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A RADON-NIKODÝM THEOREM FOR VECTOR POLYMEASURES

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Abstract

A Radon-Nikodým theorem for Banach valued polymeasures is proved.

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1. Introduction

A Radon-Nikodým theorem for (scalar) Radon polymeasures was given in [5]. The aim of this paper is to give a Radon-Nikodým theorem for (general) Banach valued polymeasures. Despite the fact that the Radon-Nikodým theorem proved here extends that of [9], Example 6 points out that a Radon-Nikodým theorem for polymeasures (inclusive, for scalar bimeasures) cannot be stated using a condition of localization which is just an extension of the Maynard's condition of localization in compacts. The integrable d-tuples used here belong to the first integrable class of [3], and their integrals coincide with that of [3].

Finally, let us point out that, as predicted in [7], polymeasures that do not determine a measure in the product space are increasingly appearing in areas as diverse as non-stationary processes, harmonic analysis, operator theory and quantum physics (see [6]).

2. Notation and preliminaries

In the following, d will be a fixed positive integer, Σ_i, $i = 1, \dots, d$,will be σ-algebras of subsets of non empty sets Ω_i, X_i, $i = 1, \dots, d$, Y and Z will be

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given Banach spaces over the same scalar field K of real or complex numbers, and $\phi : X_1 \times \cdots \times X_d \times Y \to Z$ will be a continuous multilinear mapping, that without loss of generality, can be supposed to be such that $\|\phi\| \le 1$. We also consider two fixed polymeasures (in the usual sense of [2]) $\alpha : \Sigma_1 \times \cdots \times \Sigma_d \to Y$ and $\gamma : \Sigma_1 \times \cdots \times \Sigma_d \to Z$.

Denote by $S_i(\Omega_i)$, $i = 1, \dots, d$, the normed linear space of Σ_i-simple functions $f_i : \Omega_i \to X_i$ with the sup norm $\|f_i\|_{\Omega_i} = \sup\{|f_i(t_i)| : t_i \in \Omega_i\}$. Let $f_i \in S_i(\Omega_i)$, $i = 1, \dots, d$, be of the form $f_i = \sum_{j_i=1}^{r_i} x_{i,j_i}\chi_{A_{i,j_i}}$ with $x_{i,j_i} \in X_i$, and with pairwise disjoint $A_{i,j_i} \in \Sigma_i$. Then the integral of the d-tuple $(f_1, \dots, f_d) \in S_1(\Omega_1) \times \cdots \times S_d(\Omega_d)$ over $(A_1, A_2, \dots, A_d) \in \Sigma_1 \times \cdots \times \Sigma_d$ is defined (see [2]) by

$$\int_{(A_1,A_2,\dots,A_d)} (f_1, \dots, f_d)\, d\alpha = \sum_{j_1=1}^{r_1} \cdots \sum_{j_d=1}^{r_d} \phi\big(x_{1,j_1}, \dots, x_{d,j_d}, \alpha\big(A_{1,j_1}, \dots, A_{d,j_d}\big)\big).$$

In the sequel, if there is no confusion, we will simply write S_i, $\|f_i\|$, $\int_{(A_i)} (f_i)\, d\alpha$, $(f_i) \in \times S_i$ and $(A_i) \in \times \Sigma_i$.

Following [2], we define the semivariation $\hat{\alpha} : \Sigma_1 \times \cdots \times \Sigma_d \to [0, +\infty]$ and the variation $|\alpha| : \Sigma_1 \times \cdots \times \Sigma_d \to [0, +\infty]$, of the polymeasure α, by

$$\hat{\alpha}(A_1, \dots, A_d) = \sup\left\{\left\|\int_{(A_i)} (f_i)\, d\alpha\right\| : f_i \in S_i,\ \|f_i\| \le 1,\ i = 1, \dots, d\right\}$$

and

$$|\alpha|(A_1, \dots, A_d) = \sup\left\{\sum_{j_1=1}^{r_1} \cdots \sum_{j_d=1}^{r_d} \left\|\alpha(A_{1,j_1}, \dots, A_{d,j_d})\right\|,\right.$$
$$\left. A_{i,j_i} \in \Sigma_i \text{ pairwise disjoint with } A_i = \bigcup_{j_i=1}^{r_i} A_{i,j_i}\right\}.$$

Let us also consider the set function $|||\hat{\alpha}||| : \times \Sigma_i \to [0, +\infty]$ defined by

$$|||\hat{\alpha}|||(A_1, \dots, A_d) = \sum_{i=1}^{d} |\hat{\alpha}|\, (\Omega_1, \dots, A_i, \dots, \Omega_d).$$

In the following, let us assume that $\hat{\alpha}(\Omega_1, \dots, \Omega_d) < +\infty$.

DEFINITION 1. A function $f_i : \Omega_i \to X_i$ is said to be *u-simple* if it is a uniform limit of a sequence of simple functions from Ω_i into X_i. Denote by U_i, $i = 1, \dots, d$, the space of the u-simple functions $f_i : \Omega_i \to X_i$.

If $(f_i) \in \times U_i$, then the integral of the d-tuple $(f_1, \dots, f_d) \in U_1 \times \cdots \times U_d$ over $(A_i) \in \times \Sigma_i$ is defined by

$$\int_{(A_1, A_2, \dots, A_d)} (f_1, \dots, f_d)\, d\alpha = \lim_{n\to\infty} \int_{(A_1, A_2, \dots, A_d)} (f_1^n, \dots, f_d^n)\, d\alpha,$$

where $(f_i^n)_{n\in\mathbb{N}}$, $i = 1, \dots, d$, is any sequence in S_i which is uniformly convergent to f_i. It is easily proved that this integral is well defined and it has all properties of the integral of d-tuples of simple functions.

A d-tuple of functions $f_i : \Omega_i \to X_i$, $i = 1, \dots, d$, is said to be α-measurable, if for every $\varepsilon > 0$ there exists $(A_i) \in \times \Sigma_i$ such that $|||\alpha|||(\Omega_1 - A_1, \dots, \Omega_d - A_d) < \varepsilon$ and $(f_i \chi_{A_i}) \in \times U_i$.

An α-measurable d-tuple (f_i) is said to be α-integrable, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every d-tuple $(A_i) \in \times \Sigma_i$ satisfying $\hat{\alpha}(A_1, \dots, A_d) < \delta$ and $(f_i \chi_{A_i}) \in \times U_i$, we have that

$$\left\| \int_{(A_i)} (f_i)\, d\alpha \right\| < \varepsilon.$$

If the d-tuple (f_i) is α-integrable, then the integral of (f_i) over $(A_i) \in \times \Sigma_i$ is defined by

$$\int_{(A_1, A_2, \dots, A_d)} (f_1, \dots, f_d)\, d\alpha = \lim_{\substack{(B_i) \in \times \Sigma_i \cap \wp(A_i) \\ (f_i \chi_{A_i}) \in \times U_i}} \int_{(B_i)} (f_i)\, d\alpha,$$

where $\wp(A_i)$ denotes the family of all subsets of $A_i (i = 1, \dots, d)$.

It is easily proved that the above integral is well defined. Furthermore, every α-integrable d-tuple belongs to the first integrable class, defined in [3, pp. 592, 593], and its integrals in both senses coincide. Let us also point out, that this integral coincides, when $d = 1$, with the vector integration introduced in [4].

3. A Radon-Nikodým theorem

Suppose that $0 < |\alpha|(\Omega_1, \dots, \Omega_d) < +\infty$.

DEFINITION 2. Let $(A_1, \dots, A_d) \in (\Sigma_1 \times \cdots \times \Sigma_d)^+ = \{(B_i) \in \times \Sigma_i : |\alpha|(B_1, \dots, B_d) > 0\}$. A mapping $\varphi : (0, +\infty) \to \times S_i(A_i)$ will be a *localization* for $(A_1, \dots, A_d)$ if the following assertions hold:

(2.1) $\left\| \gamma(B_1, \dots, B_d) - \int_{(B_i)} \varphi(\varepsilon)\, d\alpha \right\| \leq \varepsilon$, for every $\varepsilon \in (0, +\infty)$ and every d-tuple $(B_i) \in \times \Sigma_i$ such that $(B_i) \subseteq (A_i)$ (that is, $B_i \subseteq A_i$, $i = 1, \dots, d$).

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(2.2) $\|\pi_i\varphi(\varepsilon) - \pi_i\varphi(\varepsilon')\|_{A_i} \leq \varepsilon$, for every $0 < \varepsilon' < \varepsilon < +\infty$ and $i = 1, \dots, d$, where $\pi_i : \times S_i(A_i) \to S_i(A_i)$, $i = 1, \dots, d$, denotes the usual projection ($\pi_i(f_1, \dots, f_d) = f_i$).

We say that a d-tuple $(A_i) \in (\times \Sigma_i)^+$ is *localized*, when there is a localization for it. A sequence of d-tuples $\{(A_i^n)\}_{n\in\mathbb{N}} \subseteq (\times \Sigma_i)^+$ is said to be *localized* if $(A_1^n, \dots, A_d^n)$ is localized for every $n \in \mathbb{N}$.

If φ_j is a localization for $(A_i^j) \in (\times \Sigma_i)^+$, $j = 1, 2$, and $(A_i^1) \subseteq (A_i^2)$, we will say that φ_1 and φ_2 are *consistent* if $\|\pi_i\varphi_1(\varepsilon) - \pi_i\varphi_2(\varepsilon)\chi_{A_i^1}\|_{A_i^1} \leq \varepsilon$, $i = 1, \dots, d$, for every $\varepsilon > 0$. Let $\{(A_i^n)\}_{n\in\mathbb{N}} \subseteq (\times \Sigma_i)^+$ be an increasing localized sequence of d-tuples. If there exists a localization φ_n for (A_i^n), $n \in \mathbb{N}$, such that φ_n and φ_m are consistent for all $n, m \in \mathbb{N}$ with $m > n$, then we say that $\{(A_i^n)\}_{n\in\mathbb{N}}$ is *consistent*.

REMARK 3. A d-tuple $(A_i) \in (\times \Sigma_i)^+$ is localized if and only if there exist d non empty compact subsets $K_i \subseteq X_i$, $i = 1, \dots, d$, such that for every $\varepsilon > 0$ and every finite family $\{K_i^j\}_{j\in J_i}$ of non empty compact subsets of X_i with diameter less than or equal ε, such that $K_i = \bigcup_{j\in J_i} K_i^j$, $i = 1, \dots, d$, there exists $\lambda > 0$ (depending on ε and $\{K_i\}_{i=1}^d$) such that:

(3.1) For every $i = 1, \dots, d$, there exists a measurable and finite partition $\{E_i^j\}_{j\in J_i} \subseteq \Sigma_i$ of Ω_i and $x_i^j \in K_i^j$, $j \in J_i$, such that if $f_i = \sum_{j\in J_i} x_i^j \chi_{E_i^j}$ $(i = 1, \dots, d)$, then

$$\left\|\gamma(B_1, \dots, B_d) - \int_{(B_i)} (f_i)\, d\alpha\right\| \leq \varepsilon\lambda\, |\alpha|\, (B_1, \dots, B_d),$$

for every d-tuple $(B_i) \in \times \Sigma_i$ with $(B_i) \subseteq (A_i)$.

(3.2) If (i) $0 < \varepsilon' < \varepsilon$, $i = 1, \dots, d$, and $\{C_i^j\}_{j\in J_i'}$ is a finite family of non empty compact subsets of X_i, with diameter less than or equal ε', such that $K_i = \bigcup_{j\in J_i'} C_i^j$, and (ii) for every $i = 1, \dots, d$ and every $j \in J_i'$, there exists $h_j \in J_i$, with $C_i^j \subseteq K_i^{h_j}$, then there exists a finite partition $\{F_i^j\}_{j\in J_i'} \subseteq \Sigma_i$ of Ω_i and $y_i^j \in C_i^j$ $(j \in J_i')$ such that if $g_i = \sum_{j\in J_i'} y_i^j \chi_{F_i^j}$ $(i = 1, \dots, d)$, then $\|f_i - g_i\| \leq 2\varepsilon$ for every $i = 1, \dots, d$ and $\left\|\gamma(B_1, \dots, B_d) - \int_{(B_i)} (g_i)\, d\alpha\right\| \leq \varepsilon'\lambda|\alpha|(B_1, \dots, B_d)$, for every $(B_i) \in \times \Sigma_i$ with $(B_i) \subseteq (A_i)$.

THEOREM 4. *The polymeasure γ has a Radon-Nikodým derivative with respect to α (that is, there exists an α-integrable d-tuple of functions (f_i) such that $\gamma(A_1, \dots, A_d) = \int_{(A_i)} (f_i)\, d\alpha$, for every $(A_i) \in \times \Sigma_i$) if and only if the following assertions hold*:

(4.1) $\|\gamma\| \ll \hat{\alpha}$ (*that is,* $\lim \|\gamma(A_1, \dots, A_d)\| = 0$ *as* $\hat{\alpha}(A_1, \dots, A_d) \to 0$, $(A_i) \in \times \Sigma_i$).

(4.2) *There exists a consistent sequence* $\{(A_i^n)\}_{n\in\mathbb{N}} \subseteq (\times \Sigma_i)^+$ *such that* $\|\|\alpha\|\|\left(\Omega_1 - \bigcup_{n\in\mathbb{N}} A_1^n, \dots, \Omega_d - \bigcup_{n\in\mathbb{N}} A_d^n\right) = 0$.

PROOF. Suppose that there exists a Radon-Nikodým derivative $(f_1, \dots, f_d)$ of γ with respect to α. Since (4.1) follows immediately from the properties of the integral, let us prove (4.2). In fact, since $(f_1, \dots, f_d)$ is α-measurable, there exists $(B_i^n) \in \times \Sigma_i$, $n \in \mathbb{N}$, such that

$$|||\alpha|||(\Omega_1 - B_1^n, \dots, \Omega_d - B_d^n) < \frac{|\alpha|(\Omega_1, \dots, \Omega_d)}{1+n}$$

and $(f_i \chi_{B_i^n}) \in \times U_i$.

If $A_i^n = \bigcup_{1 \le m \le n} B_i^m$ for every $n \in \mathbb{N}$ and every $i = 1, \dots, d$, then

$$|\alpha|(A_1^n, \dots, A_d^n) \ge |\alpha|(B_1^1, \dots, B_d^1) > \tfrac{1}{2}|\alpha|(\Omega_1, \dots, \Omega_d) > 0$$

and $\{(A_i^n)\}_{n\in\mathbb{N}} \subseteq (\times \Sigma_i)^+$.

Furthermore,

$$\begin{aligned}|||\alpha|||\left(\Omega_1 - \bigcup_{n\in\mathbb{N}} A_1^n, \dots, \Omega_d - \bigcup_{n\in\mathbb{N}} A_d^n\right) &\le |||\alpha|||\left(\Omega_1 - B_1^m, \dots, \Omega_d - B_d^m\right) \\ &\le \frac{|\alpha|(\Omega_1, \dots, \Omega_d)}{1+m}\end{aligned}$$

for every $m \in \mathbb{N}$, and therefore,

$$|||\alpha|||\left(\Omega_1 - \bigcup_{n\in\mathbb{N}} A_1^n, \dots, \Omega_d - \bigcup_{n\in\mathbb{N}} A_d^n\right) = 0.$$

Thus we have only to prove that the increasing sequence $\{(A_i^n)\}_{n\in\mathbb{N}}$ is consistent. Let us prove first that $(A_1^n, \dots, A_d^n)$ is localized for every $n \in \mathbb{N}$. In fact, since $(f_i \chi_{A_i^n}) \in \times U_i$ (and therefore, $\|f_i\|_{A_i^n} < +\infty$, for $i = 1, \dots, d$), for every $\varepsilon > 0$ there exists $(g_i^{n,\varepsilon}) \in \times S_i(A_i^n)$ such that

$$\|f_i \chi_{A_i^n} - g_i^{n,\varepsilon}\|_{A_i^n} \le \min\left[1, \varepsilon \middle/ \left[2d\left(1 + \max_{1\le i\le d} \|f_i\|_{A_i^n}\right)^d (1 + \hat{\alpha}(A_1, \dots, A_d))\right]\right]$$

for every $i = 1, \dots, d$. Let $\varphi_n : (0, +\infty) \to \times S_i(A_i^n)$ be the function defined by $\varphi_n(\varepsilon) = (g_1^{n,\varepsilon}, \dots, g_d^{n,\varepsilon})$ for every $\varepsilon > 0$.

If $(B_i) \in \times \Sigma_i$ verify that $(B_i) \subseteq (A_i^n)$, then

$$\begin{aligned}&\left\|\gamma(B_1, \dots, B_d) - \int_{(B_i)} \varphi_n(\varepsilon)\, d\alpha\right\| \\ &\quad = \left\|\int_{(B_i)} (f_i)\, d\alpha - \int_{(B_i)} (g_i^{n,\varepsilon})\, d\alpha\right\|\end{aligned}$$

$$\leq \left\| \int_{(B_i)} (f_1 - g_1^{n,\varepsilon}, f_2, \dots, f_d)\, d\alpha \right\|$$
$$+ \dots + \left\| \int_{(B_i)} (g_1^{n,\varepsilon}, \dots, g_{d-1}^{n,\varepsilon}, f_d - g_d^{n,\varepsilon})\, d\alpha \right\|$$
$$\leq \frac{d \left(\prod_{i=1}^d (1 + \|f_i\|_{A_i^n}) \right) \varepsilon \hat{\alpha}(B_1, \dots, B_d)}{2d \left(1 + \max_{1 \leq i \leq d} \|f_i\|_{A_i^n}\right)^d (1 + \hat{\alpha}(A_1, \dots, A_d))}$$
$$< \varepsilon.$$

Furthermore, for every $0 < \varepsilon' < \varepsilon$, $n, m \in \mathbb{N}$ with $n < m$ and $i = 1, \dots, d$, we have

$$\left\| g_i^{n,\varepsilon} - g_i^{n,\varepsilon'} \right\|_{A_i^n} \leq \left\| g_i^{n,\varepsilon} - f_i \right\|_{A_i^n} + \left\| f_i - g_i^{n,\varepsilon'} \right\|_{A_i^n} < \varepsilon$$

and

$$\left\| g_i^{m,\varepsilon} - g_i^{n,\varepsilon} \right\|_{A_i^n} \leq \left\| g_i^{m,\varepsilon} - f_i \right\|_{A_i^m} + \left\| f_i - g_i^{n,\varepsilon} \right\|_{A_i^n} \leq \varepsilon.$$

Therefore (4.2) holds.

Conversely, assume that (4.1) and (4.2) are verified. Let $\{(A_i^n)\}_{n \in \mathbb{N}} \subseteq (\times \Sigma_i)^+$ be a consistent sequence verifying that

$$|||\alpha||| \left(\Omega_1 - \bigcup_{n \in \mathbb{N}} A_1^n, \dots, \Omega_d - \bigcup_{n \in \mathbb{N}} A_d^n \right) = 0.$$

For every $n \in \mathbb{N}$, let $\varphi_n : (0, +\infty) \to \times S_i(A_i^n)$ be a localization for (A_i^n) such that φ_n and φ_m are consistent for all $m, n \in \mathbb{N}$ with $n < m$. Let us set $\varphi_n(1/m) = (g_1^{n,m}, \dots, g_d^{n,m}) \in \times S_i(A_i^n)$. Since $\|g_i^{n,m}(t) - g_i^{n,p}(t)\| \leq 1/m$ for every $m, n, p \in \mathbb{N}$ with $m \leq p$ and every $t \in A_i^n$, it follows that $(g_i^{n,m}(t))_{m \in \mathbb{N}}$ is a Cauchy sequence for every $t \in A_i^n$. Let

$$g_i^n(t) = \lim_m g_i^{n,m}(t)$$

for every $t \in A_i^n$. The above limit is uniform on A_i^n, and thus the function $g_i^n : A_i^n \to X_i$ is u-simple. Furthermore, $g_i^m |_{A_i^n} = g_i^n$ for all $m, n \in \mathbb{N}$ with $m > n$, because

$$\left\| g_i^n(t) - g_i^m(t) \right\| \leq \left\| g_i^n(t) - g_i^{n,p}(t) \right\| + \left\| g_i^{n,p}(t) - g_i^{m,p}(t) \right\| + \left\| g_i^{m,p}(t) - g_i^m(t) \right\|$$
$$\leq 3/p$$

for every $p \in \mathbb{N}$.

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If $n \in \mathbb{N}$, $(E_i) \in \times \Sigma_i$ and $(E_i) \subseteq (A_i)$, then

$$\left\| \gamma(E_1, \dots, E_d) - \int_{(E_i)} (g_i^n)\, d\alpha \right\| = \lim_m \left\| \gamma(E_1, \dots, E_d) - \int_{(E_i)} (g_i^{n,m})\, d\alpha \right\| \leq \lim_m \left(\frac{1}{m}\right) = 0,$$

and therefore,

$$\gamma(E_1, \dots, E_d) = \int_{(E_i)} (g_i^n)\, d\alpha \tag{4.3}$$

For every $i = 1, \dots, d$, consider the function $f_i : \Omega_i \to X_i$ defined by

$$f_i(t) = \begin{cases} g_i^n(t), & \text{if there exists } n \in \mathbb{N} \text{ such that } t \in A_i^n \\ 0, & \text{if } t \in \Omega_i - \bigcup_{n \in \mathbb{N}} A_i^n. \end{cases}$$

Let us prove that $(f_1, \dots, f_d)$ is α-measurable. In fact, since

$$|||\alpha|||\left(\Omega_1 - \bigcup_{n\in\mathbb{N}} A_1^n, \dots, \Omega_d - \bigcup_{n\in\mathbb{N}} A_d^n\right) = 0$$

and $|\alpha|$ is a positive d-polymeasure, for every $\varepsilon > 0$ there exist $r_1, \dots, r_d \in \mathbb{N}$ verifying that

$$|\alpha|(\Omega_1, \dots, \Omega_i - A_i^{r_i}, \dots, \Omega_d) = |\alpha| \left(\Omega_1, \dots, \left(\bigcup_{n\in\mathbb{N}} A_i^n\right) - A_i^{r_i}, \dots, \Omega_d\right) < \frac{\varepsilon}{d}.$$

Thus setting $r = \max\{r_1, \dots, r_d\}$ we have

$$|||\alpha|||\,(\Omega_1 - A_1^r, \dots, \Omega_d - A_d^r) < \varepsilon$$

and the function $f_i\,|_{A_i^r} = g_i^r$ is u-simple, for $i = 1, \dots, d$.

Let us prove now that $(f_1, \dots, f_d)$ is α-integrable. Let $\varepsilon > 0$. Since $\|\gamma\| \ll \hat{\alpha}$, there exists $\delta > 0$ such that $\|\gamma(B_1, \dots, B_d)\| < \varepsilon/2$ for every $(B_i) \in \times \Sigma_i$ with $\hat{\alpha}(B_1, \dots, B_d) \leq \delta$. Let $(E_i) \in (\times \Sigma_i)$ be such that $\hat{\alpha}(E_1, \dots, E_d) \leq \delta$ and $(f_1\chi_{A_1}, \dots, f_d\chi_{A_d}) \in \times U_i$. Then, there exists $K > 0$ such that $\|f_i\|_{E_i} \leq K$ for all $i = 1, \dots, d$. Proceeding as before, we can find $m \in \mathbb{N}$ such that

$$|||\alpha|||\,(\Omega_1 - A_1^m, \dots, \Omega_d - A_d^m) < \frac{\varepsilon}{2K^d}.$$

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Therefore,

$$\begin{aligned}
&\left\|\int_{(E_i)}(f_i)\,d\alpha\right\| \\
&\le \left\|\int_{(E_1-A_1^m,E_2,\dots,E_d)}(f_i)\,d\alpha\right\| + \left\|\int_{(E_1\cap A_1^m,E_2-A_2^m,E_3,\dots,E_d)}(f_i)\,d\alpha\right\| \\
&\quad + \dots + \left\|\int_{(E_1\cap A_1^m,\dots,E_{d-1}\cap A_{d-1}^m,E_d-A_d^m)}(f_i)\,d\alpha\right\| + \left\|\int_{(E_1\cap A_1^m,\dots,E_d\cap A_d^m)}(f_i)\,d\alpha\right\| \\
&\le \|f_1\|_{E_1}\cdots\|f_d\|_{E_d}[\hat{\alpha}(E_1-A_1^m,E_2,\dots,E_d) \\
&\quad + \hat{\alpha}(E_1\cap A_1^m,E_2-A_2^m,E_3,\dots,E_d) \\
&\quad + \dots + \hat{\alpha}(E_1\cap A_1^m,\dots,E_{d-1}\cap A_{d-1}^m,E_d-A_d^m)] \\
&\quad + \left\|\int_{(E_1\cap A_1^m,\dots,E_d\cap A_d^m)}(g_i^m)\,d\alpha\right\| \\
&\le K^d\left(|\!|\!|\alpha|\!|\!|\left(\Omega_1-A_1^m,\dots,\Omega_d-A_d^m\right)\right) \\
&\quad + \left\|\gamma\left(E_1\cap A_1^m,E_2\cap A_2^m,\dots,E_d\cap A_d^m\right)\right\| \\
&\le K^d(\varepsilon/2K^d)+(\varepsilon/2) \\
&= \varepsilon
\end{aligned}$$

and the d-tuple $(f_1,\dots,f_d)$ is α-integrable.

Let us see now that $(f_1,\dots,f_d)$ is a Radon-Nikodým derivative of γ with respect to α. In fact, since $|\!|\!|\alpha|\!|\!|(\Omega_1-\bigcup_{n\in\mathbb{N}}A_1^n,\dots,\Omega_d-\bigcup_{n\in\mathbb{N}}A_d^n)=0$, we have that

$$\sum_{i=1}^{d}\hat{\alpha}\left(\Omega_1,\dots,\Omega_i-\bigcup\nolimits_{n\in\mathbb{N}}A_i^n,\dots,\Omega_d\right)=0.$$

And having in mind that $\|\gamma\|\ll\hat{\alpha}$, we obtain that

$$|\!|\!|\gamma|\!|\!|\left(\Omega_1-\bigcup\nolimits_{n\in\mathbb{N}}A_1^n,\dots,\Omega_d-\bigcup\nolimits_{n\in\mathbb{N}}A_d^n\right)=0.$$

Therefore, for every $(E_1,\dots,E_d)\in\times\Sigma_i$ and $m\in\mathbb{N}$ we have

$$\begin{aligned}
&\left\|\gamma\left(E_1,\dots,E_d\right)-\gamma(E_1\cap A_1^m,\dots,E_d\cap A_d^m)\right\| \\
&\quad\le|\!|\!|\gamma|\!|\!|(\Omega_1-A_1^m,\dots,\Omega_d-A_d^m)
\end{aligned}$$

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$$= |||\gamma||| \left(\Omega_1 - \bigcup_{n\in\mathbb{N}} A_1^n, \dots, \Omega_d - \bigcup_{n\in\mathbb{N}} A_d^n\right)$$
$$+ |||\gamma||| \left(\bigcup_{n\in\mathbb{N}} A_1^n - A_1^m, \dots, \bigcup_{n\in\mathbb{N}} A_d^n - A_d^m\right)$$
$$= |||\gamma||| \left(\bigcup_{n\in\mathbb{N}} A_1^n - A_1^m, \dots, \bigcup_{n\in\mathbb{N}} A_d^n - A_d^m\right).$$

Since the sequence $\left(|||\gamma||| \left(\bigcup_{n\in\mathbb{N}} A_1^n - A_1^m, \dots, \bigcup_{n\in\mathbb{N}} A_d^n - A_d^m\right)\right)_{m\in\mathbb{N}}$ is convergent to zero (because the variation $|\gamma|$ of γ is a positive polymeasure), it follows that

$$\gamma(E_1, \dots, E_d) = \lim_{m\to+\infty} \gamma\left(E_1 \cap A_1^m, \dots, E_d \cap A_d^m\right) \tag{4.4}$$

for every $(E_i) \in \times \Sigma_i$.

Furthermore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\| \int_{(L_1,\dots,L_d)} (f_i)\, d\alpha \right\| < \frac{\varepsilon}{d}$$

for every $(L_i) \in \times \Sigma_i$ verifying that $\hat{\alpha}(L_1, \dots, L_d) \le \delta$. Then, noting that the sequence $\left(\sum_{i=1}^d \hat{\alpha}(\Omega_1, \dots, \Omega_i - A_i^m, \dots, \Omega_d)\right)_{m\in\mathbb{N}}$ converges to zero (because $\left(|||\alpha|||(\Omega_1 - A_1^m, \dots, \Omega_d - A_d^m)\right)_{m\in\mathbb{N}}$ is convergent to zero), it results the existence of $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^d \hat{\alpha}(\Omega_1, \dots, \Omega_i - A_i^n, \dots, \Omega_d) < \delta$$

for every $n \ge n_0$. Therefore, for every $(E_i) \in \times \Sigma_i$ we have

$$\left\| \int_{(E_1,\dots,E_d)} (f_i)\, d\alpha - \int_{(E_1\cap A_1^n,\dots,E_d\cap A_d^n)} (f_i)\, d\alpha \right\|$$
$$\le \left\| \int_{(E_1 - A_1^n, E_2,\dots,E_d)} (f_i)\, d\alpha \right\| + \left\| \int_{(E_1\cap A_1^n, E_2 - A_2^n, E_3,\dots,E_d)} (f_i)\, d\alpha \right\|$$
$$+ \cdots + \left\| \int_{(E_1\cap A_1^n,\dots,E_{d-1}\cap A_{d-1}^n, E_d - A_d^n)} (f_i)\, d\alpha \right\|$$
$$\le d\frac{\varepsilon}{d} = \varepsilon,$$

and consequently,

$$\int_{(E_i)} (f_i)\, d\alpha = \lim_{n\to+\infty} \int_{(E_1\cap A_1^n,\dots,E_d\cap A_d^n)} (f_i)\, d\alpha. \tag{4.5}$$

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Now it follows from(4.5), (4.3) and (4.4) that

$$\begin{aligned}\int_{(E_1,\dots,E_d)} (f_i)\, d\alpha &= \lim_{n\to+\infty} \int_{(E_1\cap A_1^n,\dots,E_d\cap A_d^n)} (f_i)\, d\alpha \\ &= \lim_{n\to+\infty} \int_{(E_1\cap A_1^n,\dots,E_d\cap A_d^n)} (g_i^n)\, d\alpha \\ &= \lim_{n\to+\infty} \gamma(E_1\cap A_1^n,\dots,E_d\cap A_d^n) \\ &= \gamma(E_1,\dots,E_d)\end{aligned}$$

for every $(E_i) \in \times \Sigma_i$. Consequently, the d-tuple (f_i) is a Radon-Nikodým derivative of the polymeasure γ with respect to the polymeasure α. □

REMARK 5. In view of condition (iii) of [9, Theorem 2.1] and Remark 3, it is natural to consider the possibility of replacing condition (4.2) in Theorem 4 by the following:

(4.6) For every $(A_i) \in (\times \Sigma_i)^+$, there exists a localized d-tuple $(B_i) \in (\times \Sigma_i)^+$ such that $(B_i) \subseteq (A_i)$.

Conditions (4.2) and (4.6) are equivalent if $d = 1$, but the following example shows that it is not possible, in general, to replace (4.2) by (4.6) in Theorem 4 when $d \geq 2$.

EXAMPLE 6. Let us consider $\Omega_i = [-1, 1]$, Σ_i the Borel σ-algebra on $[-1, 1]$, $i = 1, 2$, $X_1 = X_2 = Y = Z = \mathbb{R}$, $\phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ the ordinary product of scalars and $\alpha : \Sigma_1 \times \Sigma_2 \to \mathbb{R}$ the restriction to $\Sigma_1 \times \Sigma_2$ of the Lebesgue measure α_1 on Bor $(\Omega_1 \times \Omega_2)$.

Let γ_1 be the measure defined on the Borel σ-algebra of $\Omega_1 \times \Omega_2$, by

$$\gamma_1(B) = \int_B f\, d\alpha_1,$$

where$f = 1 - \chi_{[0,1]\times[0,1]}$.

If γ denotes the restriction of γ_1 to $\Sigma_1 \times \Sigma_2$, then it is easily proved that γ is a bimeasure verifying (4.1) and (4.6).

Let us prove that γ has no Radon-Nikodým derivative with respect to α. In fact, if there exists an α-integrable 2-tuple (f_1, f_2) such that

$$\gamma(A_1, A_2) = \int_{(A_1,A_2)} (f_1, f_2)\, d\alpha$$

for every $(A_1, A_2) \in \Sigma_1 \times \Sigma_2$, then $f_1 f_2$ is α_1-integrable and

$$\int_{A\times B} f\, d\alpha_1 = \int_{A\times B} (f_1 f_2)\, d\alpha_1$$

for every$(A, B) \in \Sigma_1 \times \Sigma_2$.

It follows now from the Fubini theorem that there exists $D \in \Sigma_1$ such that $\lambda(D) = 0$, where λ denotes the Lebesgue measure on $[-1, 1]$, $f_1(x)f_2$ is α-integrable for every $x \in \Omega_1 \setminus D$ and

$$\int_B (f(x, \cdot) - f_1(x)f_2)dy = 0$$

for every $B \in \Sigma_2$ and every $x \in \Omega_1 \setminus D$. Therefore, there exists $E \in \Sigma_2$ such that $\lambda(E) = 0$ and $f(x, y) = f_1(x)f_2(y)$ for every $(x, y) \in (\Omega_1 \times \Omega_2) \setminus (D \times E)$.

Since $\lambda(D) = \lambda(E) = 0$, there exists $x_1 \in [0, 1] \setminus D$, $y_1 \in [0, 1] \setminus E$, $x_2 \in [-1, 0] \setminus D$ and $y_2 \in [-1, 0] \setminus E$, verifying that

$$f_1(x_1)f_2(y_2) = f(x_1, y_2) = 1 = f(x_2, y_1) = f_1(x_2)f_2(y_1).$$

Therefore, $f_1(x_1) \neq 0 \neq f_2(y_1)$, and we have a contradiction because

$$f_1(x_1)f_2(y_1) = f(x_1, y_1) = 0.$$

□

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