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ITERATIVE SOLUTION OF NONLINEAR EQUATIONS OF THE MONOTONE TYPE IN BANACH SPACES

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Let E be a real Banach space with a uniformly convex dual, and let K be a nonempty closed convex and bounded subset of E. Suppose $T\colon K \to K$ is a continuous monotone map. Define $S\colon K \to K$ by $Sx = f - Tx$ for each x in K and define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 \in K$, $x_{n+1} = (1 - C_n)x_n + C_n Sx_n$, $n \geqslant 0$, where $\{C_n\}_{n=0}^{\infty}$ is a real sequence satisfying appropriate conditions. Then, for any given f in K, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of $x + Tx = f$ in K. Explicit error estimates are also computed. A related result deals with iterative solution of nonlinear equations of the dissipative type.

1. INTRODUCTION

Let E be an arbitrary real Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *monotone* [10] if the inequality

$$\|x - y\| \leqslant \|x - y + t(Tx - Ty)\|, \tag{1}$$

holds for each x, y in $D(T)$ and some $t > 0$. If the inequality (1) holds for *all* $t > 0$, then T is called *accretive* [2]. The accretive operators were introduced independently by Browder [2] and Kato [10]. If $E = H$, a Hilbert space, one of the earliest problems in the theory of monotone operators was to solve the equation $x + Tx = f$ for x, given an element f of H and a monotone operator T (see, for example, [4, 6, 7, 9, 14, 15, 23, 24]). In [2], Browder proved that if T is locally Lipschitzian and accretive then T is *m-accretive*, that is, $(I + T)E = E$, so that for any given $f \in E$, the equation $x + Tx = f$ has a solution. This result was subsequently generalised by Martin [13] to the continuous accretive operators. Zarantonello [24] also proved that if H is a Hilbert space and T is a monotone and Lipschitzian mapping of H into itself then $x + Tx = f$ has a *unique* solution in H.

A class of operators closely related to the class of accretive operators is the class of *dissipative* operators. An operator A is dissipative if and only if $(-A)$ is accretive and A on E is called *m-dissipative* if $(I - \lambda A)E = E$ for $\lambda > 0$. Browder [2] proved the following theorem:

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Theorem B. *Let A be a single-valued dissipative operator on E which is locally Lipschitzian on $E = D(A)$. Then A is m-dissipative.*

Recently, Ray [**16**] gave an elementary proof of Theorem B by employing a fixed point theorem of Caristi, [**5**].

Iterative methods for approximating a solution (when one exists) of the equation

$$x + Tx = f \tag{2}$$

have been studied by various authors. In [**9**], Dotson showed that if $T\colon H \to H$ is monotone and Lipschitzian with Lipschitz constant 1 (in this case T is called *nonexpansive* in the terminology of [**11**]), an iteration process of the type introduced by Mann [**12**], under suitable conditions, converges strongly to the unique solution of equation (2). In [**6**], the author constructed an approximation method which converges strongly to a solution of the equation (2) where $T\colon K \to H$ is a monotone Lipschitzian operator with Lipschitz constant $L \geqslant 1$ and K is a nonempty closed convex subset of H and T has a fixed point in K. The results of the author [**6**] thus generalise the above result of Dotson both in the domain of the operator T and in the range of its Lipschitz constant. In [**4**], Bruck Jr., considered an iteration process, in Hilbert space, for approximating a solution of the equation $f \in x + Tx$ where $T\colon H \to H$ is a multivalued monotone operator. He proved that if $D(T)$ is an open domain of T in H and $f \in R(I+T)$ then there exist a neighbourhood N in $D(T)$ of $\overline{x} = (I+T)^{-1}f$ and a real number $\sigma_1 > 0$ such that for any $\sigma \geqslant \sigma_1$, any initial guess $x_1 \in N$ and any single-valued section T_0 of T, the sequence generated from x_1 by $x_{n+1} = x_n - (n+\sigma)^{-1}(x_n + T_0 x_n - f)$ remains in $D(T)$ and converges to $\overline{x}$ with estimate $\|x_n - \overline{x}\| = O(n^{-1/2})$. No continuity assumption was made on the map T. This result has recently been extended by the author [**7**] to L_p spaces for $2 \leqslant p < \infty$. The method used in [**7**] could not be modified to yield any convergence result for L_p spaces when $1 < p < 2$. More recently, the author [**8**] again considered the equation (2) in L_p spaces for $2 \leqslant p < \infty$ when T is single-valued. Suppose $E = L_p$ (or ℓ_p), $p \geqslant 2$, and K is a nonempty closed convex subset of E. Suppose $T\colon K \to K$ is a monotone Lipschitzian mapping with Lipschitz constant $L \geqslant 1$. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by $x_0 \in K$, $x_{n+1} = (1-\lambda)x_n + \lambda(f - Tx_n)$ for $n \geqslant 0$, where $\lambda = [(p-1)(1+L)^2]^{-1}$. The author [**8**] proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of equation (2). Moreover, convergence is at least as fast as a geometric progression with ratio $(1-\lambda)^{1/2}$. Unfortunately, no convergence theorem could be proved in [**8**] for L_p spaces with $1 < p < 2$.

It is our purpose in this paper to prove convergence theorems for an iterative method for approximating a solution of the equation

$$x + Tx = f \tag{3}$$

in real Banach spaces much more general than L_p spaces for $2 \leqslant p < \infty$. In particular, our results will include all L_p (and ℓ_p) spaces for $1 < p < \infty$. In fact, we shall prove that if E is any real Banach space with a uniformly convex dual E^*, and if $T: K \to K$ is a *continuous* monotone map where K is a nonempty closed convex and bounded subset of E, an iteration process of the Mann-type converges strongly to a solution of equation (3) for any starting point x_0 in K. We shall also prove some related convergence theorems for the iterative approximation of a solution of the equation

$$x - \lambda Ax = f, \qquad \lambda > 0, \tag{4}$$

where A is a Lipschitzian dissipative operator on E.

2. Preliminaries

Let E be a real Banach space and E^* its dual. We shall denote by J the normalised duality mapping from E to 2^{E^*} given by

$$Jx = \{f^* \in E^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^*\rangle\},$$

where $\langle\,,\,\rangle$ denotes the generalised duality pairing. It is well known that if E^* is uniformly convex then J is single-valued and is uniformly continuous on bounded sets (see for example, [**22**]). Thus, by a single-valued duality mapping we shall mean a mapping $j: E \to E^*$ such that for each $x \in E$, $j(x)$ is an element of E^* which satisfies the following two conditions:

$$\langle u, j(u)\rangle = \|j(u)\| . \|u\|, \quad \|j(u)\| = \|u\| .$$

In terms of the single-valued duality mapping j, a map A with domain $D(A)$ and range $R(A)$ in a real Banach space E is called *monotone* if

$$\langle Ax - Ay, j(x-y)\rangle \geqslant 0, \tag{5}$$

for all x, y in $D(A)$, (see for example, [**10**]). In the sequel we shall need the following remarks:

REMARK 1. In [**21**, p.89], Reich proved that if E^* is uniformly convex then there exists a continuous nondecreasing function

$$b: [0, \infty) \to [0, \infty)$$

such that

$$b(o) = 0, \quad b(ct) \leqslant cb(t) \text{ for all } c \geqslant 1,$$

and

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\,\|y\|\, b(\|y\|), \tag{6}$$

for all x, y in E.

REMARK 2. Nevanlinna and Reich [**17**] have shown that for any given continuous non-decreasing function $b(t)$ with $b(o) = 0$, sequences $\{\lambda_n\}_{n=0}^{\infty}$ always exist satisfying:

(i) $0 < \lambda_n < 1$ for all $n \geqslant 0$;

(ii) $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$, and

(iii) $\sum\limits_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$.

If $E = L_p$, $1 < p < \infty$, we can choose any sequence $\{\lambda_n\}_{n=0}^{\infty}$ in $\ell^s \setminus \ell^1$ with $s = p$ if $1 < p \leqslant 2$ and, $s = 2$, if $p \geqslant 2$.

3. MAIN RESULTS

We prove the following theorems:

THEOREM 1. *Let E be a real Banach space with a uniformly convex dual space, E^*, and let K be a nonempty closed convex and bounded subset of E. Suppose $T: K \to K$ is a continuous monotone map. Define $S: K \to K$ by $Sx = f - Tx$ for each x in K. Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 \in K$,*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n S x_n, \tag{7}$$

for $n \geqslant 0$, where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying:

(i) $0 < \lambda_n < 1$ *for all* $n \geqslant 0$;

(ii) $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$, *and*

(iii) $\sum\limits_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$.

Then, for any given $f \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of $x + Tx = f$ in K.

PROOF: The existence of a solution of $x + Tx = f$ follows from [**13**]. Let q denote a solution of this equation. Observe that q is a fixed point of S. Moreover, the montonicity of T implies,

$$\langle Sx - Sy, j(x - y)\rangle = -\langle Tx - Ty, j(x - y)\rangle \leqslant 0, \tag{8}$$

for each x, y in K. Furthermore, using equation (7) and inequalities (6) and (8), we

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obtain:

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\lambda_n)(x_n-q)+\lambda_n(Sx_n-Sq)\|^2 \\
&\leqslant (1-\lambda_n)^2\|x_n-q\|^2+2\lambda_n(1-\lambda_n)\langle Sx_n-Sq, j(x_n-q)\rangle \\
&\quad+\max\{(1-\lambda_n)\|x_n-q\|,1\}\lambda_n\|Sx_n-Sq\|\,b(\lambda_n\|Sx_n-Sq\|) \\
&\leqslant (1-\lambda_n)^2\|x_n-q\|^2 \\
&\quad+\max\{(1-\lambda_n)\|x_n-q\|,1\}\lambda_n\|Sx_n-Sq\|\max\{\|Sx_n-Sq\|,1\}b(\lambda_n) \\
&\leqslant (1-\lambda_n)\|x_n-q\|^2+M\lambda_n b(\lambda_n), \quad \text{for some } M>0,
\end{aligned}$$

since K is bounded. Hence we have:

(9) $$\|x_{n+1}-q\|^2 \leqslant (1-\lambda_n)\|x_n-q\|^2+M\lambda_n b(\lambda_n).$$

Set $\rho_n=|x_n-q\|^2$ and $M\lambda_n b(\lambda_n)=\sigma_n$. Then, inequality (9) becomes:

(10) $$\rho_{n+1}\leqslant(1-\lambda_n)\rho_n+\sigma_n.$$

A simple induction on inequality (10) yields:

(11) $$0\leqslant\rho_{n+1}\leqslant\prod_{j=1}^{n}(1-\lambda_j)\rho_1+\beta_{n+1},$$

where

$$\beta_{n+1}=\begin{cases}\sigma_1, & n=1\\ \sigma_n+\sum_{i=1}^{n-1}\sigma_i\prod_{j=i+1}^{n}(1-\lambda_j), & n>1.\end{cases}$$

For any fixed integer k, with $1<k\leqslant n-1$, we obtain:

$$\beta_{n+1}=\sigma_n+\sum_{i=1}^{k}\sigma_i\prod_{j=i+1}^{n}(1-\lambda_j)+\sum_{i=k+1}^{n-1}\sigma_i\prod_{j=i+1}^{n}(1-\lambda_j).$$

Since $\delta_n\in[0,1]$, the above equation yields,

$$0\leqslant\beta_{n+1}\leqslant\left(\sum_{i=1}^{k}\sigma_i\right)\prod_{j=k+1}^{n}(1-\lambda_j)+\sum_{i=k+1}^{n}\sigma_i.$$

Condition (ii) implies $\lim_{n\to\infty}\prod_{j=k+1}^{n}(1-\lambda_j)=0$, $k\geqslant1$, so that

(12) $$0\leqslant\lim_{n\to\infty}\inf\beta_n\leqslant\lim_{n\to\infty}\sup\beta_n\leqslant\sum_{i=k+1}^{\infty}\sigma_i.$$

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Since inequaltiy (12) holds for arbitrary $k \geqslant 1$, and since condition (iii) implies $\lim_{k\to\infty} \sum_{i=k+1}^{\infty} \sigma_i = 0$, it follows that

$$\liminf_{n\to\infty} \beta_n = \limsup_{n\to\infty} \beta_n = \lim_{n\to\infty} \beta_n = 0. \tag{13}$$

From inequality (11) and equation (13) we obtain that $\rho_n \to 0$ as $n \to \infty$ so that $x_n \to q$ as $n \to \infty$. ∎

COROLLARY 1. *Let $E = L_p$ (or ℓ_p), $1 < p \leqslant 2$, and let K, T and S be as in Theorem 1. Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n S x_n,$$

for $n \geqslant 0$, where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying:

(i) $0 < \lambda_n < 1$ *for all* $n \geqslant 0$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$; *and*

(iii) $\sum_{n=0}^{\infty} \lambda_n^p < \infty$.

Then for any given f in K the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of $x + Tx = f$ in K.

PROOF: The existence of a solution to $x+Tx = f$ follows from [**13**]. Remark 2 and conditions (ii) and (iii) imply $\sum \lambda_n b(\lambda_n) < \infty$. The result follows from Theorem 1. ∎

COROLLARY 2. *Let $E = L_p$ (or ℓ_p), $2 \leqslant p < \infty$, and let K, T and S be as in Theorem 1. Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n S x_n,$$

for $n \geqslant 0$, where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying:

(i) $0 < \lambda_n < 1$ *for all* $n \geqslant 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$, *and*

(iii) $\sum_{n=0} \lambda_n^2 < \infty$.

Then for any given $f \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of $x + Tx = f$ in K.

PROOF: The proof follows exactly as in the proof of Corollary 1. ∎

REMARK 3. The only use we have made of the continuity of T in Theorem 1 and Corollaries 1 and 2 above is to obtain the *existence* of a solution to the equation $x + Tx = f$.

ERROR ESTIMATES: For the error estimates we shall need the following definition and lemma:

DEFINITION: The *modulus of convexity* of a real Banach space E is the function

$$\delta_E\colon [0, 2] \to [0, 1]$$

defined by the following formula:

$$\delta_E(\varepsilon) = \inf\left[1 - \|(x+y)/2\| : x, y \in E, \|x\| = 1 = \|y\|, \|x - y\| \geqslant \varepsilon\right].$$

LEMMA. Reich, [18]. *If $\delta_{E^*}(\varepsilon) \geqslant K\varepsilon^r$ for some $K > 0$ and $r \geqslant 2$ then for $t \leqslant M$, $b(t) \leqslant ct^{s-1}$ with $s = r/(r-1)$. If $E = L_p$, $1 \leqslant p < \infty$, then $s = p$ if $1 < p \leqslant 2$; and $s = 2$ if $2 \leqslant p < \infty$.*

The above lemma enables us to obtain a convergence rate in the setting of Theorem 1. If we set $\lambda_n = s/(n+1)$ then $\|x_n - q\| = O\left(n^{(-(s-1))/2}\right)$. To see this, observe that by the above lemma and inequality (9) we have:

$$\|x_{n+1} - q\|^2 \leqslant \left(1 - \frac{s}{n+1}\right)\|x_n - q\|^2 + M\left(\frac{s}{n+1}\right)c\left(\frac{s}{n+1}\right)^{s-1}$$

so that

$$\|x_{n+1} - q\|^2 \leqslant \left(1 - \frac{s}{n+1}\right)\|x_n - q\|^2 + M^*\frac{1}{(n+1)^s}, \tag{14}$$

for some constant $M^* > 0$. Let $M_2 = \max\{\|x_1 - q\|, M^*\}$. Then clearly, $\|x_1 - q\| \leqslant M_2$

CLAIM:

$$\|x_n - q\|^2 \leqslant M_2\frac{1}{n^{s-1}}.$$

Proof of this claim is by induction. For $n = 1$, the claim follows from the definition of M_2. Assume (15) holds for $n = k$. Then from (14),

$$\begin{aligned}\|x_{k+1} - q\|^2 &\leqslant \left(1 - \frac{s}{k+1}\right)\|x_k - q\|^2 + M^*\frac{1}{(k+1)^s}\\ &\leqslant \left(1 - \frac{1}{k+1}\right)^s M_2\frac{1}{k^{s-1}} + M^*\frac{1}{(k+1)^s}\\ &\leqslant \left(1 - \frac{1}{k+1}\right)^s M_2\frac{1}{k^{s-1}} + M_2\frac{1}{(k+1)^s}\\ &= M_2\frac{1}{(k+1)^{s-1}},\end{aligned}$$

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and so, by induction, (15) holds for all positive integers, and establishes that

$$\|x_n - q\| = 0\left(n^{-(s-1)/2}\right).$$

In particular, if $E = L_p$, $1 < p < \infty$, we have:

$$\|x_n - q\| = 0\left(n^{-(p-1)/2}\right) \text{ if } 1 < p \leqslant 2$$

and,

$$\|x_n - q\| = 0\left(n^{-1/2}\right) \qquad \text{if } 2 \leqslant p < \infty.$$

We now consider iterative methods for single-valued dissipative and Lipschitz maps, and prove the following theorem:

THEOREM 2. *Let E be a real Banach space with a uniformly convex dual space, E^*. Let K be a nonempty closed convex bounded subset of E. Suppose $A\colon K \to K$ is a single-valued dissipative and Lipschitzian mapping of K into itself with Lipschitz constant $L \geqslant 1$. Define $G\colon K \to K$ by $Gx = \lambda Ax + f$ for arbitrary $x \in K$ and fixed f in K. Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 \in K$,*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Gx_n,\ n \geqslant 0,$$

where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying:

(i) $0 < \lambda_n < 1$ *for all* $n \geqslant 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$, *and*

(iii) $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of the equation

$$x - \lambda Ax = f \text{ in } K. \tag{16}$$

PROOF: The existence of a solution to equation (16) follows from Theorem B. Let q denote a solution. Observe that q is a fixed point of G. Moreover, since A is dissipative we have

$$\langle Gx_n - Gq, j(x_n - q)\rangle = \lambda\langle Ax_n - Aq, j(x_n - q)\rangle \leqslant 0.$$

So,

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\lambda_n)(x_n - q) + \lambda_n(Gx_n - Gq)\|^2 \\
&\leqslant (1-\lambda_n)^2 \|x_n - q\|^2 + 2\lambda_n(1-\lambda_n)\langle Gx_n - Gq, j(x_n - q)\rangle \\
&\quad + \max\{(1-\lambda_n)\|x_n - q\|, 1\}\lambda_n \|Gx_n - Gq\|\, b(\lambda_n \|Gx_n - Gq\|) \\
&\leqslant (1-\lambda_n)^2 \|x_n - q\|^2 \\
&\quad + \max\{(1-\lambda_n)\|x_n - q\|, 1\}\lambda_n \|Gx_n - Gq\| \max\{\|Gx_n - Gq\|, 1\} b(\lambda_n) \\
&\leqslant (1-\lambda_n)\|x_n - q\|^2 + M\lambda_n b(\lambda_n), \qquad \text{for some constant } M > 0.
\end{aligned}$$

Now, set $\rho_n = \|x_n - q\|^2$, so that the above inequality becomes

$$\rho_{n+1} \leqslant (1-\lambda_n)\rho_n + \sigma_n$$

which is exactly inequality (10). The rest of the argument now follows exactly as in the proof of Theorem 1 to yield that $x_n \to q$ as $n \to \infty$, completing the proof of Theorem 2. ∎

COROLLARY 1. *Let $E = L_p$, $1 < p \leqslant 2$ and let K, A and G be as in Theorem 2. Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n G x_n, \quad n \geqslant 0$$

where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying:

(i) $0 < \lambda_n < 1$, *for all* n,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$, *and*

(iii) $\sum_{n=0}^{\infty} \lambda_n^p < \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of the equation $x - \lambda A x = f$ in K.

PROOF: Observe that conditions (ii) and (iii) imply that $\{\lambda_n\}_{n=0}^{\infty} \in \ell^p \setminus \ell^i$ and so by Remark 2, $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$. Corollary 1 then follows from Theorem 2. ∎

COROLLARY 2. *Let $E = L_p$, $2 \leqslant p < \infty$, and let K, A and G be as in Theorem 2. Define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n G x_n, \quad n \geqslant 0,$$

where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying:

(i) $0 < \lambda_n < 1$ *for all* n,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$, *and*

(iii) $\sum_{n=0}^{\infty} \lambda_n^2 < \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of the equation $x - \lambda A x = f$ in K.

PROOF: Since $\{\lambda_n\}_{n=0}^{\infty} \in \ell^2 \setminus \ell^1$, Remark 2 implies $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$, and the Corollary follows from Theorem 2. ∎

REMARK 4. The error estimates for Theorem 2 and Corollaries 1 and 2 of this theorem are obtained as in Theorem 1 (and Corollaries 1 and 2 of Theorem 1).

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REMARK 5. Corollaries 1 and 2 of Theorem 1 generalise the main result of Dotson [9] and of the author [6] in several ways. In both [9] and [6] convergence theorems were proved in Hilbert spaces and the operator T was assumed to be monotone and *Lipschitzian*. In Corollaries 1 and 2 of Theorem 1 above, convergence of the iteration scheme is proved for the much larger class of Banach spaces L_p spaces, $1 < p < \infty$, and also for the much larger class of monotone *continuous* maps. Moreover, the rate of convergence established in Corollaries 1 and 2 of Theorem 1 above agrees with that established in both [6] and [9].

REMARK 6. It is a consequence of the proof of Theorem 1 that, under the hypotheses of the theorem, the solution of the given equation must be unique. The element $q \in F(S)$, where $F(S)$ denotes the set of fixed points of S, was arbitrarily chosen. Suppose now there is a $q^* \in F(S)$ with $q^* \neq q$. Repeating the argument of the theorem relative to q^*, one sees that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to both q and q^*, showing that $F(S) = \{q\}$.

A similar argument shows that the solution of the equation in *Theorem 2* is *unique.*

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