = H(A) + iH(A)$ and also (see, for example, [3, Proposition 2.9.1, Theorem 2.9.5]) $M = \text{ext } B$, so that $K_B = K$ and, if $x \in K$, then $x = y + iz$ ($y, z \in H(A)$), which gives, for all $f$ in $B$, $f(y) = \text{Re} f(x) \geq 0$, $f(z) = \text{Im} f(x) \geq 0$, and so $K \subseteq K(A) + iK(A)$. Thus $A$ is a $C^*$-algebra if and only if $K(A) + iK(A) = K$.

The set $M$ also arises in a somewhat different context. In connection with the notion of numerical ranges, F. F. Bonsall [2] considers the subset of $B$ given by

$$D = \{ f \in B : L_f \text{ is an m.l.i. of } A \},$$

where

$$L_f = \{ x \in A : f(J_x) = (0) \},$$

$J_x$ being the principal left ideal generated by $x$. It is easy to show that $M = D$. We remark that the definition of $M$ is more direct than that of $D$.

Since $0 \notin S$, $C'$ and $K'$ are cones; $C'$ is contained in the set of all those functionals $f$ which support the unit ball at $e$; the intersection of the associated hyperplanes ($\{ x : f(x) = ||f|| = f(e) \}$) is $\{ e \}$ (Bohnenblust and Karlin [1]), that is, $e$ is a vertex of the unit ball.

The set $S' = \text{co } \{ S \cup (-iS) \}$ is a w*-compact base for $K'$. Since $w^*$-convergent nets are norm bounded and $S, S'$ are $w^*$-closed, it follows that such nets in $C'$ or $K'$ are contained in suitable nonnegative multiples of $S$ or $S'$ which are evidently $w^*$-closed and therefore contain the limits of these nets. Thus $C'$ and $K'$ are $w^*$-closed. Excepting $M$, the sets discussed so far are also norm-closed. Using the facts that $e$ and $ie$ are in $K$ (§ 2), it can be shown that $C'$ and $K'$ are also normal. It is clear that $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda \geq 0$, that is, $K$ is a wedge. Here, in letting $K'$ induce $K$, we are following the definitions of H. H. Schaefer [7, p. 214].

2. Properties of the wedge $K$. Let $x \in A$. Then

(1) $x \in K$ if and only if $\text{Re} f(x) \geq 0$ and $\text{Im} f(x) \geq 0$ for all $f$ in $C'$ (or, equivalently, $S$ or $\Omega$ or $M$); (2) $x \in (-K) \cap K$ if and only if $f(x) = 0$ for all $f$ in $K'$ (or, equivalently, $C'$ or $S$ or $\Omega$ or $M$); (3) since $f(e) = ||f||$ for all $f$ in $C'$, (1) shows that $e$, $ie$ and (hence) $u = e + ie$ are in $K$;

(4) $\text{Re} f(u) = \text{Im} f(u) = 1$ for all $f$ in $M$;

(5) $K$ is weakly closed and (hence) norm-closed.

Let $f = f_1 - if_2 \in K'$, so that $f_1$ and $f_2$ are in $C'$. From (4) we have, for all $x$ in $A$,

$$\text{Re} f(u + x) = f_1(e) + f_2(e) + \text{Re} f_1(x) + \text{Im} f_2(x) \geq (||f_1|| + ||f_2||)(1 - ||x||).$$

This shows that $K$ contains the $u$-translate of the unit ball. Therefore $u$ is in int $K$ (interior of $K$); hence it is an order unit, and $K$ generates $A$. 

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It is easy to see that, if \( txu - x \in K \) for all \( \alpha > 0 \), then \(-x \in K\), so that \( K \) induces an Archimedean ordering denoted by \( \leq \).

If \( x \in K \), then \( x - \lambda u \in K \) for some \( \lambda > 0 \); hence, from (1) and (4), we have, for all \( f \) in \( M \), \( f(x) \neq 0 \) (indeed, \( 0 < \lambda \leq \min \{\text{Re}(f(x)), \text{Im}(f(x))\} \)). Since, for each m.l.i. \( J \) of \( A \), \( f(J) = 0 \) for some \( f \) in \( M \), one has that \( x \) is not in any m.l.i. Therefore \( x \) is left regular. Thus if \( K \) is in the principal component of the set of all left regular elements of \( A \).

For each \( x \in A \), the set \( \hat{x}(M) = \{f(x) : f \in M\} \) contains the set \( \sigma_J(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not left regular}\} \), which may be called the left spectrum of \( x \), and which, in the commutative case, is the spectrum of \( x \).

Since \( ieK \) and \( -e = (i\alpha)^2 \notin K \), \( K \) is not closed under multiplication; in contrast, Kung-Fu Ng [5] obtains representation theorems for Banach algebras with given cones closed under multiplication.

If \( x \) is in every m.l.i., then \( f(x) = 0 \) for all \( f \) in \( M \), so that, by (2) above, \( x \in (-K) \cap K \). This shows that, if \( K \) is a cone, that is, if \( (0) = (-K) \cap K \), then \( A \) is semisimple.

3. Two seminorms on \( A \) and their properties. Since \( u \) is an order unit for \( A \) and (therefore) for \( A_0 \), the underlying real space of \( A \), the following is an order unit seminorm on \( A_0 \):

\[
\| x \|_u = \inf \{\alpha > 0 : -\alpha u \leq x \leq \alpha u\}.
\]

Since the ordering is Archimedean, this seminorm is a norm on \( A_0 \) if and only if \( A_0 \) is a cone. For the same reason, the \( \| \cdot \|_u \)-unit ball is the order-interval \([-u, u]\). Since \( K \) contains the \( u \)-translate of the unit ball, \( \| x \|_u \leq \| x \| \), so that \([-u, u]\) contains the given unit ball.

Since \( \Omega \) is \( w^* \)-compact, the usual embedding of \( A \) into \( C(\Omega) \) and the restriction of the sup norm to \( A \), yield the following function seminorm on \( A \):

\[
\| x \|_{\omega} = \sup \{|f(x)| : f \in \Omega\}.
\]

Here \( \Omega \) may be replaced by \( M \) or \( S \). This is a norm if and only if \( K' - K' \) is \( w^* \)-dense in \( A' \), which is so (by property (2) of the wedge \( K \)) if and only if \( K \) is a cone. We also have \( \| x \|_{\omega} \leq \| x \|_{\omega} \), for all \( x \) in \( A \). However, unlike the order unit seminorm, the function seminorm is absolutely homogeneous for complex scalars.

**Proposition 1.** On \( A_0 \), \( \| \cdot \|_u \) and \( \| \cdot \|_{\omega} \) are equivalent seminorms.

**Proof.** For all \( f \) in \( \Omega \), \( \text{Re} f(u) = f(e) = 1 \); hence, if \(-\alpha u \leq x \leq \alpha u \) for some \( \alpha > 0 \), then \( \alpha u \pm x \in K \), or, equivalently, \( |\text{Re}(f(x))| \leq \alpha \) and \( |\text{Im}(f(x))| \leq \alpha \). Thus \( |f(x)| \leq \sqrt{2} \alpha \) and

\[
\| x \|_{\omega} \leq \sqrt{2} \| x \|_u.
\]

On the other hand, for all \( f \) in \( \Omega \), \( |\text{Re}(f(x))| \leq |f(x)| \leq \| x \|_{\omega} \); equivalently, \( \text{Re}(\| x \|_{\omega} u) \leq \text{Re}(f(x)) \leq \text{Re}(\| x \|_{\omega} u) \). These inequalities hold with \( \text{Im} \) in place of \( \text{Re} \); hence \( \| x \|_{\omega} u \leq x \leq \| x \|_{\omega} u \), that is,

\[
\| x \|_u \leq \| x \|_{\omega} \cdot u.
\]

The assertion follows.

The next proposition is, essentially, well known. We are indebted to the referee for the proof given here.
**Proposition 2.** Let the wedge $K$ be a cone. Then the following statements are equivalent:

(a) $K$ is normal with respect to the given norm.
(b) $\| \cdot \|$ and $\| \cdot \|_u$ are equivalent norms on $A_0$.
(c) The $\| \cdot \|$ and $\| \cdot \|_u$ duals of $A_0$ are identical.

**Proof.** Since $K$ is a cone and the ordering is Archimedean, $\| \cdot \|_u$ is a norm on $A_0$. Hence, if $K$ is normal, $[-u, u]$ is norm bounded, i.e., $\|x\| \leq m \|x\|_u$ for all $x$ in $A_0$ and for some constant $m > 0$. Since we already have $\|x\|_u \leq \|x\|$, (a) $\Rightarrow$ (b). Conversely, since $\| \cdot \|_u$ is monotonic on $K$, if $\| \cdot \|$ is equivalent to $\| \cdot \|_u$, then $K$ is normal.

That (b) $\Rightarrow$ (c) is evident. Conversely, if the dual spaces are identical, then the dual norms are equivalent (by the closed graph theorem and the fact that there is an inequality one way) and hence, by the Hahn–Banach theorem, the norms on $A_0$ are equivalent.

4. The Representation Theorem.

**Theorem.** Let $A$ be a complex Banach algebra with unit $e$ of norm one and let the sets defined in § 1 be constructed. Then there is a mapping $\phi$ from $A$ into $\mathcal{C}(\Omega)$ such that

(a) $\phi$ is a continuous linear homomorphism from $A$ to a subspace of $\mathcal{C}(\Omega)$;
(b) with respect to the cone $P$ of functions in $\mathcal{C}(\Omega)$ with nonnegative real and imaginary parts, the homomorphism $\phi$ is also an order homomorphism;
(c) the homomorphism $\phi$ is an isomorphism if and only if $K$ is a cone;
(d) if $K$ is a cone, then $\phi$ is a homeomorphism if and only if $K$ is normal.

**Proof.** Let $\phi$ be the evaluation mapping of $A$ into $\mathcal{C}(\Omega)$; i.e., let $\phi(x) = \hat{x}$, where $\hat{x}(f) = f(x)$. Then use the fact that $\| \cdot \|_\Omega \leq \| \cdot \|$ to establish that $\phi$ is continuous and hence that (a) holds.

(b) follows by observing that $x \in K$ if and only if $0 \leq \text{Re} \hat{x}(f)$ and $0 \leq \text{Im} \hat{x}(f)$ for all $f$ in $\Omega$, which is so if and only if $\hat{x} \in P$ (as an element of $\mathcal{C}(\Omega)$).

In § 2, property (2) of $K$ shows that $K$ is a cone if and only if $x = 0$ whenever $\hat{x}(f) = 0$ for all $f \in \Omega$, which is so if and only if $x = 0$ whenever $\hat{x}|_{\Omega} = 0$; (c) follows.

Let $K$ be a cone. From (c), the evaluation mapping is a continuous isomorphism. It is a homeomorphism if and only if the norms $\| \cdot \|$ and $\| \cdot \|_\Omega$ are equivalent, which is so if and only if $\| \cdot \|$ and $\| \cdot \|_u$ are equivalent norms on $A_0$ (since $\| \cdot \|_u$ and $\| \cdot \|_\Omega$ are equivalent on $A_0$, by Proposition 1). By Proposition 2, this is so if and only if $K$ is normal, and (d) follows.

**Remark.** If $A$ is commutative, then $M$ is precisely the set of nonzero multiplicative linear functionals, which is $w^*$-closed, so that $M = \Omega$. The evaluation mapping reduces to the Gelfand transform and we thus have a generalization of Gelfand's representation theorem.

5. Examples. These are drawn from the algebra $E = M_n(\mathbb{C})$ of all $n \times n$ ($n \geq 2$) complex matrices. It can be considered as the space of linear operators on $\mathbb{C}^n$, the space of $n$-tuples of complex numbers.

In what follows, an element of $E$ will be denoted either by its entries or by its column vectors enclosed within square brackets. Thus $x = (a_{ij}) = [b_1, b_2, \ldots, b_n]$ denotes an arbitrary
element $x$ of $E$ with column vectors $b_1, b_2, \ldots, b_n$ and $f = (p_{ij})$ denotes an arbitrary element of $E'$, the algebraic dual of $E$; unless the contrary is stated, the indices range from 1 to $n$.

One can verify that the operator norm on $E$ induced by the maximum modulus norm on $\mathbb{C}^n$, its dual norm, and $f(x)$ are given by

$$
||x|| = \max \left\{ \sum_j |a_{ij}| \right\},
$$

$$
||f|| = \sum_j \max \{ |p_{ij}| \},
$$

$$
f(x) = \sum_j a_{ij} p_{ij}.
$$

With respect to this norm, $E$ is a noncommutative Banach algebra with unit $I$ (the identity matrix) of norm one.

The maximal left ideals (m.l.i.) of $E$ are its principal left ideals generated by elements of rank $(n-1)$ (see, for example, [4], pp. 230–231). These generating elements can be chosen in “row-reduced echelon form” as follows:

$$
x_0 = [0, e_1, \ldots, e_{n-1}],
$$

$$
x_s = \left[ e_1, \ldots, e_s, \sum_{t=s+1} e_t, e_{s+1}, \ldots, e_{n-1} \right],
$$

(1)

where the unit basis vectors $e_t$ are regarded as column vectors, $1 \leq t \leq s \leq n-1$, and $\alpha_t \in \mathbb{C}$.

In Propositions 3, 4, and 5 below, we show that certain properties of the set $M$ which hold in commutative complex algebras are not true in general.

**Proposition 3.** *In general, the set $M$ and the m.l.i.'s are not in one-to-one correspondence.*

**Proof.** The set $B$ (see § 1) for the Banach algebra $E$ is given by

$$
B = \left\{ f = (p_{ij}) \in E' : \max |p_{ij}| \leq p_{ii} \leq 1 = \sum_r p_{rr} \right\}.
$$

The extreme points of $B$ are given by

$$
g_t = [\alpha_1 e_1, \alpha_2 e_2, \ldots, e_i, \ldots, e_n^t],
$$

where $\alpha_i = 1 = |\alpha_t|$ ($t \neq i$).

The assertion follows by proving that each $g_t$ determines and belongs properly to a convex subset of $B$, each element of which annihilates the same m.l.i.

It is enough to consider $g_1$, for which $\alpha_1 = 1$ and $|\alpha_t| = 1$ ($2 \leq t \leq n$). Define $\lambda_1$ by $\lambda_1 = (1/\alpha_n)$, $\lambda_i = \lambda_1 \alpha_1$ ($2 \leq i \leq n-1$), and $\lambda_n = 1$. Thus $|\lambda_i| = 1$ for all $i$. The following are $n$ elements of ext $B$ (the set of extreme points of $B$):

$$
f_r = \lambda_r^{-1} [\lambda_1 e_r, \lambda_2 e_r, \ldots, \lambda_n e_r] \quad (r = 1, 2, \ldots, n).
$$

(2)

The element

$$
x = [e_1, e_2, \ldots, e_{n-1}, -\sum_{j=1}^{n-1} \lambda_j e_j]
$$

(3)
has rank \((n - 1)\), so that the principal left ideal \(J_x\) generated by \(x\) is maximal. An element \(c\) of \(J_x\) is given by
\[
c = [c_1, c_2, \ldots, c_{n-1}, -\sum_{j=1}^{n-1} \lambda_j c_j],
\]
where \(c_j\) are arbitrary vectors of \(C^n\). Regarding \(e_r\) in (2) as linear functionals on \(C^n\), one has
\[
f_r(c) = \lambda_r^{-1} \sum_{j=1}^{n-1} \lambda_j (e_r c_j) - (e_r, \sum_{j=1}^{n-1} \lambda_j c_j) = 0.
\]
Therefore each \(f_r\) and all linear combinations of \(f_1, \ldots, f_n\) annihilate \(J_x\). Conversely, each m.l.i. generated by an element such as \(x\) in (3) above is annihilated by elements such as \(f_r\) in (2) above, and their linear combinations.

The assertion follows.

**Remarks.** (a) In the above example, some m.l.i.'s have unique annihilators in \(B\). The annihilator of \(J_{x_0}\) (see (1) above) is \(f = (p_{ij})\), whose only nonzero entry is \(p_{11} = 1\); that of \(J_{x_0'}\), for \(|\alpha_t| < 1\) for all \(t\), is \(f = (p_{ij})\) such that \(p_{(s+1)t} = \alpha_t (1 \leq t \leq s), p_{(s+1)(s+1)} = 1,\) and \(p_{ij} = 0\) otherwise.

(b) Since \(\text{ext} B \subseteq M\), we have \(S = B, C' = \text{the cone generated by } B\), and \(K' = C' - iC'\). The wedge \(K\) is described by
\[
K = \{x = (a_{ij}) \in E : \sum_{j \neq i}^n |a_{ij}| \leq \min \{\text{Re } a_{ii}, \text{Im } a_{ii}\}\}.
\]
It is clear that \(K\) is a cone. Hence \(\| \cdot \|_u\) and \(\| \cdot \|_\Omega\) are norms. Indeed, the latter is the given norm and
\[
\| x \|_u = \max \left\{ \sum_{j \neq i}^n |a_{ij}| + \max \{\text{Re } a_{ii}, \text{Im } a_{ii}\} : i = 1, 2, \ldots, n \right\}.
\]

Hence the representation of \(E\) on \(\Omega\) is an isometric order isomorphism of \(E\) into \(C(\Omega)\) and \(K\) is a normal, closed, generating cone.

**Proposition 4.** In general, the set \(M\) depends on the norm.

**Proof.** For \(M_2(C)\) with the above norm, it is readily verified that \(f_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\) annihilates the principal left ideal generated by \(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\) and is in \(M\). If, now, we consider \(M_2(C)\) with the \(l_1\)-operator norm, \(f_0\) is no longer of norm 1 in the dual space and so is not in the new set \(M\). Conversely, \(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\) is in the new set but not in the former one.

**Remark.** This example corrects the oversight in [6] by which it was assumed that \(M\) is independent of all equivalent Banach algebra norms \(p\) such that \(p(e) = 1\). This oversight made it appear that \(K\) is a cone if and only if \(A\) has no nonzero topologically nilpotent elements. Clearly, \(E = M_n(C)\) has such elements and we have seen that the wedge \(K\) (above) is a cone.
PROPOSITION 5. *In general, the set $M$ is not w*-closed.*

*Proof.* We show that $M$ can even be dense in $B$. Let $E$ be the noncommutative algebra of all $2 \times 2$ upper triangular complex matrices. This is a Banach algebra for the $l_{\infty}$-operator norm. The m.l.i.'s are of the following form:

$J_1$: with elements of the form

$$
\begin{bmatrix}
0 & b \\
0 & c
\end{bmatrix}
$$

$J_{2k}$: with elements of the form

$$
\begin{bmatrix}
a & -ka \\
0 & 0
\end{bmatrix} (k \in \mathbb{C}).
$$

The annihilators of these in $B$ are respectively given by

$$f_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad g_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (|k| < 1)$$

and

$$\text{co} \left\{ \begin{bmatrix} k & 1/k \\ 0 & 0 \end{bmatrix}, g_0 \right\} (|k| \geq 1).$$

Therefore no point of $\text{int} \text{ co} \{f_0, g_0\}$ annihilates either $J_1$ or $J_{2k}$. The assertion follows, as $M$ is the complement of this interior in $B$.

We observe that throughout the preceding discussion maximal left ideals can be replaced by maximal right ideals (m.r.i.). Corresponding to these two cases, we now let the sets $M$ be denoted by $M_l$ and $M_r$, respectively.

PROPOSITION 6. (a) *In general, $M_l \neq M_r$;* (b) however, there exist unital noncommutative Banach algebras for which $M_l = M_r$.

*Remark.* If $M_l = M_r$, then, whether or not the algebra is commutative, the spectrum of each element $x$ of the algebra is contained in the set $\{f(x) : f \in M\}$ and $\text{int} K$ is contained in the principal component of the set of all regular elements.

*Proof.* (a) This follows from the previous example, whose m.r.i.'s are $I_1$ and $I_{2k}$ with elements respectively of the forms

$$\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \text{ and } \begin{bmatrix} 0 & a \\ 0 & -ka \end{bmatrix} (a, k \in \mathbb{C}).$$

Let $f = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \in B$. If $\gamma = 0$, then $f$ annihilates $I_1$; if $\gamma \neq 0$, then it annihilates $I_{2k}$ with $k = \beta \gamma^{-1}$. Hence $M_r = B$, which properly contains $M_l$.

(b) This is true for $M_2(\mathbb{C})$ with the $l_{\infty}$-operator norm. For its m.r.i.'s are $I_1$ and $I_{2k}$ with elements respectively of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix} (a, b, k \in \mathbb{C}).$$
The annihilators in $B$ of these ideals are respectively of the forms

$$(I_1): \begin{bmatrix} 1 & q \\ 0 & 0 \end{bmatrix} (|q| \leq 1);$$

$$(I_{20}): \begin{bmatrix} 0 & 0 \\ r & 1 \end{bmatrix} (|r| \leq 1);$$

$$(I_{2k}): \begin{bmatrix} p & q \\ r & s \end{bmatrix}_{(k \neq 0)}$$

where $p = \frac{|k|}{|k'|}$, $r = \frac{|k|}{kk'}$, $q = k/k'$, $s = 1/k'$, $k' = 1 + |k|$. $I_{2k}$ ($k \neq 0$) have, unlike $I_1$ and $I_{20}$, unique annihilators.

Therefore

$$M_r = \bigcup M_i (1 \leq i \leq 3),$$

where

$$M_1 = \text{co} \left\{ \begin{bmatrix} 1 & q \\ 0 & 0 \end{bmatrix} : |q| = 1 \right\},$$

$$M_2 = \text{co} \left\{ \begin{bmatrix} 0 & 0 \\ r & 1 \end{bmatrix} : |r| = 1 \right\},$$

$$M_3 = \bigcup_{|k| = 1} M_{3k},$$

with

$$M_{3k} = \text{int co} \left\{ \begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ k & 1 \end{bmatrix} : |k| = 1 \right\}$$

($\kappa$ is the conjugate of $k$).

$M_1$ can be computed from the general case discussed above and shown to be equal to $M_r$.

REFERENCES


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