DUAL INTEGRAL EQUATIONS WITH BESSEL FUNCTION AND TRIGONOMETRICAL KERNELS

by K. N. SRIVASTAVA
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1. In a recent note Sneddon (3) has proved that the solution of the dual integral equations

$$\left.\begin{aligned} \int_0^\infty y^{-1}g(y)\cos(xy)dy &= f(x), \quad 0 \leqq x \leqq 1, \\ \int_0^\infty g(y)\cos(xy)dy &= 0, \qquad\quad x > 1, \end{aligned}\right\} \quad (A)$$

where $f(x)$ can be represented in a series of Jacobi polynomials in the form

$$f(x) = \sum_{n=1}^{\infty} a_n \mathscr{F}_n(0, \tfrac{1}{2}, x^2), \quad \ldots\ldots(1)$$

and satisfies the condition

$$\int_0^1 (1-x^2)^{-\frac{1}{2}} f(x)dx = 0, \quad \ldots\ldots(2)$$

is

$$g(x) = 2\sum_{n=1}^{\infty} na_n J_{2n}(x). \quad \ldots\ldots(3)$$

If in (A), cos (xy) is replaced by sin (xy) and $f(0) = 0$, the solution is analogous to the above. Here

$$\mathscr{F}_n(\alpha, \beta, x) = {}_2F_1(-n, \alpha+n;\ \beta;\ x) \quad \ldots\ldots(4)$$

is Jacobi's polynomial (2).

2. By the analysis used by Sneddon, we can demonstrate that the solution of the dual integral equations

$$\left.\begin{aligned} \int_0^\infty y^{2k-1}g(y)\cos(xy)dy &= f(x), \quad 0 \leqq x < 1, \\ \int_0^\infty g(y)\cos(xy)dy &= 0, \qquad\quad x > 1, \end{aligned}\right\} \quad (B)$$

where $f(x)$ can be represented in a series of Jacobi polynomials in the form

$$f(x) = \sum_{n=1}^{\infty} a_n \mathscr{F}_n(k, \tfrac{1}{2}, x^2), \quad \ldots\ldots(5)$$

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and satisfies the condition

$$\int_0^1 (1-x^2)^{k-\frac{1}{2}} f(x)dx = 0, \qquad (6)$$

is

$$g(x) = \sum_{n=1}^{\infty} 2^{1-k}\Gamma(n+1)[\Gamma(n+k)]^{-1} a_n x^{-k} J_{2n+k}(x), \qquad (7)$$

provided $-\frac{1}{2}<k<{}^3/_2$.

The solution of the dual integral equations

$$\left.\begin{aligned} &\int_0^\infty y^{2k-1} g(y) \sin(xy)dy = f(x), \quad 0 \leqq x < 1, \\ &\int_0^\infty g(y) \sin(xy)dy = 0, \qquad x > 1, \end{aligned}\right\} \qquad (C)$$

where $f(x)$ can be represented in a series of Jacobi polynomials

$$f(x) = \sum_{n=1}^{\infty} a_n x \mathscr{F}_n(k+1, \tfrac{3}{2}, x^2) \qquad (8)$$

and satisfies the conditions

$$\text{(i) } f(0) = 0, \quad \text{(ii) } \int_0^1 x(1-x^2)^{k-\frac{1}{2}} f(x)dx = 0, \qquad (9)$$

is

$$g(x) = \sum_{n=1}^{\infty} 2^{1-k}\Gamma(n+1)[\Gamma(n+k+1)]^{-1} a_n x^{-k} J_{k+2n+1}(x), \qquad (10)$$

provided $-\frac{1}{2}<k<{}^3/_2$.

For $k = 0$, we get the results obtained by Sneddon.

3. Next consider the dual integral equations with Bessel function kernel. Given the dual integral equations

$$\left.\begin{aligned} &\int_0^\infty y^{2k-2} g(y)(xy)^{\frac{1}{2}} J_\nu(xy)dy = f(x), \quad 0 \leqq x < 1, \\ &\int_0^\infty g(y)(xy)^{\frac{1}{2}} J_\nu(xy)dy = 0, \qquad x > 1; \end{aligned}\right\} \qquad (D)$$

if $f(x)$ can be represented in a series of Jacobi polynomials in the form

$$f(x) = \sum_{n=1}^{\infty} a_n x^{\nu+\frac{1}{2}} \mathscr{F}_n(\nu+k, \nu+1, x^2), \qquad (11)$$

and satisfies the conditions

$$\text{(i) } f(0) = 0, \quad \text{(ii) } \int_0^1 x^{\nu+\frac{1}{2}}(1-x^2)^{k-1} f(x)dx = 0, \qquad (12)$$

then the solution to (D) is

$$g(x) = \sum_{n=1}^{\infty} a_n 2^{1-k}\Gamma(\nu+1)\Gamma(n+1)[\Gamma(n+k+1)]^{-1} x^{\frac{1}{2}-k} J_{\nu+2n+k}(x), \qquad (13)$$

provided $0<k<2$ and $\nu>-\frac{1}{2}$.

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In the case when $\nu = -\frac{1}{2}$, the condition $f(0) = 0$ is dropped. The above result is true even in the case $-1<\nu<-\frac{1}{2}$, provided in the first equation of (D) the condition $0 \leqq x<1$ is replaced by $0<x<1$.

It is interesting to note that the results obtained by Sneddon and those given in the Section **2** are particular cases of the results given above. The results in Section **2** are obtained by replacing k by $k+\frac{1}{2}$ and ν by $\pm\frac{1}{2}$.

4. A formal verification of the solution given in Section **3** is given below. Taking $\nu>-1$, n a positive integer and k real and positive, the integral

$$\int_0^\infty y^{1-k} J_{\nu+2n+k}(y) J_\nu(xy)\,dy$$

converges and its value is given by Watson (**4**(a)). For $x>1$ the integral vanishes and the value of $g(x)$ as given in (13) automatically satisfies the second equation of (D). Again from Watson (**4**(a)), we have

$$\int_0^\infty y^{k-1} J_{\nu+2n+k}(y) J_\nu(xy)\,dy = \frac{2^{k-1}\Gamma(\nu+n+k)}{\Gamma(\nu+1)\Gamma(n+1)} x^\nu \mathscr{F}_n(\nu+k,\, \nu+1,\, x^2)$$

for $0<k<2$, $\nu>-1$ and $0<x<1$. Since

$$f(x) = \sum_{n=1}^\infty a_n x^{\nu+\frac{1}{2}} \mathscr{F}_n(\nu+k,\, \nu+1,\, x^2),$$

the first equation of (D) is also satisfied. The cases when $\nu \geqq -\frac{1}{2}$, $0<k<2$ and $x = 0$ or 1, can be varied by using the tables of integral transforms (**1**, 6.8 (1), 6.8 (11), (33)). Multiplying the first equation of (a) by $x^{\nu+\frac{1}{2}}(1-x^2)^{k-1}$, integrating between the limits (0, 1) and interchanging the order of integration, we get

$$\begin{aligned}
&\int_0^1 x^{\nu+\frac{1}{2}}(1-x^2)^{k-1} f(x)\,dx \\
&= \int_0^\infty y^{2k-\frac{1}{2}} g(y)\,dy \int_0^{\pi/2} J_\nu(y \sin\theta)(\sin\theta)^{\nu+1}(\cos\theta)^{2k-1}\,d\theta \\
&= 2^{k-1}\Gamma(k) \int_0^\infty y^{k-\frac{1}{2}} g(y) J_{\nu+k}(y)\,dy \\
&= \sum_{n=1}^\infty \Gamma(k)\Gamma(\nu+1)\Gamma(n+1)[\Gamma(\nu+k+n)]^{-1} a_n \int_0^\infty y^{-1} J_{\nu+2n+k}(y) J_{\nu+k}(y)\,dy \\
&= 0.
\end{aligned}$$

Here use has been made of the following results given in Watson (**4**(b), **4**(c)):

$$\int_0^\infty y^{-1} J_{\nu+2n+k}(y) J_{\nu+2m+k}(y)\,dy = 0, \quad m \neq n,$$

$$\int_0^{\pi/2} J_\nu(y \sin\theta)(\sin\theta)^{\nu+1}(\cos\theta)^{2k-1}\,d\theta = \frac{2^{k-1}\Gamma(k) J_{\nu+k}(y)}{y^k}.$$

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With the value of $g(y)$ given in (13) all the conditions are satisfied. The results given in Section **2** can be verified in the same way.

The results of Section **2** are true even in the case when the condition $0 \leqq x < 1$ in the first equations of (B) and (C) is replaced by $0 \leqq x \leqq 1$, provided that $-\frac{1}{2} < k < \frac{1}{2}$. Similarly the result of Section **3** is true when the condition $0 \leqq x < 1$ in the first equation of (D) is replaced by $0 \leqq x \leqq 1$, provided $0 < k < 1$.

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M.A. College of Technology
Bhopal (M.P.), India