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ON A GENERALIZED DIVISOR PROBLEM I

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Abstract. We give a discussion on the properties of $\Delta_a(x)$ $(-1 < a < 0)$, which is a generalization of the error term $\Delta(x)$ in the Dirichlet divisor problem. In particular, we study its oscillatory nature and investigate the gaps between its sign-changes for $-1/2 \le a < 0$.

§1. Introduction

Let $\sigma_a(n) = \sum_{d|n} d^a$ and define for $-1 < a < 0$ and $x \ge 1$,

$$\Delta_a(x) = {\sum_{n\le x}}' \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a}x^{1+a} + \frac{1}{2}\zeta(-a), \tag{1.1}$$

where the last term in the sum is halved when x is an integer. The limit function $\lim_{a\to 0-} \Delta_a(x)$ is the same as the classical error term $\Delta(x)$. The determination of its precise order of magnitude, called the Dirichlet's divisor problem, remains open to date. On the other hand, the highly oscillatory behaviour of $\Delta(x)$ has attracted the attention of many authors. There are numerous papers devoted to the study of properties of $\Delta(x)$, such as its power moments, $\Omega_\pm$-results, gaps between sign-changes and etc.. Correspondingly not many results are known for $\Delta_a(x)$. The mean square result is a mature one among them. In 1995, Meurman [9] proved that

$$\int_2^T \Delta_a(x)^2\,dx = \begin{cases} c_1 T^{3/2+a} + O(T) & \text{for } -1/2 < a < 0,\\ c_2 T\log T + O(T) & \text{for } a = -1/2,\\ O(T) & \text{for } -1 < a < -1/2, \end{cases} \tag{1.2}$$

where $c_1 = (6+4a)^{-1}\pi^{-2}\zeta(3/2-a)\zeta(3/2+a)\zeta(3/2)^2\zeta(3)^{-1}$, $c_2 = \zeta(3/2)^2/(24\zeta(3))$ and the constants implied by the O-symbols may depend on a. He established the result by using a weighted Voronoi type formula

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with an explicit error term of his own. The O-terms in the first and the third case are the best possible. The first one was confirmed by Lam and Tsang [7] while, in fact, there exists an asymptotic formula for the third case $(-1 < a < -1/2)$ in an old paper of Chowla [1] where it is proved that

$$\int_2^T \Delta_a(x)^2\,dx = c_3 T + O(T^{3/2+a}\log T)$$

with $c_3 = \zeta(-2a)\zeta(1-a)^2/(12\zeta(2-2a))$. These mean square results are definitely important. From them, one can see that $a = -1/2$ is a 'critical' point for the behaviour of $\Delta_a(x)$, for example, the 'average' order of $\Delta_a(x)$ is $O(x^{1/4+a/2})$ for $-1/2 < a < 0$, $O(\sqrt{\log x})$ for $a = -1/2$ and $O(1)$ for $-1 < a < -1/2$. Analogous to the case $\Delta(x)$, we expect that $\Delta_a(x) \ll x^{1/4+a/2+\epsilon}$ $(-1/2 \le a < 0)$ and $\Delta_a(x) \ll x^{\epsilon}$ $(-1 < a \le -1/2)$, both of which are still open. In the opposite direction, we can find the following $\Omega_\pm$-results: for $-1/2 \le a < 0$,

$$\Delta_a(x) = \Omega_\pm(x^{1/4+a/2}\log^{1/4+|a|/2} x), \tag{1.3}$$

and for $-1 < a < -25/38$,

$$\Delta_a(x) = \Omega_\pm\left(\exp\left((1+o(1))\frac{1}{1-|a|}\left(\frac{|a|}{2}\right)^{1-|a|}\frac{(\log x)^{1-|a|}}{\log\log x}\right)\right)$$

due to Hafner [2] and Pétermann [10] respectively.

As mentioned, the behaviour of $\Delta_a(x)$ changes at $a = -1/2$. It seems that $\Delta_a(x)$ behaves like $\Delta(x)$ only for the range $-1/2 < a < 0$ (or $-1/2 \le a < 0$). We shall investigate the properties of $\Delta_a(x)$ at different values of a.

In this paper, we shall consider the case $-1/2 \le a < 0$ (and the other case in the sequel paper [8]). Our first result is about the difference $\Delta_a(x+h) - \Delta_a(x)$ for $-1/2 \le a < 0$ in the mean. In [5], Jutila proved that for $T^\epsilon \ll h \le \sqrt{T}/2$,

$$\int_T^{2T} (\Delta(t+h) - \Delta(t))^2\,dt \asymp Th\log^3(\sqrt{T}/h).$$

Results of this kind can reveal their oscillatory natures. Parallel to this, we prove the following:

THEOREM 1. *Let $T \geq 2$ and suppose $a \in [-1/2, -\delta]$ where δ is an arbitrarily small positive number. Then, for $1 \ll h \leq \sqrt{T}$, we have*

$$\int_T^{2T} (\Delta_a(t+h) - \Delta_a(t))^2 \, dt \ll_\delta T h^{1+2a} \min\left(\frac{1}{1/2 - |a|}, \log h\right)$$

where the implied constant depends only on δ.

Remark. A recent paper of Kiuchi and Tanigawa [6] includes the result of the case $-1/2 < a < 0$ here, but not $a = -1/2$.

An application of Theorem 1 is to yield the width of gaps between sign-changes of $\Delta_a(x)$.

THEOREM 2. *For $-1/2 \leq a < 0$, we can find a sequence $\{T_n\}$ tending to infinity such that $\Delta_a(x)$ has no sign-changes in the interval $[T_n, T_n + c_a H_n]$ where c_a is a constant depending only on a, $H_n = T_n^{1/2-|a|(1/2+|a|)}(\log T_n)^{-1}$ if $-1/2 < a < 0$ and $H_n = (\log T_n)^{3/2}/(\log\log T_n)^2$ if $a = -1/2$.*

Concerning the sign-changes of $\Delta_a(x)$, by taking $g(n) = \pi^{a/2}\sigma_a(n)$, $\mu(n) = \pi n$, $r = 1+a$ and $\alpha = 1$ in Ivić [4, Theorem 1], it follows immediately an upper bound result for the length of gaps between sign-changes.

THEOREM 3. *Let $-1/2 \leq a \leq 0$ and T be any sufficiently large number. Then, $\Delta_a(x)$ has a sign-change in $[T, T + c_a\sqrt{T}]$ for some constant c_a depending on a only.*

For the case $a = 0$, Heath-Brown and Tsang [3] showed in the opposite direction that one can find a sequence $\{T_n\}$ which tends to infinity and $\Delta(x)$ has no sign-changes in the interval $[T_n, T_n + c'\sqrt{T_n}/\log^5 T_n]$ for some constant c'. These almost determined the exact order of magnitude of the gaps between sign-changes of $\Delta(x)$. However results of opposite direction for other cases are not known yet. Our Theorem 2, based on the method of Heath-Brown and Tsang, is to furnish this part. It can be seen that the value of H_n in Theorem 2 deviates away from $\sqrt{T_n}$ as a decreases from 0 to $-1/2$. The true order of magnitude of the gaps between sign-changes is still mysterious for such cases. At present, there are not enough information to predict the right order of magnitude.

Finally we want to mention that the $\Omega_\pm$-result in (1.3) for the case $a = -1/2$ can be improved.

THEOREM 4. *We have*

$$\Delta_{-1/2}(x) = \Omega_{\pm}\left(\exp\left((1+o(1))\frac{(\log x)^{1/2}}{\log\log x}\right)\right).$$

It comes from [8, Theorem 1] with $a = -1/2$. However, one should note that the definition of $\Delta_a(x)$ in [8] is different from here. Let us extend the definition of $\sigma_a(n)$ by defining $\sigma_a(x) = 0$ when x is not a positive integer. Then [8, Theorem 1] gives

$$\Delta_{-1/2}(x) + \frac{1}{2}\sigma_{-1/2}(x) = \Omega_{\pm}\left(\exp\left((1+o(1))\frac{(\log x)^{1/2}}{\log\log x}\right)\right).$$

From (1.1), it is apparent that

$$\Delta_{-1/2}(x) \le \Delta_{-1/2}(x) + \frac{1}{2}\sigma_{-1/2}(x) \le \Delta_{-1/2}(x+1) + O(1)$$

and hence Theorem 4 follows. It should be remarked that unlike the case of $-1/2 \le a \le 0$, Theorem 1 in [8] is derived by another tool instead of the Voronoi-type formula (thus, so is Theorem 4); nonetheless, the method used there in the discussion of sign-changes cannot yield results for the case $a = -1/2$. (Note that the results in Theorems 2 and 3 here include this case.) These altogether perhaps give a further support of the peculiarity of $\Delta_{-1/2}(x)$.

§2. Proof of Theorem 1

From [9, Lemma 1] with $X = 2T$, $Z = 4T$, we have for $t \in [T, 3T]$,

$$\Delta_a(t) = \Delta_a(t,T) + R_a(t,T) + O(T^{-1/4+a/2})$$

where

$$\Delta_a(t,T) = \frac{1}{\pi\sqrt{2}} t^{1/4+a/2} \sum_{n\le 4T} \frac{\sigma_a(n)}{n^{3/4+a/2}} w_T(n) \cos\left(4\pi\sqrt{nt} - \frac{\pi}{4}\right)$$

and

$$R_a(t,T) = \frac{1}{2\pi}\sum_{n\le 4T}\sigma_a(n)\int_1^2\int_{2uT}^{\infty} v^{-1}\sin(4\pi(\sqrt{t}-\sqrt{n})\sqrt{v})\,dv\,du$$

with $w_T(u) = 1$ for $1 \le u \le 2T$ and $w_T(u) = 2 - u/(2T)$ for $2T \le u \le 4T$. Then,

$$\int_T^{2T} (\Delta_a(t+h) - \Delta_a(t))^2\, dt \ll T + \int_T^{2T} (\Delta_a(t+h,T) - \Delta_a(t,T))^2\, dt \tag{2.1}$$

where the mean square value of $R_a(\cdot, T)$ is estimated by [9, (2.3)]. Now,

$$\begin{aligned}
&\int_T^{2T} (\Delta_a(t+h,T) - \Delta_a(t,T))^2\, dt \\
&\ll \int_T^{2T} t^{1/2+a} |\sum_{n\le 4T} \frac{\sigma_a(n)}{n^{3/4+a/2}} w_T(n)(e(2\sqrt{n(t+h)}) - e(2\sqrt{nt}))|^2\, dt \\
&\qquad + \int_T^{2T} ((t+h)^{1/4+a/2} - t^{1/4+a/2})^2 \\
&\qquad\qquad \times |\sum_{n\le 4T} \frac{\sigma_a(n)}{n^{3/4+a/2}} w_T(n) e(2\sqrt{n(t+h)})|^2\, dt \\
&= I_1 + I_2, \text{ say.}
\end{aligned} \tag{2.2}$$

We split I_1 into three parts as follows.

$$\begin{aligned}
I_1 \ll\ & \\
&T^{1/2+a} \int_T^{2T} |\sum_{n\le T/(2h^2)} \frac{\sigma_a(n)}{n^{3/4+a/2}} (e(2\sqrt{n}(\sqrt{t+h} - \sqrt{t})) - 1) e(2\sqrt{nt})|^2\, dt \\
&+ T^{1/2+a} \int_T^{2T} |\sum_{T/(2h^2)<n\le 4T} \frac{\sigma_a(n)}{n^{3/4+a/2}} w_T(n) e(2\sqrt{n(t+h)})|^2\, dt \\
&+ T^{1/2+a} \int_T^{2T} |\sum_{T/(2h^2)<n\le 4T} \frac{\sigma_a(n)}{n^{3/4+a/2}} w_T(n) e(2\sqrt{nt})|^2\, dt \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{2.3}$$

Following the arguments for the estimate of $J^{\pm}$ in [9, p.354-355], we see that $I_2 \ll T$ and $I_{12}, I_{13} \ll Th^{1+2a} \min((1/2 - |a|)^{-1}, \log h) + T$. With $e(2\sqrt{n}(\sqrt{t+h} - \sqrt{t})) - 1 \ll h\sqrt{n/T}$, Second Mean Value Theorem for integrals, we get for some $\xi \in [T, 2T]$,

$$I_{11} \ll T^{1/2+a} h^2 \sum_{n\le T/(2h^2)} \frac{\sigma_a(n)^2}{n^{1/2+a}} + T^{1+a} \left| \int_\xi^{2T} \sum (t) e(2(\sqrt{m} - \sqrt{n})\sqrt{t})\, \frac{dt}{\sqrt{t}} \right|$$

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where

$$\sum(t) = \sum_{m \neq n \leq T/(2h^2)} \frac{\sigma_a(m)\sigma_a(n)}{(mn)^{3/4+a/2}} (e(2\sqrt{m}(\sqrt{t+h}-\sqrt{t}))-1) \times \overline{(e(2\sqrt{n}(\sqrt{t+h}-\sqrt{t}))-1)}.$$

The first summand in the right-hand side is $\ll Th^{1+2a}$. By integration by parts, we see that the second summand is $\ll Th^{2a}$ and so $I_{11} \ll Th^{1+2a}$. Our theorem follows from (2.1)–(2.3).

Remark. A careful treatment, following the same line of arguments in Jutila [5], can furthermore lead to $\int_T^{2T} (\Delta_a(t+h)-\Delta_a(t))^2\, dt \gg_\delta Th^{1+2a}$. This was not done here for simplicity.

§3. Proof of Theorem 2

Following the method in [3], we first show that for $1 \ll H \leq \sqrt{T}$,

$$\int_T^{2T} \max_{h \leq H} (\Delta_a(t+h)-\Delta_a(t))^2\, dt \ll T(H \log H \min(\frac{1}{1/2-|a|}, \log H))^{1/(1+|a|)}. \tag{3.1}$$

To prove it, let us write $H = 2^\lambda b$, $\lambda \in \mathbf{N}$. Since for $v \leq u$,

$$\Delta_a(u) - \Delta_a(v) = \sum_{v<n\leq u}{}' \sigma_a(n) - \zeta(1-a)(u-v) - \frac{\zeta(1+a)}{1+a}(u^{1+a}-v^{1+a}) \geq -O(|u-v|),$$

we have for $jb < h \leq (j+1)b$,

$$\Delta_a(t+jb)-\Delta_a(t)-O(b) \leq \Delta_a(t+h)-\Delta_a(t) \leq \Delta_a(t+(j+1)b)-\Delta_a(t)+O(b).$$

Hence, for a fixed t, let $|\Delta_a(t+h)-\Delta_a(t)|$ attain a maximum at $h_0 = h_0(t)$ over $[0, H]$, we then have

$$\max_{h\leq H} |\Delta_a(t+h)-\Delta_a(t)| = |\Delta_a(t+h_0)-\Delta_a(t)| \leq \max_{1\leq j\leq 2^\lambda} |\Delta_a(t+jb)-\Delta_a(t)| + O(b).$$

Now, let $\max_{1\le j\le 2^\lambda} |\Delta_a(t+jb)-\Delta_a(t)| = |\Delta_a(t+j_0b)-\Delta_a(t)|$ for some $j_0 = j_0(t)$, by writing $j_0 = 2^\lambda \sum_\mu 2^{-\mu}$ where the sum runs over a certain set S_t of non-negative integers $\mu \le \lambda$, we can express it as

$$\Delta_a(t+j_0b)-\Delta_a(t) = \sum_\mu (\Delta_a(t+(\nu+1)2^{\lambda-\mu}b)-\Delta_a(t+\nu 2^{\lambda-\mu}b))$$

where $0 \le \nu = \nu_{t,\mu} < 2^\mu$ is an integer. (To be specific, $\nu = 2^\mu \sum_\alpha 2^{-\alpha}$ where α runs over S_t and satisfies $\min S_t \le \alpha < \mu$.) By Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
&(\Delta_a(t+j_0b)-\Delta_a(t))^2\\
&\le \left(\sum_\mu 1\right)\left(\sum_\mu (\Delta_a(t+(\nu+1)2^{\lambda-\mu}b)-\Delta_a(t+\nu 2^{\lambda-\mu}b))^2\right)\\
&\le (\lambda+1)\sum_\mu \sum_{0\le\nu<2^\mu} (\Delta_a(t+(\nu+1)2^{\lambda-\mu}b)-\Delta_a(t+\nu 2^{\lambda-\mu}b))^2,
\end{aligned}$$

after including all other integers $\nu \in [0, 2^\mu)$. Thus, by taking $b = (H\log H\min((1/2-|a|)^{-1}, \log H))^{1/(2+2|a|)}$, we obtain with Theorem 1,

$$\begin{aligned}
&\int_T^{2T} \max_{h\le H}(\Delta_a(t+h)-\Delta_a(t))^2\,dt\\
&\ll \lambda \sum_{\mu\le\lambda}\sum_{0\le\nu<2^\mu}\int_{T+\nu 2^{\lambda-\mu}b}^{2T+\nu 2^{\lambda-\mu}b} (\Delta_a(t+2^{\lambda-\mu}b)-\Delta_a(t))^2\,dt + Tb^2\\
&\ll T\left(H\log H\min\left(\frac{1}{|a|-1/2}, \log H\right)\right)^{1/(1+|a|)}.
\end{aligned}$$

Applying (3.1) with $H = c_aT^{(1/2+a)(1+|a|)}/\log T$ for $-1/2 < a < 0$ and $H = c_a(\log T)^{3/2}/(\log\log T)^2$ for $a = -1/2$ for some suitable small constant $c_a > 0$, we see together with (1.2) that the integral

$$\int_T^{2T} (\Delta_a(t)^2 - \max_{h\le H}(\Delta_a(t+h)-\Delta_a(t))^2)\,dt$$

is positive. Our assertion then follows.

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