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# ON QUASI-ESSENTIAL SUBGROUPS OF PRIMARY ABELIAN GROUPS

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All groups considered in this paper are abelian. A subgroup $N$ of a group $G$ is said to be a quasi-essential subgroup of $G$ if $G = \langle H, K \rangle$ whenever $H$ is an $N$-high subgroup of $G$ and $K$ is a pure subgroup of $G$ containing $N$. We started the study of such subgroups in [**5**]; in particular, we characterized subsocles of a primary group which were both quasi-essential and centres of purity. In this paper we show that quasi-essential subsocles of a primary group are necessarily centres of purity answering thus in the affirmative a question raised in [**5**].

We obtain the following theorem: *A subsocle $S$ of a $p$-group $G$ is quasi-essential if and only if either $S \subset G^1$ or $(p^nG)[p] \supset S \supset (p^{n+1}G)[p]$ for some non-negative integer $n$.* The notation is that of [**1**]. If $G$ is a group, then

$$G^1 = \bigcap_{n=1}^{\infty} nG \qquad \text{and} \qquad p^{\omega}G = \bigcap_{n=1}^{\infty} p^nG,$$

where $p$ is a prime integer. $G_t$ is the torsion subgroup of $G$. For $p$-groups we have $G^1 = p^{\omega}G$. Let $N$ be a subgroup of $G$. If all $N$-high subgroups of $G$ are pure, we say that $N$ is a centre of purity. All topological references are to the $p$-adic topology.

**1. The algebra of $N$-high subgroups of a group.** In this section we list without proof some general facts about $N$-high subgroups of a group. These properties are of a set-theoretic nature and can be visualized by appropriate Venn diagrams. Group-theoretic properties of $N$-high subgroups such as neatness, purity, etc., can be found in the papers listed in the references (see, e.g. [**4**; **6**]).

*Definition* 1.1. Let $G$ be a group and $N$ a subgroup of $G$. A subgroup $H$ of $G$ is said to be an $N$-high subgroup of $G$ if it is a maximal subgroup of $G$ with respect to the property $H \cap N = 0$.

LEMMA 1.2. *Let $G$ be a group and $M \subset N$ subgroups of $G$. Let $H$ be an $M$-high subgroup of $G$. Then $H \cap N$ is an $M$-high subgroup of $N$. Conversely, let $K$ be an $M$-high subgroup of $N$; then $K = H \cap N$ whenever $H$ is an $M$-high subgroup of $G$ containing $K$.*

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LEMMA 1.3. *Let $G$ be a group and $M \subset N$ subgroups of $G$. Let $H$ be an $M$-high subgroup of $G$. Then every $(H \cap N)$-high subgroup of $H$ is an $N$-high subgroup of $G$.*

LEMMA 1.4. *Let $G$ be a group, $M \subset N$ subgroups of $G$, and $K$ an $M$-high subgroup of $N$. Then a subgroup $H$ containing $K$ is an $M$-high subgroup of $G$ if and only if $H/K$ is an $(N/K)$-high subgroup of $G/K$.*

LEMMA 1.5. *Let $G$ be a group and let $N$ and $K$ be subgroups of $G$ such that $N \cap K = 0$. Then a subgroup $H$ containing $K$ is an $N$-high subgroup of $G$ if and only if $H/K$ is an $((N \oplus K)/K)$-high subgroup of $G/K$.*

**2. Some results on pure subgroups of a group.** We prove here a few simple lemmas which might be of independent interest and will be used throughout the paper.

LEMMA 2.1. *Let $G$ be a group, $R \subset S \subset T$ subgroups of $G$, where $S$ is a pure subgroup of $G$. Then $T/R$ is a pure subgroup of $G/R$ if and only if $T$ is a pure subgroup of $G$.*

*Proof.* Let $f: G/R \to G/S$ be the canonical map. Then $f(T/R) = T/S$ and $\ker f \subset T/R$, and therefore $T/S$ is a pure subgroup of $G/S$. Since $S$ is a pure subgroup of $G$, so is $T$. The converse is obvious.

COROLLARY. *Let $G$ be a group and $R$ a subgroup of $G_t$. Suppose that $K/R$ is a pure subgroup of $G/R$. Then $\langle K, G_t\rangle$ is a pure subgroup of $G$.*

*Proof.* Since $G_t/R$ is the torsion subgroup of $G/R$, it follows that $\langle (K/R), (G_t/R)\rangle = \langle K, G_t\rangle/R$ is a pure subgroup of $G/R$ (see [**1**, p. 94]). Letting $G_t = S$ and $\langle K, G_t\rangle = T$ in Lemma 2.1, the result follows.

A somewhat surprising and very useful fact is the following.

LEMMA 2.2. *Let $G$ be a group, $H$ a subgroup of $G$, and $B$ a pure subgroup of $H$ such that $G/B = (H/B) \oplus (K/B)$. Then $K$ is a pure subgroup of $G$.*

*Proof.* Suppose that $ng \in K$ where $g \in G$ and $n \in Z^+$. We have $g + B = (h + B) + (k + B)$ for some $h \in H$ and $k \in K$, and for some $b \in B$, $g - h - k = b$ and thus $nh = ng - nk - nb \in H \cap K = B$. Since $B$ is a pure subgroup of $H$, there exists $b' \in B$ such that $nh = nb'$, therefore $ng = n(b' + b + k)$ and $b' + b + k \in K$, and $K$ is a pure subgroup of $G$ as claimed.

COROLLARY. *Let $G$ be a $p$-group, $H$ a subgroup of $G$, and $B$ a pure dense subgroup of $H$. Then there exists a pure dense subgroup $K$ of $G$ such that $K \cap H = B$.*

*Proof.* $G/B = (H/B) \oplus (K/B)$ since $H/B$ is divisible. $K$ is pure by Lemma 2.2 and $G/K \simeq H/B$. Note that $K$ can be chosen to contain any subgroup of $G$ disjoint from $H$.

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This last corollary generalizes a result in [**2**] about cobounded subgroups of primary groups. The next lemma is a very useful one; it has numerous applications.

LEMMA 2.3. *Let $G$ be a group and $S$ a subgroup of $G[p]$ such that $S \not\subset p^\omega G$. Let $K$ be a maximal pure subgroup of $G$ such that $K[p] \subset S$. Then $\langle S, K\rangle/K$ is contained in the reduced part of $p^\omega(G/K)$.*

*Proof.* Since $S$ has an element of finite $p$-height, there exists at least one pure subgroup $K$ of $G$ such that $K[p] \subset S$ and by Zorn's Lemma there exists a maximal one. Clearly $\langle S, K\rangle/K \subset (G/K)[p]$ and if it had an element of finite $p$-height, $G/K$ would have a summand $K'/K$ such that

$$(K'/K)[p] \subset \langle S, K\rangle/K$$

but this implies that $K'[p] \subset S$, and since $K'$ is a pure subgroup of $G$ this would contradict the maximality of $K$. Therefore $\langle S, K\rangle/K \subset \bigcap p^\omega(G/K)$. Using the absolute summand property of divisible groups, a similar argument shows that the subgroup in question is actually contained in the reduced part of $p^\omega(G/K)$.

**3. Quasi-essential subgroups of primary groups.** Our aim in this section is to answer in the affirmative a question raised in [**5**], namely: Are quasi-essential subsocles of a $p$-group $G$ also centres of purity of $G$?

*Definition* 3.1. Let $G$ be a group and $N$ a subgroup of $G$. We say that $N$ is a quasi-essential subgroup of $G$ (q.-e.) if $G = \langle H, K\rangle$ whenever $H$ is an $N$-high subgroup of $G$ and $K$ is a pure subgroup of $G$ containing $N$.

We next establish a characterization of quasi-essential subgroups which is needed in the proof of later results.

LEMMA 3.2. *Let $G$ be a group and $N$ a subgroup of $G$. Then $N$ is a quasi-essential subgroup of $G$ if and only if $K/H$ is an absolute summand of $G/H$ whenever $K$ is a pure subgroup of $G$ containing $N$ and $H$ is an $N$-high subgroup of $K$.*

*Proof.* Let $M/H$ be a $(K/H)$-high subgroup of $G/H$. Then by Lemma 1.4, $M$ is an $N$-high subgroup of $G$ and if $N$ is q.-e., we have $\langle M, K\rangle = G$; therefore $G/H = K/H \oplus M/H$. Conversely, let $M$ be an $N$-high subgroup of $G$. Then $M \cap K$ is an $N$-high subgroup of $K$ by Lemma 1.2. Thus $K/(M \cap K)$ is an absolute summand of $G/(M \cap K)$ and by Lemma 1.4, $M/(M \cap K)$ is a $(K/(M \cap K))$-high subgroup of $G/(M \cap K)$; therefore $G/(M \cap K) = (M/(M \cap K)) \oplus (K/(M \cap K))$ and $G = \langle M, K\rangle$.

The following lemma is needed for the next theorem, which has some topological implications.

LEMMA 3.3. *Let $G$ be a group and $N$ a subgroup of $G$ such that*
(*) *$G/H$ is divisible for all $N$-high subgroups $H$ of $G$. Then every subgroup of $N$ has the property* (*).

*Proof.* Let $M$ be a subgroup of $N$ and let $H$ be an $M$-high subgroup of $G$. By Lemma 1.3, $H$ contains an $N$-high subgroup of $G$, therefore $G/H$ is divisible being a homomorphic image of a divisible group.

The next theorem is useful to solve the question on quasi-essential subsocles of primary groups, although it is stated for arbitrary groups.

THEOREM 3.4. *Let $G$ be a group and $N$ a subgroup of $G$ such that $G/H$ is divisible for every $N$-high subgroup $H$ of $G$. Then $N[p] \subset p^\omega G$.*

*Proof.* Suppose that $x \in N[p]$; then by Lemma 3.3, $\langle x \rangle$ has the same property as $N$. If $x \notin p^\omega G$, then there exists a summand $K$ of $G$ such that $G = K \oplus H$ and $K[p] = \langle x \rangle$ and clearly $H$ is an $\langle x \rangle$-high subgroup of $G$ and $G/H$ is not divisible, contrary to the hypothesis. Therefore $x \in p^\omega G$ and $N[p] \subset p^\omega G$.

COROLLARY. *Let $G$ be a $p$-group and $N$ a subgroup of $G$. Then $G/H$ is divisible for every $N$-high subgroup $H$ if and only if $N[p] \subset p^\omega G$.*

THEOREM 3.5. *Let $G$ be a $p$-group and $N$ a quasi-essential subgroup of $G$ where $N[p] \not\subset p^\omega G$. Then every pure subgroup $K$ of $G$ containing $N$ is a cobounded summand of $G$.*

*Proof.* Let $K$ be a pure subgroup of $G$ containing $N$. Then by Lemma 3.2, $K/H$ is an absolute summand of $G/H$ for every $N$-high subgroup $H$ of $K$. Since $N[p] \not\subset p^\omega G$, Theorem 3.4 implies that $K/H$ is not divisible for some $N$-high subgroup $H$ of $K$, because $p^\omega K \subset p^\omega G$. Now by [**5**, Theorem 4.4], there exists $n \in Z^+$ such that

$$p^{n+1}(G/H)[p] \subset (K/H)[p] \subset p^n(G/H)[p];$$

in particular, $p^{n+1}G[p] \subset K$ and it follows that $K$ is a cobounded summand of $G$.

The previous theorem suggests the following question: What subgroups $N$ of a group $G$ have the property: $K$ is a summand of $G$ whenever $K$ is a pure subgroup of $G$ containing $N$? The next theorems answer this question for subsocles of primary groups.

THEOREM 3.6. *Let $G$ be a $p$-group, $S$ a subsocle of $G^1$. Then every pure subgroup of $G$ containing $S$ is a summand of $G$ if and only if $G$ is the direct sum of a bounded group and a divisible group.*

*Proof.* Let $H$ be a high subgroup of $G$. Suppose that $H$ is unbounded; then $H$ contains a proper basic subgroup $B$ of $H$, and $G/B = (H/B) \oplus (M/B)$, where $M$ can be chosen to contain $G^1$. By Lemma 2.2, $M$ is a pure subgroup of $G$, and $M$ contains $S$; therefore $G = M \oplus K$ and $K$ is divisible which is a contradiction. Thus $H$ is bounded and therefore $G = H \oplus D$, where $D$ is a divisible group. For the converse, refer to Lemma 4.2.

THEOREM 3.7. *Let $G$ be a $p$-group and $S$ a subsocle of $G$. Then the following are equivalent:*

(i) *$S \supset G^1[p]$ and every pure subgroup of $G$ containing $S$ is a summand of $G$;*

(ii) *Every pure subgroup of $G$ containing $S$ is a cobounded summand of $G$;*

(iii) *$S \supset p^nG[p]$ for some $n \in Z^+$.*

*Proof.* We show: (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Suppose that $x \in G^1[p]$ and $x \notin S$. Let $H$ be an $\langle x\rangle$-high subgroup of $G$ containing $S$. Then $H$ is a pure subgroup of $G$ and $G/H$ is divisible, a contradiction; therefore $x \in S$ and $G^1[p] \subset S$.

(i) $\Rightarrow$ (iii). If $S = G^1$, then by Theorem 3.6, $G = B \oplus D$, where $B$ is bounded and $D$ is divisible and clearly $S \supset p^nG[p]$ if $p^nB = 0$. If $S \neq G^1$, let $K$ be a maximal pure subgroup of $G$ such that $K[p] \subset S$; then $(K + S)/K \subset (G/K)^1$ by Lemma 2.3. Now every pure subgroup $M/K$ of $G/K$ containing $(K + S)/K$ is a summand of $G/K$ since $M$ is a pure subgroup of $G$ containing $M$. Therefore $G/K$ is the direct sum of a bounded group and a divisible group. Thus $G/K$ is pure complete which leads to a contradiction. Therefore $K[p] = S$ and $G/K$ is bounded. We conclude that $S \supset p^nG[p]$, where $p^n(G/K) = 0$.

(iii) $\Rightarrow$ (ii). Let $K$ be a pure subgroup of $G$ containing $S$. Then $K \supset p^nG$, and thus $K$ is a cobounded summand of $G$.

COROLLARY 1. *Let $G$ be a reduced $p$-group and $S$ a subsocle of $G$; then every pure subgroup $K$ of $G$ containing $S$ is a summand of $G$ if and only if $S \supset p^nG[p]$ for some $n \notin Z^+$.*

*Proof.* It suffices to show that $S \supset G^1[p]$. Let $x \in G^1[p]$ and suppose that $x \notin S$. Let $H$ be an $\langle x\rangle$-high subgroup of $G$ containing $S$. Then $H$ is a pure subgroup of $G$ and $G = H \oplus D$, where $D \simeq G/H$ is divisible but $G$ is reduced. Therefore $x \in S$ and $S \supset G^1[p]$.

COROLLARY 2. *Let $G$ be a $p$-group and $S$ a subsocle of $G$ satisfying any of the three conditions in Theorem 3.7. Then $S$ supports a pure subgroup of $G$.*

*Proof.* $S \supset p^nG[p]$ for some $n \in Z^+$ and the result follows from [**5**, corollary to Lemma 4.3].

COROLLARY 3. *Let $G$ be a $p$-group and $S$ a quasi-essential subsocle of $G$. Then $S$ supports a pure subgroup of $G$ or $S \subset G^1$.*

*Proof.* Suppose that $S \not\subset G^1$. Then by Theorem 3.5, $S$ satisfies condition (ii) of Theorem 3.7; thus $S$ supports a pure subgroup by Corollary 2.

This last corollary shows that quasi-essential subsocles of primary groups are centres of purity. In fact, they support pure subgroups which are absolute summands. Thus we have the following complete characterization.

THEOREM 3.8. *A subsocle of a $p$-group $G$ is quasi-essential if and only if either $S \subset G^1$ or $(p^nG)[p] \supset S \supset (p^{n+1}G)[p]$ for some non-negative integer $n$.*

The next section contains some applications of the results in §§ 2 and 3.

**4. Some applications.** We first prove a result due to Hill and Megibben [**3**, Theorem 2] as an application of our results. We will need the next lemma, the proof of which relies heavily upon Lemma 2.2.

LEMMA 4.1. *Let $G$ be a $p$-group and $N$ a subgroup of $G$ such that no proper pure subgroup of $G$ contains $N$. Then every pure subgroup containing $N[p]$ is a cobounded summand of $G$.*

*Proof.* Let $H$ be a subgroup of $G$ such that $H \cap N = 0$. Then $H$ is bounded, otherwise $H$ has a proper basic subgroup $B$ and $G/B = (H/B) \oplus (K/B)$, where, by Lemma 2.2, $K$ is a proper pure subgroup of $G$ and $K \supset N$, a contradiction. Now let $K$ be a pure subgroup of $G$ containing $N[p]$; then $G/K$ has a bounded basic subgroup. Otherwise let $B/K$ be an unbounded basic subgroup of $G/K$; then $B = K \oplus M$ where $M \simeq B/K$ (see [**8**, Theorem 5]) and $M \cap N = 0$, a contradiction since $M$ is unbounded. Therefore $G/K = (B/K) \oplus (D/K)$, where $B/K$ is a bounded subgroup of $G/K$ and $D/K$ is a divisible subgroup of $G/K$. Now we show that $D/K = 0$ otherwise $G/B$ is divisible, and $B$ a pure subgroup of $G$ implies that $B[p]$ is proper dense in $G[p]$ and $B[p] \supset N[p]$, which is a contradiction. Therefore $G/K$ is bounded and $K$ is a summand of $G$.

This lemma has the following interesting corollary about minimal pure subgroups containing a given subgroup of a primary group.

COROLLARY [**3**, Theorem 2]. *Let $N$ be a subgroup of $G$ and $H$ a minimal pure subgroup of $G$ containing $N$. Then $H = B \oplus K$, where $B$ is bounded and $K[p] = N[p]$.*

*Proof.* From Lemma 4.1 and Theorem 3.7, $N[p]$ supports a pure subgroup $K$ of $H$ and $H/K$ is bounded; therefore $H = B \oplus K$.

The proof of Lemma 4.1 suggests the following question: What primary groups $G$ have the property

(**) $G/K = (B/K) \oplus (D/K)$, where $B/K$ is bounded and $D/K$ is divisible whenever $K$ is a pure subgroup of $G$ containing $G^1$?

If $G^1$ is divisible, clearly $G = B \oplus D$, where $D = G^1$ and $B$ is a bounded subgroup of $G$. In this section we answer the question for certain classes of groups.

LEMMA 4.2. *Let $G = B \oplus D$ be a $p$-group, where $B$ is bounded and $D$ is divisible. Then every pure subgroup $K$ of $G$ is the direct sum of a bounded and a divisible group.*

*Proof.* Let $K$ be a pure subgroup of $G$; then $K \cap D = K^1$. Let $H$ be a high subgroup of $K$; then $H \cap D = 0$ and $H$ is therefore bounded. It follows that $K = H \oplus (K \cap D)$, where $K \cap D$ is divisible.

A primary group $G$ is said to be essentially finitely indecomposable (e.f.i.) if it has no unbounded direct sum of cyclic groups summand. The next theorem shows that groups with property (**) must be e.f.i.

THEOREM 4.3. *Let $G$ be a $p$-group. If $G$ satisfies property* (**), *then every pure subgroup of $G$ containing $G^1$ is e.f.i.*

*Proof.* Let $M$ be a pure subgroup of $G$ containing $G^1$. Then $M$ satisfies (**). For, let $K$ be a pure subgroup of $M$ containing $M^1 = G^1$; then $M/K$ is a pure subgroup of $G/K$ and the assertion follows from Lemma 4.2. Suppose that $M$ is not e.f.i.; then $M = S \oplus T$ where $S$ is an unbounded direct sum of cyclic groups. Therefore $T$ is a pure subgroup of $M$ containing $M^1$ and $M/T$ is unbounded, a contradiction.

COROLLARY. *Let $G$ be a $p$-group satisfying* (**) *and suppose that $|G^1| \leqq \aleph_0$. Then $G = B \oplus D$, where $B$ is a bounded group and $D$ is divisible.*

*Proof.* Embed $G^1$ in a pure subgroup $K$ such that $|K| = \aleph_0$. Let $K = R \oplus D$, where $D$ is the divisible subgroup of $K$ and $R$ is a reduced subgroup of $K$. We claim that $R$ is bounded for otherwise $R$ is an unbounded reduced countable $p$-group and by [**1**, p. 143, Exercise 19 (a)], $R$ is not e.f.i., contradicting Theorem 4.3. Therefore $R$ is bounded and $G^1 = K^1 = D$. Now $G = B \oplus D$ and $B$ must be bounded since it is reduced.

After reading this last corollary, one might be tempted to conjecture that a $p$-group $G$ satisfies (**) if and only if $G = B \oplus D$ where $B$ is bounded and $D$ is divisible; however, the next theorem shows that this is not the case.

THEOREM 4.4. *Let $G$ be a $p$-group such that $G/G^1$ is a quasi-closed group and some high subgroup of $G$ is countable. Then $G$ satisfies* (**).

*Proof.* Let $K$ be a pure subgroup of $G$ containing $G^1$ and let $H$ be a high subgroup of $G$. Then $G = H + K$ by [**5**, Theorem 2.4]; thus

$$G/K = (H + K)/K \simeq H/(H \cap K)$$

but $H$ is countable since one high subgroup is countable (see [**6**]). Therefore $H/(H \cap K)$ is countable. Now $G/K \simeq (G/G^1)/(K/G^1)$ and as such it is a homomorphic image of a quasi-closed group with unbounded pure kernel so that $G/K$ is the direct sum of a closed group and a divisible group (see [**9**]), but a countable closed group is necessarily bounded.

*Example.* Theorem 4.4 would be meaningless without a concrete example. In [**10**], the following group is studied. Let $B = \sum C(p^n)$ and consider $\bar{B}$, the torsion completion of $B$, that is to say the torsion subgroup of the product of $C(p^n)$. Let $G = \bar{B}/(B[p])$; then $G^1 = (\bar{B}[p]/(B[p])$ and a high subgroup of $G$ is $B/(B[p])$ which is countable. Furthermore, $G/G^1 \simeq p\bar{B}$; now $p\bar{B}$ is a torsion complete group and thus it is quasi-closed. This group $G$ satisfies the hypothesis of Theorem 4.4.

**5. Examples.** In discussing quasi-essential subgroups of a primary group $G$ we have obtained a complete characterization of the quasi-essential subgroups of $G$ which are contained in $G[p]$. However, we have not considered arbitrary subgroups of $G$. Trivially, the fact that the socle of a subgroup is quasi-essential implies that the subgroup is quasi-essential, but the converse is not true as the following example shows.

*Example* 5.1. Let $G$ be an unbounded reduced $p$-group and let $\langle x\rangle$ be a finite cyclic summand of $G$ such that $O(x) = n + 1$, $n \in Z^+$, and $n > k = \text{Min}\{h(g): g \in G[p]\}$. Now since $h(x) < \infty$ and $G$ is unbounded, $\langle x\rangle$ is not a centre of purity of $G$, therefore there exists an $\langle x\rangle$-high subgroup $H$ of $G$ which is not pure in $G$. This subgroup $H$ is quasi-essential because whenever $K$ is a pure subgroup containing $H$, $K$ contains $G[p]$ and therefore $K = G$. Now $H[p]$ is not quasi-essential because $H[p] \oplus \langle p^n x\rangle = G[p] = (p^k G)[p]$ and if $H[p]$ were quasi-essential, then $H[p] \supset (p^{k+1}G)[p]$, a contradiction.

It is remarkable that if $N$ is a quasi-essential subgroup of a reduced primary group $G$ such that $N[p] \not\subset G^1$, then $N[p] \supset G^1[p]$ and every $N$-high subgroup of $G$ is bounded. This can be shown as follows. From Theorem 3.5, every pure subgroup containing $N$ is a cobounded summand of $G$. Now let $H$ be an $N$-high subgroup of $G$ and suppose that $H$ is not bounded. Then $H$ has a proper basic subgroup $B$ and $G/B = (H/B) \oplus (K/B)$, where $K$ is a pure subgroup of $G$ and can be chosen to contain $N$. Now then $G = K \oplus D$ where $D$ is divisible, and this is a contradiction to the fact that $K$ is cobounded. Thus all $N$-high subgroups of $G$ are bounded. Note that $N[p] \supset G^1[p]$ is established using a technique similar to the one used in Corollary 1 of Theorem 3.7.

In view of the preceding paragraph one might ask the following question. Let $N$ be a subgroup of $G$ and let $H$ be an $N$-high subgroup of $G$. If $H$ is bounded, are all $N$-high subgroups of $G$ bounded?

The answer is no; in fact, the next example shows that $N$-high subgroups can be bounded by different bounds and also unbounded for a fixed subgroup $N$ of $G$.

*Example* 5.2. Let $P$ be the Prüfer group, let $N$ be a high subgroup of $P$ and let $P^1 = \langle x\rangle$. Then $G[p] = N[p] \oplus \langle x\rangle$. Now let $\{y_n\}$ be a sequence of elements of $N[p]$ such that $h(y_n) = n$, $n \in Z^+$. Then

$$\langle x + y_n\rangle \oplus N[p] = G[p]$$

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and $\langle x + y_n \rangle$ supports a finite summand $K_n$ of $G$ since $h(x + y_n) < \infty$. Clearly $K_n$ is an $N$-high subgroup of $G$ for each $n$ (see [**5**, Lemma 1]).

Let $G = \sum P_i$ be a countable direct sum of copies of $P$ and let $K_n$ be the $N$-high subgroup of $P$ chosen in $P_n$. Then $\sum K_i$ is $(\sum N_i)$-high in $G$ where $N_i = N$ for each $i$, whereas $K_i$ is of order $p^i$ for each $i$. Thus $\sum K_i$ is an unbounded $\sum N_i$-high subgroup of $G$. However, by choosing all $K_i$ to be the same we obtain a $(\sum N_i)$-high subgroup of $G$ which is bounded.

Note that the construction in Example 5.2 can be carried out in any reduced group $G$ by taking $N$ to be a subgroup such that $G/N = Z(p^\infty)$ and $N[p] \neq G[p]$.

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