FOURIER–STIELTJES TRANSFORMS WHICH VANISH AT INFINITY OFF CERTAIN SETS

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0. Introduction. In this paper G is a nondiscrete compact abelian group with character group Γ and $M(G)$ the usual convolution algebra of Borel measures on G. We designate the following subspaces of $M(G)$ employing the customary notations: $M_a(G)$ those measures which are absolutely continuous with respect to Haar measure; $M_s(G)$ the space of measures concentrated on sets of Haar measure zero and $M_d(G)$ the discrete measures.

The Fourier–Stieltjes transform of the measure $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G \gamma(-x)\, d\mu(x) \qquad (\gamma \in \Gamma).$$

The ideal of measures whose transforms vanish at infinity will be denoted by $M_0(G)$.

Let $L^p(G)$ $(1 \leq p \leq \infty)$ be the Lebesgue space of index p formed with respect to Haar measure on G and $C(G)$ those $f \in L^\infty(G)$ which are continuous. For any subspace $B(G)$ of $M(G)$ and subset E of Γ put

$$B_E(G) = \{\mu \in B(G) : \hat{\mu} = 0 \text{ off } E\}.$$

In section 1 we give several extensions of a classical theorem due to Rajchman [**23**] which we state for the circle group $\mathbf{T}$ as

THEOREM 1. *Suppose* $\mu \in M(\mathbf{T})$ *and* $\hat{\mu}(n) = o(1)$ *for all* $n > 0$ *or all* $n < 0$. *Then* $\mu \in M_0(\mathbf{T})$.

An analogue of Theorem 1 is valid for any compact abelian group with ordered dual in the sense of relating the behavior of a Fourier–Stieltjes transform at "$+\infty$" with its behavior at "$-\infty$", see [**4**, p. 230] and especially the example.

A subset $\mathfrak{R}$ of Γ will be called a *Rajchman set* if whenever $\mu \in M(G)$ and $\hat{\mu} \in C_0(\Gamma \setminus \mathfrak{R})$, then $\mu \in M_0(G)$. Here $C_0(\Gamma \setminus \mathfrak{R})$ means those complex-valued functions on Γ which vanish at infinity off $\mathfrak{R}$. In section 1 we prove the following two theorems which we now cite.

(i) The union of a Rajchman set and a Sidon set is a Rajchman set;

(ii) The union of a Rajchman set and any set $\mathscr{E}$ satisfying $M_a(G)^\wedge|_{\mathscr{E}} \subset M_d(G)^\wedge|_{\mathscr{E}}$ is a Rajchman set.

Let E be a Rider set contained in $\mathbb{Z}^+$, the positive part of $\mathbb{Z}$. We also prove in section 1 that:

(iii) $\mathbb{Z}^- \cup E^n$ is a Rajchman set.

In section 2 we present some examples of Rajchman sets in Γ. The sum sets in example (III) are especially interesting from an arithmetic point of view. In section 3 we

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exhibit a connection between Rajchman sets in the integer group $\mathbb{Z}$ and idempotent measures on the circle. The result of this section provided the original motivation for our study of Rajchman sets.

1. Union problems for Rajchman sets. A subset $\mathbf{S}$ of Γ is said to be a *Sidon set* if whenever $f \in L^\infty_{\mathbf{S}}(G)$, then $\sum |\hat{f}(\alpha)| < \infty$. Our first result is:

THEOREM 2. *The union of a Rajchman set and a Sidon set is a Rajchman set.*

Proof. We adapt the proof of [**5**]. Let $\hat{\mu} \in C_0(\Gamma \backslash \mathfrak{R} \cup S)$, where $\mathfrak{R}$ is Rajchman and $\mathbf{S}$ is Sidon. By a result of Drury, given a natural number m there is a measure $\nu_m \in M(G)$ such that

$$\hat{\nu}_m(\alpha) = 1 \qquad (\alpha \in \mathbf{S}), \tag{1}$$

$$|\hat{\nu}_m(\alpha)| < \frac{1}{m} \qquad (\alpha \notin \mathbf{S}). \tag{2}$$

Since $\mathfrak{R}$ is Rajchman it follows from (1) that the measure

$$\mu - \nu_m * \mu \in M_0(G). \tag{3}$$

On letting $m \to \infty$ in (3) we gather from (2) that $\hat{\mu} \in C_0(\Gamma \backslash \mathbf{S})$. Our result now follows from Theorem 2.20 of [**16**, p. 30].

We remark that Theorem 2.20 of [**16**, p. 30] shows that the method of proof of [**5**] works for any compact abelian group to establish that the union of a small p set and a Sidon set is a small $2p$ set. For if Γ is not ordered or the positive cone is not Rajchman then the first part of the argument of [**5**] shows that $\hat{\mu} \in C_0(\Gamma \backslash \mathbf{S})$. An improvement of [**5**] due to the author and S. Saeki can be found in [**22**, p. 91]. The reader is also referred to example I of the present paper.

Next, let $\mathscr{E}$ satisfy

$$M_a(G)^\wedge|_{\mathscr{E}} \subset M_d(G)^\wedge|_{\mathscr{E}}. \tag{$*$}$$

The interpolation property $(*)$ has been studied by many authors; the reader is referred to [**2**], [**11**], [**14**], and [**21**]. It is known that the set of prime powers and the sets $\{r^n + r^m : n, m \in \mathbb{Z}^+\}$ where $r \in \mathbb{Z}^+$ and $r \geq 2$ satisfy $(*)$ for $\Gamma = \mathbb{Z}$; see [**10**].

THEOREM 3. *The union of a Rajchman set $\mathfrak{R}$ and any set $\mathscr{E}$ satisfying $(*)$ is a Rajchman set.*

Proof. Let $\mu \in M(G)$ and suppose

$$\hat{\mu} \in C_0(\Gamma \backslash \mathscr{E} \cup \mathfrak{R}). \tag{1}$$

Let $\alpha_0 \in \mathscr{E}$ and choose $\nu \in M_a(G)$ such that

$$\hat{\nu}(\alpha_0) = 1. \tag{2}$$

Since $\mathscr{E}$ satisfies $(*)$ we gather there is a measure $\mu_d \in M_d(G)$ such that

$$\hat{\mu}_d = \hat{\nu} \quad \text{on} \quad \mathscr{E}. \tag{3}$$

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By (1)

$$(\mu_d * \mu - \nu * \mu)\hat{} \in C_0(\Gamma \backslash \mathfrak{R});$$

thus

$$\mu_d * \mu - \nu * \mu \in M_0(G). \tag{4}$$

In light of the Riemann–Lebesgue Lemma, (4) implies that

$$\mu_d * \mu \in M_0(G). \tag{5}$$

Put

$$M_0^\perp(G) = \{\rho \in M(G) : \rho \text{ is singular with each } \tau \in M_0(G)\}.$$

Then, as is well known, $M(G) = M_0(G) \oplus M_0^\perp(G)$. Since $M_0(G)$ is translation invariant and since $M_0^\perp(G)$ is translation invariant and closed, it follows that if $\rho_d \in M_d(G)$, then

$$\rho_d * M_0^\perp(G) \subset M_0^\perp(G). \tag{6}$$

Write $\mu = \mu_0 + \mu_\perp$ where $\mu_0 \in M_0(G)$ and $\mu_\perp \in M_0^\perp(G)$. We infer from (5) that $\mu_d * \mu_\perp \in M_0(G)$, so we obtain via (6) that

$$\mu_d * \mu_\perp = 0. \tag{7}$$

But (2) and (3) in combination with (7) yield $(\mu_\perp)\hat{}(\alpha_0) = 0$. Thus

$$(\mu * \mu_\perp)\hat{} \in C_0(\Gamma \backslash \mathfrak{R}) \tag{8}$$

and so $\mu * \mu_\perp \in M_0(G)$. It is now evident that $\mu_\perp * \mu_\perp \in M_0(G)$ and this is possible if and only if $\mu_\perp = 0$.

Theorem 3 implies the following result which we state without proof.

Every non-Sidon subset of a discrete abelian group contains a Rajchman set which is non-Sidon.

For the remainder of this section our notation will be for the most part that of [**16**]. In what follows Γ is ordered by the positive cone $\mathcal{P}$.

A subset S of Γ is called asymmetric if $0 \notin S$ and $\alpha \in S$ imply $-\alpha \notin S$. For any subset E of Γ and integer $s \geq 0$, $R_s(E)$ denotes the number of asymmetric subsets S of $E \cup -E$ satisfying $|S| = s$ and $\sum_{\alpha \in S} \alpha = 0$. The set E is called a *Rider set* if there is a constant $B > 0$ such that $R_s(E) \leq B^s$ for all s. For $k \in \mathbb{Z}^+$ let E_k consist of all characters of the form $\sum_{\alpha \in S} \alpha$ where S is an asymmetric subset of $E \cup -E$ and $|S| = k$.

Before presenting our result we shall need the next two propositions, which we now state for the readers' convenience.

Proposition 1. (Bonami [**3**]). *Let E be a Rider set in Γ. Then E_k is a $\Lambda(q)$ set for all $1 \leq q < \infty$ and all $k > 0$.*

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Recall that a set A is called a $\Lambda(q)$ set if there exists a $p<q$ and a constant K_p such that

$$\|t\|_q < K_p \|t\|_p$$

for all trigonometric polynomials t on G with $\hat{t}=0$ off A.

PROPOSITION 2. (Rudin [**24**]). *Let* $\mu \in M(\mathbf{T})$ *and* A *a set of type* $\Lambda(1)$. *If* supp $\hat{\mu} \subset \mathbb{Z}^- \cup A$, *then* $\mu \in M_a(\mathbf{T})$.

For a generalization of Proposition 2 see [**18**]. In this connection we point out that for the corollary of [**18**, p. 369] to be valid we must replace the word infinity by "$-\infty$" in both the statement of the corollary and its proof. We shall call a subset R of Γ a *weak Rajchman set* if whenever supp $\hat{\mu} \subset R$ then $\mu \in M_0(G)$. The set of Proposition 2 is a weak Rajchman set which is (to the author's knowledge) not known to be Rajchman. The method of Theorem 2 shows that the union of a weak Rajchman set and a Sidon set is a weak Rajchman set.

We shall be interested in showing that for certain $\Lambda(1)$ sets A (which are not Sidon) $\mathbb{Z}^- \cup A$ is a Rajchman set. For any subset E of Γ and natural number $n \geq 2$ put

$$E^n = \{n_{j_1} + n_{j_2} + \ldots + n_{j_n} : n_{j_i} \neq n_{j_k}, n_{j_i} \in E\}$$

and $E^1 = E$. We now present this extension of Rajchman's Theorem:

THEOREM 4. *Let E be a Rider set in* $\mathbb{Z}$ *such that* $E \subset \mathbb{Z}^+$. *Then* $\mathbb{Z}^- \cup E^n$ *is a Rajchman set.*

Proof. Let $\mu \in M(\mathbf{T})$ and $\hat{\mu} \in C_0(\mathbb{Z} \setminus \mathbb{Z}^- \cup E^n)$. We shall suppose $\hat{\mu} \notin C_0(\mathbb{Z})$ and force a contradiction.

If $\hat{\mu} \notin C_0(\mathbb{Z})$ then there exists by Theorem 1 a $\delta > 0$ such that the set

$$\mathcal{S} = \{\alpha \in E^n : |\hat{\mu}(\alpha)| \geq \delta\}$$

is infinite. Choose a sequence $\langle \alpha_j \rangle_1^\infty$ with the α's distinct and in $\mathcal{S}$.

Let ν be any weak-star cluster point of $\{-\alpha_j \mu\}$. Since the α's are distinct it follows from the Helson Translation Lemma that

$$\nu \in M_s(\mathbf{T}). \tag{1}$$

Observe that

$$\hat{\nu}(0) \neq 0 \tag{2}$$

since the α's belong to $\mathcal{S}$.

We shall fix $\alpha \in \mathbb{Z}^+$ and calculate $\hat{\nu}(\alpha)$. If $\alpha + \alpha_j$ meets E^n only finitely many times then, since $\hat{\mu} \in C_0(\mathbb{Z} \setminus \mathbb{Z}^- \cup E^n)$, it follows that $\hat{\nu}(\alpha) = 0$. Thus if $\alpha \in \mathbb{Z}^+$, then $\hat{\nu}(\alpha) \neq 0$ implies that

$$\alpha \in \overline{\lim_j}(E^n - \alpha_j) = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} (E^n - \alpha_k).$$

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Well, it is easy to see that

$$\bigcup_{j=1}^{\infty}(E^n-\alpha_j)\subset A, \tag{3}$$

where

$$A=\{0\}\cup E_2\cup E_4\ldots\cup E_{2n}\qquad (n\geq 1).$$

By Proposition 1 and [**23**, p. 217] this set A is a $\Lambda(q)$ set for all finite q. Inasmuch as ν is singular and supp $\hat{\nu}\subset\mathbb{Z}^-\cup A$ we obtain via Proposition 2:

$$\hat{\nu}(0)=0. \tag{4}$$

Since (4) contradicts (2) we conclude that $\hat{\mu}\,\dot{\in}\,C_0(\mathbb{Z})$; i.e. $\mu\in M_0(\mathbf{T})$ and this finishes our proof.

COROLLARY. *If E and F are disjoint with $E\cup F$ dissociate and $E\cup F\subset\mathbb{Z}^+$, then $\mathbb{Z}^-\cup(E+F)$ is a Rajchman set.*

For some interesting properties of $E+F$ where E and F are disjoint with dissociate union see [**1**].

2. Examples of Rajchman Sets. In this section we present some examples of Rajchman sets in Γ. Most of the examples appear in the literature implicitly.

EXAMPLES (I) A subset $\mathscr{E}$ of $\mathbb{Z}^+$ is said to satisfy the *lacunary condition* ($\mathscr{P}$) if for every increasing sequence $n_1, n_2, \ldots, \in E$, $\mathbb{Z}^+\cap\underline{\lim}\,(\mathscr{E}-n_j)$ is a finite set.

Then if $\mathscr{E}$ satisfies ($\mathscr{P}$), $\mathbb{Z}^-\cup\mathscr{E}$ is a Rajchman set. To confirm this assertion it suffices to repeat the proof of Theorem 3 of [**8**]. A proof for arbitrary discrete Γ that a Sidon set is a Rajchman set can be based on Theorem 1.4 of [**16**, p. 8] and the method of proof of the present example.

(II) A subset $\mathscr{F}$ of $\mathbb{Z}$ is said to be a set of *uniform convergence* or *UC*-set if every $f\in C_{\mathscr{F}}(\mathbf{T})$ has uniformly convergent Fourier series. Non-Sidon *UC*-sets were first exhibited by Figà–Talamanca; see [**16**, pp. 82–86]. Careful scrutiny of the proof of Theorem 5 of [**7**] shows that $\mathbb{Z}^-\cup\mathscr{F}$ is a Rajchman set.

(III) A subset $\mathfrak{E}$ of $\mathbb{Z}$ is defined to be a *strong Rajchman set* if $\bar{\mathfrak{E}}$ is a Rajchman set. Here $\bar{\mathfrak{E}}$ is a closure of $\mathfrak{E}$ in $\mathbb{Z}$, where $\mathbb{Z}$ has the relative topology of its Bohr compactification, $\bar{\mathbb{Z}}$. A subset $\mathscr{R}$ of $\mathbb{Z}$ is said to be a *Riesz set* if $M_{\mathscr{R}}(\mathbf{T})=L^1_{\mathscr{R}}(\mathbf{T})$. A subset $\mathscr{E}$ of $\mathbb{Z}$ is called a *strong Riesz set* if $\bar{\mathscr{E}}$ is a Riesz set.

Replacing the decomposition $M(\mathbf{T})=M_a(\mathbf{T})\oplus M_s(\mathbf{T})$ by $M_0(\mathbf{T})\oplus M_0^{\perp}(\mathbf{T})$ in [**17**] and adapting the methods of Meyer we can easily prove: *If $\mathfrak{R}$ is a Rajchman set and $\mathfrak{E}$ is a strong Rajchman set then $\mathfrak{R}\cup\mathfrak{E}$ is a Rajchman set.* In particular we have that the union of a Rajchman set and a strong Riesz set is a Rajchman set. The following examples of strong Riesz sets can be found in [**6**] and [**17**] respectively:

(i) The set of integers expressible as the sum of two perfect squares is a strong Riesz set;

(ii) Let n_k be a sequence of positive integers such that n_{k+1}/n_k is an integer >2 for all k.

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Then the set of all finite sums of the form

$$\sum_{k\geq 0} t_k n_k \quad \text{with} \quad t_k \in \{0, 1\}, \qquad n_i \neq n_j$$

is a strong Riesz set.

Here is another example due to the author and S. Saeki:

Fix any two sequences $\langle p_n \rangle_1^\infty$ and $\langle q_n \rangle_1^\infty$ of natural numbers ≥ 2. Let

$$\mathscr{E}_n = \{p_1 p_2 \ldots p_n k : k = 0, \pm 1, \ldots, \pm q_n\}$$

and put $\mathscr{E} = \bigcup_1^\infty \mathscr{E}_n$. Then $\mathscr{E} + \ldots + \mathscr{E}$ (any finite number of summands) is a strong Riesz set.

To see this let

$$\mathbf{D} = \{e^{2\pi i m/p_1 \cdots p_n} : m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+\}.$$

Consider $\mathbf{D}$ with the discrete topology and $\hat{\mathbf{D}}$ the compact dual of $\mathbf{D}$. It is easy to prove that the only accumulation point of $\mathscr{E}^+$ in $\hat{\mathbf{D}}$ is 0; see [**13**, p. 107] and [**13**, p. 403]. Since $\hat{\mathbf{D}}$ is a factor group of $\bar{\mathbb{Z}}$ and $\mathbf{D}$ is dense in $\mathbf{T}$ we gather that the set of accumulation points of $\mathscr{E}^+$ in $\mathbb{Z}$ (with the relative Bohr topology) is a subset of $\{0\}$. Thus $\mathscr{E}^+$ is strong Riesz and moreover since $\mathscr{E}$ is symmetric we have that $\mathscr{E}^-$ is strong Riesz. Thus $\mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^-$ is strong Riesz.

Now the set of accumulation points for $(\mathscr{E}+\mathscr{E})^+$ in $\hat{\mathbf{D}}$ is a subset of $\{0\} \cup \mathscr{E}$. So since $\mathscr{E}$ is strong Riesz it follows from [**17**, p. 90] that $(\mathscr{E}+\mathscr{E})^+$ is strong Riesz. Thus $\mathscr{E}+\mathscr{E} = (\mathscr{E}+\mathscr{E})^+ \cup (\mathscr{E}+\mathscr{E})^-$ is strong Riesz and the proof for any finite number of summands follows inductively.

(IV) Let $\mathfrak{R}$ be a Rajchman set in Γ. Suppose $\mathfrak{A} \subset \Gamma$ has the property that

$$\{(\mathfrak{A} - \alpha) \cap \mathfrak{A}\} \text{ is a Rajchman set}$$

for all $\alpha \notin \mathfrak{R}$. Then $\mathfrak{A}$ is a Rajchman set.

The proof of the above statement is similar to the proof of Theorem 2 of [**19**, p. 77] and we omit the details.

3. Weak Rajchman sets and idempotent measures. Using a remarkable result of K. de Leeuw and Y. Katznelson, we establish a connection between idempotent measures and Rajchman sets.

THEOREM 5. *Suppose R is a weak Rajchman set in $\mathbb{Z}$ and $\mu \in M(\mathbf{T})$ such that $\hat{\mu} = \hat{\mu}^2$ off R. Then there is an idempotent measure $\nu \in M(\mathbf{T})$ such that $\hat{\mu} = \hat{\nu}$ off R.*

Proof. Let $\mu \in M(\mathbf{T})$ with

$$\hat{\mu} = \hat{\mu}^2 \quad \text{off} \quad R. \tag{1}$$

Via (1)

$$\limsup_{n \notin R} |\hat{\mu}(n) - \hat{\mu}^2(n)| = 0$$

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and since R is weak Rajchman, we gather that

$$\limsup_{n\in\mathbf{Z}} |\hat{\mu}(n)-\hat{\mu}^2(n)|=0. \tag{2}$$

As a consequence of Theorem 2 of [**4**, pp. 220–221] (2) gives:

$$\mu=\mu_1+\mu_2, \qquad \hat{\mu}_1 \text{ periodic with } \mu_1 \text{ idempotent} \tag{3}$$

and

$$\limsup_{n\in\mathbf{Z}} |\hat{\mu}_2(n)|=0. \tag{4}$$

As a consequence of (1), (3), and (4), we deduce that $\hat{\mu}=\hat{\mu}_1$ off $R\backslash F$ where F is some finite set. Thus, since $\hat{\mu}_1$ is periodic, we are done.

Theorem 5 is an extension of a theorem of H. Helson [**12**]; see also [**15**] and [**20**]. We conclude our paper with some open questions.

(i) If $\mathcal{R}$ is a Riesz set must $\mathcal{R}$ be a Rajchman set?

(ii) Is the union of a Rajchman set and a UC-set a Rajchman set? In this connection see example (II) of section 2.

A subset $\mathbf{A}\subset\Gamma$ is called a *Rosenthal set* if $L^\infty_{\mathbf{A}}(G)=C_{\mathbf{A}}(G)$. It is known that the sum sets $\mathscr{E}+\ldots+\mathscr{E}$ in example (III) of section 2 are Rosenthal sets. The following question suggests itself:

(iii) Is the union of a Rajchman set and a Rosenthal set a Rajchman set? An analogous result can be found in [**9**].

After our manuscript had been accepted for publication the author generalized Theorem 5 to compact abelian groups.

Note added in proof. The author has recently learned of the work of Keiji Izuchi, Sidon sets and small M_0-sets, *Sci. Rep. Tokyo Kyoiku Daigaku Sect. A* **12** (1974), 146–148. Izuchi proves Theorem 2 of the present paper using the method of [**5**]. Theorem 2 was also known to the author in 1974.

REFERENCES

1. R. C. Blei, A tensor approach to interpolation phenomena in discrete abelian groups, *Proc. Amer. Math. Soc.* **49** (1975), 175–177.

2. R. C. Blei, On Fourier–Stieltjes transforms of discrete measures, *Math. Scand.* **35** (1974), 211–214.

3. A. Bonami, Etude des coefficients de Fourier des fonctions de $L^p(G)$, *Ann. Inst. Fourier* (*Grenoble*) **20** (1970), 335–402.

4. K. de Leeuw and Y. Katznelson, The two sides of a Fourier–Stieltjes transform and almost idempotent measures, *Israel J. Math.* **8** (1970), 213–229.

5. R. E. Dressler, W. Parker and L. Pigno, Sidon sets and small p sets, *Quart, J. Math. Oxford Ser.* 2, **24** (1973), 79–80.

6. R. E. Dressler and L. Pigno, On strong Riesz sets, *Colloq. Math.* **29** (1974), 157–158.

7. R. E. Dressler and L. Pigno, Sets of uniform convergence and strong Riesz sets, *Math. Ann.* **211** (1974), 227–231.

8. R. E. Dressler and L. Pigno, Some lacunary conditions for Fourier–Stieltjes transforms, *Ark. Mat.* **13** (1975), 73–77.

9. R. E. Dressler and L. Pigno, Une remarque sur les ensembles de Rosenthal et Riesz, *C.R. Acad. Sci. Paris*, **280** (1975), 280–281.

10. R. E. Dressler and L. Pigno, Modification sets and transforms of discrete measures, *Acta. Sci. Math.* (*Szeged*) **38** (1976), 13–16.

11. S. Hartman and C. Ryll–Nardzewski, Almost periodic extensions of functions, II, *Colloq. Math.* **15** (1966), 79–86.

12. H. Helson, On a theorem of Szegö, *Proc. Amer. Math. Soc.* **6** (1955), 235–242.

13. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, Vol. 1* (Springer Verlag, 1963).

14. J.-P. Kahane, Ensembles de Ryll–Nardjewski et ensembles de Helson, *Colloq. Math.* **15** (1966), 87–92.

15. I. Kessler, Semi-idempotent measures on abelian groups, *Bull. Amer. Math. Soc.* **73** (1967), 258–260.

16. J. M. López and K. A. Ross, *Sidon Sets* (Marcel Dekker, Inc., New York, 1975).

17. Y. Meyer, Spectres des mesures et mesures absolument continues, *Studia Math.* **30** (1968), 87–99.

18. L. Pigno, A variant of the F. and M. Riesz theorem, *J. London Math. Soc.* (2), **9** (1974), 368–370.

19. L. Pigno, Convolution products with small Fourier–Stieltjes transforms, *Illinois J. Math.* **19** (1975), 77–78.

20. L. Pigno, Integer-valued Fourier–Stieltjes transforms, Technical Report No. 53 Kansas State University (1976).

21. L. Pigno and S. Saeki, Interpolation by transforms of discrete measures, *Proc. Amer. Math. Soc.* **52** (1975), 156–158.

22. L. Pigno and S. Saeki, Fourier–Stieltjes Transform which vanish at infinity, *Math. Z.* **141** (1975), 83–91.

23. A. Rajchman, Une classe de series trigonometriques . . . *Math. Ann.* **101** (1929), 686–700.

24. W. Rudin, Trigonometric series with gaps, *J. Math. and Mech.* **9** (1970), 203–228.

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