FOURIER-STIELTJES TRANSFORMS WHICH VANISH AT INFINITY OFF CERTAIN SETS

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0. Introduction. In this paper $G$ is a nondiscrete compact abelian group with character group $\Gamma$ and $M(G)$ the usual convolution algebra of Borel measures on $G$. We designate the following subspaces of $M(G)$ employing the customary notations: $M_a(G)$ those measures which are absolutely continuous with respect to Haar measure; $M_c(G)$ the space of measures concentrated on sets of Haar measure zero and $M_d(G)$ the discrete measures.

The Fourier–Stieltjes transform of the measure $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G \gamma(-x) \, d\mu(x) \quad (\gamma \in \Gamma).$$

The ideal of measures whose transforms vanish at infinity will be denoted by $M_0(G)$.

Let $L^p(G)$ ($1 \leq p \leq \infty$) be the Lebesgue space of index $p$ formed with respect to Haar measure on $G$ and $C(G)$ those $f \in L^\infty(G)$ which are continuous. For any subspace $B(G)$ of $M(G)$ and subset $E$ of $\Gamma$ put

$$B_E(G) = \{\mu \in B(G) : \hat{\mu} = 0 \text{ off } E\}.$$

In section 1 we give several extensions of a classical theorem due to Rajchman [23] which we state for the circle group $T$ as

**Theorem 1.** Suppose $\mu \in M(T)$ and $\hat{\mu}(n) = o(1)$ for all $n > 0$ or all $n < 0$. Then $\mu \in M_0(T)$.

An analogue of Theorem 1 is valid for any compact abelian group with ordered dual in the sense of relating the behavior of a Fourier–Stieltjes transform at "$+\infty$" with its behavior at "$-\infty$", see [4, p. 230] and especially the example.

A subset $\mathcal{R}$ of $\Gamma$ will be called a Rajchman set if whenever $\mu \in M(G)$ and $\hat{\mu} \in C_0(\Gamma \setminus \mathcal{R})$, then $\mu \in M_0(G)$. Here $C_0(\Gamma \setminus \mathcal{R})$ means those complex-valued functions on $\Gamma$ which vanish at infinity off $\mathcal{R}$. In section 1 we prove the following two theorems which we now cite.

(i) The union of a Rajchman set and a Sidon set is a Rajchman set;

(ii) The union of a Rajchman set and any set $\mathcal{S}$ satisfying $M_d(G)^\uparrow \subseteq M_d(G)^\uparrow_{\mathcal{S}}$ is a Rajchman set.

Let $E$ be a Rider set contained in $\mathbb{Z}^+$, the positive part of $\mathbb{Z}$. We also prove in section 1 that:

(iii) $\mathbb{Z}^- \cup E^\circ$ is a Rajchman set.

In section 2 we present some examples of Rajchman sets in $\Gamma$. The sum sets in example (III) are especially interesting from an arithmetic point of view. In section 3 we
exhibit a connection between Rajchman sets in the integer group $\mathbb{Z}$ and idempotent measures on the circle. The result of this section provided the original motivation for our study of Rajchman sets.

1. Union problems for Rajchman sets. A subset $S$ of $\Gamma$ is said to be a Sidon set if whenever $f \in L^\infty_\mathbb{R}(G)$, then $\sum |\hat{f}(\alpha)| < \infty$. Our first result is:

**Theorem 2.** The union of a Rajchman set and a Sidon set is a Rajchman set.

**Proof.** We adapt the proof of [5]. Let $\hat{\mu} \in C_0(\Gamma \setminus \mathcal{H} \cup S)$, where $\mathcal{H}$ is Rajchman and $S$ is Sidon. By a result of Drury, given a natural number $m$ there is a measure $\nu_m \in M(G)$ such that

$$\hat{\nu}_m(\alpha) = \begin{cases} 1 & (\alpha \in S), \\ \frac{1}{m} & (\alpha \notin S). \end{cases}$$

Since $\mathcal{H}$ is Rajchman it follows from (1) that the measure $\mu - \nu_m \ast \mu \in M_0(G)$. On letting $m \to \infty$ in (3) we gather from (2) that $\hat{\mu} \in C_0(\Gamma \setminus S)$. Our result now follows from Theorem 2.20 of [16, p. 30].

We remark that Theorem 2.20 of [16, p. 30] shows that the method of proof of [5] works for any compact abelian group to establish that the union of a small $p$ set and a Sidon set is a small $2p$ set. For if $\Gamma$ is not ordered or the positive cone is not Rajchman then the first part of the argument of [5] shows that $\hat{\mu} \in C_0(\Gamma \setminus S)$. An improvement of [5] due to the author and S. Saeki can be found in [22, p. 91]. The reader is also referred to example 1 of the present paper.

Next, let $G$ satisfy

$$M_a(G)^\mathcal{E} \subseteq M_d(G)^\mathcal{E}. \quad (*)$$

The interpolation property ($\ast$) has been studied by many authors; the reader is referred to [2], [11], [14], and [21]. It is known that the set of prime powers and the sets $\{r^n + r^m : n, m \in \mathbb{Z}^+\}$ where $r \in \mathbb{Z}^+$ and $r \geq 2$ satisfy ($\ast$) for $\Gamma = \mathbb{Z}$; see [10].

**Theorem 3.** The union of a Rajchman set $\mathcal{H}$ and any set $\mathcal{E}$ satisfying ($\ast$) is a Rajchman set.

**Proof.** Let $\mu \in M(G)$ and suppose

$$\hat{\mu} \in C_0(\Gamma \setminus \mathcal{E} \cup \mathcal{H}).$$

Let $\alpha_0 \in \mathcal{E}$ and choose $\nu \in M_a(G)$ such that

$$\hat{\nu}(\alpha_0) = 1.$$  

Since $\mathcal{E}$ satisfies ($\ast$) we gather there is a measure $\mu_d \in M_d(G)$ such that

$$\hat{\mu}_d = \hat{\nu} \quad \text{on} \quad \mathcal{E}.$$
By (1)

\[(\mu_d * \mu - \nu * \mu)^\wedge \in C_0(\Gamma \setminus \mathfrak{H})\;,
\]

thus

\[\mu_d * \mu - \nu * \mu \in M_0(G). \tag{4}\]

In light of the Riemann–Lebesgue Lemma, (4) implies that

\[\mu_d * \mu \in M_0(G). \tag{5}\]

Put

\[M_0^0(G) = \{ \rho \in M(G) : \rho \text{ is singular with each } \tau \in M_0(G) \}.\]

Then, as is well known, \(M(G) = M_0(G) \oplus M_0^0(G)\). Since \(M_0(G)\) is translation invariant and since \(M_0^0(G)\) is translation invariant and closed, it follows that if \(\rho_d \in M_d(G)\), then

\[\rho_d * M_0^0(G) \subset M_0^0(G) \tag{6}\]

Write \(\mu = \mu_0 + \mu_\perp\) where \(\mu_0 \in M_0(G)\) and \(\mu_\perp \in M_0^0(G)\). We infer from (5) that \(\mu_d * \mu_\perp \in M_0(G)\), so we obtain via (6) that

\[\mu_d * \mu_\perp = 0. \tag{7}\]

But (2) and (3) in combination with (7) yield \((\mu_\perp)^\wedge(\alpha_0) = 0\). Thus

\[(\mu * \mu_\perp)^\wedge \in C_0(\Gamma \setminus \mathfrak{H}) \tag{8}\]

and so \(\mu * \mu_\perp \in M_0(G)\). It is now evident that \(\mu_\perp * \mu_\perp \in M_0(G)\) and this is possible if and only if \(\mu_\perp = 0\).

Theorem 3 implies the following result which we state without proof.

Every non-Sidon subset of a discrete abelian group contains a Rajchman set which is non-Sidon.

For the remainder of this section our notation will be for the most part that of [16]. In what follows \(\Gamma\) is ordered by the positive cone \(\mathfrak{P}\).

A subset \(S\) of \(\Gamma\) is called asymmetric if \(0 \notin S\) and \(\alpha \in S\) imply \(-\alpha \notin S\). For any subset \(E\) of \(\Gamma\) and integer \(s \geq 0\), \(R_s(E)\) denotes the number of asymmetric subsets \(S\) of \(E \cup -E\) satisfying \(|S| = s\) and \(\sum \alpha = 0\). The set \(E\) is called a Rider set if there is a constant \(B > 0\) such that \(R_s(E) \leq B^s\) for all \(s\). For \(k \in \mathbb{Z}^+\) let \(E_k\) consist of all characters of the form \(\sum \alpha\) where \(S\) is an asymmetric subset of \(E \cup -E\) and \(|S| = k\).

Before presenting our result we shall need the next two propositions, which we now state for the readers’ convenience.

Proposition 1. (Bonami [3]). Let \(E\) be a Rider set in \(\Gamma\). Then \(E_k\) is a \(\Lambda(q)\) set for all \(1 \leq q < \infty\) and all \(k > 0\).
Recall that a set $A$ is called a $\Lambda(q)$ set if there exists a $p < q$ and a constant $K_p$ such that

$$\|t\|_q < K_p \|t\|_p$$

for all trigonometric polynomials $t$ on $G$ with $t = 0$ off $A$.

**Proposition 2.** (Rudin [24]). Let $\mu \in M(T)$ and $A$ a set of type $\Lambda(1)$. If $\text{supp } \hat{\mu} \subset \mathbb{Z}^- \cup A$, then $\mu \in M_s(T)$.

For a generalization of Proposition 2 see [18]. In this connection we point out that for the corollary of [18, p. 369] to be valid we must replace the word infinity by $\infty$ in both the statement of the corollary and its proof. We shall call a subset $R$ of $\Gamma$ a weak Rajchman set if whenever $\text{supp } \hat{\mu} \subset R$ then $\mu \in M_0(G)$. The set of Proposition 2 is a weak Rajchman set which is (to the author's knowledge) not known to be Rajchman. The method of Theorem 2 shows that the union of a weak Rajchman set and a Sidon set is a weak Rajchman set.

We shall be interested in showing that for certain $\Lambda(1)$ sets $A$ (which are not Sidon) $\mathbb{Z}^- \cup A$ is a Rajchman set. For any subset $E$ of $\Gamma$ and natural number $n \geq 2$ put

$$E^n = \{n_1 + n_2 + \ldots + n_k : n_i \neq n_j, n_i \in E\}$$

and $E^1 = E$. We now present this extension of Rajchman's Theorem:

**Theorem 4.** Let $E$ be a Rider set in $\mathbb{Z}$ such that $E \subset \mathbb{Z}^+$. Then $\mathbb{Z}^- \cup E^n$ is a Rajchman set.

**Proof.** Let $\mu \in M(T)$ and $\hat{\mu} \in C_0(\mathbb{Z} \setminus \mathbb{Z}^- \cup E^n)$. We shall suppose $\hat{\mu} \notin C_0(\mathbb{Z})$ and force a contradiction.

If $\hat{\mu} \notin C_0(\mathbb{Z})$ then there exists by Theorem 1 a $\delta > 0$ such that the set

$$\mathcal{S} = \{\alpha \in E^n : |\hat{\mu}(\alpha)| \geq \delta\}$$

is infinite. Choose a sequence $\langle \alpha_i \rangle^\infty_{i=1}$ with the $\alpha$'s distinct and in $\mathcal{S}$.

Let $\nu$ be any weak-star cluster point of $\{-\alpha_i \mu\}$. Since the $\alpha$'s are distinct it follows from the Helson Translation Lemma that

$$\nu \in M_s(T). \quad (1)$$

Observe that

$$\hat{\nu}(0) \neq 0 \quad (2)$$

since the $\alpha$'s belong to $\mathcal{S}$.

We shall fix $\alpha \in \mathbb{Z}^+$ and calculate $\hat{\nu}(\alpha)$. If $\alpha + \alpha_i$ meets $E^n$ only finitely many times then, since $\hat{\mu} \in C_0(\mathbb{Z} \setminus \mathbb{Z}^- \cup E^n)$, it follows that $\hat{\nu}(\alpha) = 0$. Thus if $\alpha \in \mathbb{Z}^+$, then $\hat{\nu}(\alpha) \neq 0$ implies that

$$\alpha \in \liminf_j (E^n - \alpha_i) = \bigcap_{m=1}^\infty \bigcup_{k=1}^\infty (E^n - \alpha_k).$$
Well, it is easy to see that

$$\bigcup_{j=1}^{\infty} (E^n - \alpha_j) \subseteq A,$$

where

$$A = \{0\} \cup E_2 \cup E_4 \ldots \cup E_{2n}, \quad (n \geq 1).$$

By Proposition 1 and [23, p. 217] this set $A$ is a $\Lambda(q)$ set for all finite $q$. Inasmuch as $\nu$ is singular and $\nu \subseteq \mathbb{Z}^+ \cup A$ we obtain via Proposition 2:

$$\hat{\nu}(0) = 0. \quad (4)$$

Since (4) contradicts (2) we conclude that $\hat{\mu} \in C_0(\mathbb{Z})$; i.e. $\mu \in M_0(T)$ and this finishes our proof.

**Corollary.** If $E$ and $F$ are disjoint with $E \cup F$ dissociate and $E \cup F \subseteq \mathbb{Z}^+$, then $\mathbb{Z}^+ \cup (E + F)$ is a Rajchman set.

For some interesting properties of $E + F$ where $E$ and $F$ are disjoint with dissociate union see [1].

**2. Examples of Rajchman Sets.** In this section we present some examples of Rajchman sets in $T$. Most of the examples appear in the literature implicitly.

**Examples (I)** A subset $\mathcal{E}$ of $\mathbb{Z}^+$ is said to satisfy the lacunary condition $(\mathcal{P})$ if for every increasing sequence $n_1, n_2, \ldots, \in E$, $\mathbb{Z}^+ \cap \lim (\mathcal{E} - n_i)$ is a finite set. Then if $\mathcal{E}$ satisfies $(\mathcal{P})$, $\mathbb{Z}^+ \cup \mathcal{E}$ is a Rajchman set. To confirm this assertion it suffices to repeat the proof of Theorem 3 of [8]. A proof for arbitrary discrete $T$ that a Sidon set is a Rajchman set can be based on Theorem 1.4 of [16, p. 8] and the method of proof of the present example.

**II** A subset $\mathcal{F}$ of $\mathbb{Z}$ is said to be a set of uniform convergence or UC-set if every $f \in C_0(T)$ has uniformly convergent Fourier series. Non-Sidon UC-sets were first exhibited by Figà-Talamanca; see [16, pp. 82–86]. Careful scrutiny of the proof of Theorem 5 of [7] shows that $\mathbb{Z}^+ \cup \mathcal{F}$ is a Rajchman set.

**III** A subset $\mathcal{G}$ of $\mathbb{Z}$ is defined to be a strong Rajchman set if $\mathcal{G}$ is a Rajchman set. Here $\overline{\mathcal{G}}$ is a closure of $\mathcal{G}$ in $\mathbb{Z}$, where $\mathbb{Z}$ has the relative topology of its Bohr compactification, $\tilde{\mathbb{Z}}$. A subset $\mathcal{H}$ of $\mathbb{Z}$ is said to be a Riesz set if $M_{\mathcal{H}}(T) = L^1_{\mathcal{H}}(T)$. A subset $\mathcal{E}$ of $\mathbb{Z}$ is called a strong Riesz set if $\overline{\mathbb{E}}$ is a Riesz set.

Replacing the decomposition $M(T) = M_a(T) \oplus M_s(T)$ by $M_0(T) \oplus M_0^c(T)$ in [17] and adapting the methods of Meyer we can easily prove: If $\mathcal{H}$ is a Rajchman set and $\mathcal{E}$ is a strong Rajchman set then $\mathcal{H} \cup \mathcal{E}$ is a Rajchman set. In particular we have that the union of a Rajchman set and a strong Riesz set is a Rajchman set. The following examples of strong Riesz sets can be found in [6] and [17] respectively:

(i) The set of integers expressible as the sum of two perfect squares is a strong Riesz set;

(ii) Let $n_k$ be a sequence of positive integers such that $n_{k+1}/n_k$ is an integer $> 2$ for all $k$.
Then the set of all finite sums of the form
\[ \sum_{k \geq 0} t_k n_k \] with \( t_k \in \{0, 1\}, \quad n_i \neq n_j \)
is a strong Riesz set.

Here is another example due to the author and S. Saeki:
Fix any two sequences \( (p_n)_1^\infty \) and \( (q_n)_1^\infty \) of natural numbers \( \geq 2 \). Let
\[ \mathcal{E}_n = \{ p_1 p_2 \ldots p_k : k = 0, \pm 1, \ldots, \pm q_n \} \]
and put \( \mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n \). Then \( \mathcal{E} + \ldots + \mathcal{E} \) (any finite number of summands) is a strong Riesz set.

To see this let
\[ D = \{ e^{2\pi i m/p : \ldots p_n : m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+} \}. \]
Consider \( D \) with the discrete topology and \( \hat{D} \) the compact dual of \( D \). It is easy to prove that the only accumulation point of \( \mathcal{E}^+ \) in \( \hat{D} \) is 0; see [13, p. 107] and [13, p. 403]. Since \( \hat{D} \) is a factor group of \( \hat{\mathbb{Z}} \) and \( D \) is dense in \( T \) we gather that the set of accumulation points of \( \mathcal{E}^+ \) in \( \mathbb{Z} \) (with the relative Bohr topology) is a subset of \{0\}. Thus \( \mathcal{E}^+ \) is strong Riesz and moreover since \( \mathcal{E} \) is symmetric we have that \( \mathcal{E}^- \) is strong Riesz. Thus \( \mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- \) is strong Riesz.

Now the set of accumulation points for \( (\mathcal{E} + \mathcal{E})^+ \) in \( \hat{D} \) is a subset of \{0\} \cup \mathcal{E}. So since \( \mathcal{E} \) is strong Riesz it follows from [17, p. 90] that \( (\mathcal{E} + \mathcal{E})^+ \) is strong Riesz. Thus \( \mathcal{E} + \mathcal{E} = (\mathcal{E} + \mathcal{E})^+ \cup (\mathcal{E} + \mathcal{E})^- \) is strong Riesz and the proof for any finite number of summands follows inductively.

(IV) Let \( \mathcal{R} \) be a Rajchman set in \( \Gamma \). Suppose \( \mathcal{U} \subset \Gamma \) has the property that
\[ \{(\mathcal{U} - \alpha) \cap \mathcal{U}\} \text{ is a Rajchman set} \]
for all \( \alpha \in \mathcal{R} \). Then \( \mathcal{U} \) is a Rajchman set.

The proof of the above statement is similar to the proof of Theorem 2 of [19, p. 77] and we omit the details.

3. Weak Rajchman sets and idempotent measures. Using a remarkable result of K. de Leeuw and Y. Katznelson, we establish a connection between idempotent measures and Rajchman sets.

**Theorem 5.** Suppose \( R \) is a weak Rajchman set in \( \mathbb{T} \) and \( \mu \in M(\mathbb{T}) \) such that \( \hat{\mu} = \hat{\mu}^2 \) off \( R \). Then there is an idempotent measure \( \nu \in M(\mathbb{T}) \) such that \( \hat{\mu} = \hat{\nu} \) off \( R \).

**Proof.** Let \( \mu \in M(\mathbb{T}) \) with
\[ \hat{\mu} = \hat{\mu}^2 \text{ off } R. \] Via (1)
\[ \limsup_{n \notin R} |\hat{\mu}(n) - \hat{\mu}^2(n)| = 0 \]
and since $R$ is weak Rajchman, we gather that

$$
\limsup_{n \in \mathbb{Z}} |\hat{\mu}(n) - \hat{\mu}^2(n)| = 0. 
$$

(2)

As a consequence of Theorem 2 of [4, pp. 220–221] (2) gives:

$$
\mu = \mu_1 + \mu_2, \quad \hat{\mu}_1 \text{ periodic with } \mu_1 \text{ idempotent}
$$

(3)

and

$$
\limsup_{n \in \mathbb{Z}} |\hat{\mu}_2(n)| = 0. 
$$

(4)

As a consequence of (1), (3), and (4), we deduce that $\hat{\mu} = \hat{\mu}_1$ off $R \setminus F$ where $F$ is some finite set. Thus, since $\hat{\mu}_1$ is periodic, we are done.

Theorem 5 is an extension of a theorem of H. Helson [12]; see also [15] and [20]. We conclude our paper with some open questions.

(i) If $R$ is a Riesz set must $R$ be a Rajchman set?

(ii) Is the union of a Rajchman set and a UC-set a Rajchman set? In this connection see example (II) of section 2.

A subset $A \subseteq \Gamma$ is called a Rosenthal set if $L_A^\infty(G) = C_A(G)$. It is known that the sum sets $\mathcal{S} + \ldots + \mathcal{S}$ in example (III) of section 2 are Rosenthal sets. The following question suggests itself:

(iii) Is the union of a Rajchman set and a Rosenthal set a Rajchman set? An analogous result can be found in [9].

After our manuscript had been accepted for publication the author generalized Theorem 5 to compact abelian groups.


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