RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_4$

TAKAO WATANABE

Introduction

Let $G = Sp_4$ be the symplectic group of degree two defined over an algebraic number field $F$ and $K$ the standard maximal compact subgroup of the adele group $G(A)$. By the general theory of Eisenstein series ([14]), one knows that the Hilbert space $L^2(G(F) \backslash G(A))$ has an orthogonal decomposition of the form

$$L^2(G(F) \backslash G(A)) = L^2(G) \oplus L^2(B) \oplus L^2(P_1) \oplus L^2(P_2),$$

where $B$ is a Borel subgroup and $P_i$ are standard maximal parabolic subgroups in $G$ for $i = 1, 2$. The purpose of this paper is to study the space $L^2_2(B)$ associated to discrete spectrums in $L^2(B)$.

In order to obtain such discrete spectrums, we follow Langlands’ way. To be more precise, let $T = \{ t(a, b) = \text{diag}(a, b, a^{-1}, b^{-1}) \}$ be a maximal split torus and $B = NT$ a Levi decomposition. For any quadratic character $\mu$ of $F^* \backslash A^*$, the character $\chi(\mu, \mu)$ of $T(A)$ is defined to be $\chi(\mu, \mu)(t(a, b)) = \mu(ab)$ for $t(a, b) \in T(A)$. Let $I(\chi(\mu, \mu) \backslash A)$ be the space of functions $\Phi$ on $G(A)$ satisfying $\Phi(ntg) = \chi(\mu, \mu)(t)\Phi(g)$ for any $n \in N(A)$, $t \in T(A)$ and $g \in G(A)$, and let

$$I(\mu, \beta_1) = \text{Ind} (B(A) \uparrow G(A); e^{<\beta_1, H>} \chi(\mu, \mu))$$

be the normalized induced representation of $G(A)$, where $\beta_1$ is a fundamental weight of $G$. For an admissible vector $\Phi \in I(\chi(\mu, \mu) \backslash A)$, one can define the Eisenstein series $E(g, \Phi, \Lambda)$ and take its iterated residue $\text{Res}_{\Lambda = \beta_1} E^1(g, \Phi, \Lambda)$, $E^1(g, \Phi, \Lambda) = \text{Res}_{\Lambda = \beta_1} E(g, \Phi, \Lambda)$. Through this procedure, one obtains a mapping

$$\mathcal{R}_\mu : \Phi \mapsto \text{Res}_{\Lambda = \beta_1} E^1(g, \Phi, \Lambda)$$

Received April 23, 1990
from the space of admissible vectors in \( I(\mu, \beta_1) \) into the space of automorphic forms on \( G(A) \). In this situation, the following are conjectured.

**Conjecture.** For each nontrivial quadratic character \( \mu \) of \( F^* \backslash A^* \),

1. \( \mathcal{R}_\mu \) is nontrivial.
2. The image of \( \mathcal{R}_\mu \) is contained in \( L^2(B) \), so that \( \mathcal{R}_\mu \) is extended to an intertwining operator from \( I(\mu, \beta_1) \) into \( L^2(B) \).
3. \( \pi(\mu) \), the image of this intertwining operator, is irreducible and is of multiplicity one in \( L^2(B) \).

On condition that these conjectures are true, it seems \( I(\mu) \) has an irreducible decomposition

\[
I(\mu) \cong \bigoplus_{\mu} \lambda(\mu)
\]

where \( \mu \) runs over all nontrivial quadratic characters of \( F^* \backslash A^* \) and \( \lambda(\mu) \) denotes the space of constant functions. In this paper, we will show that a part of the conjecture is true for \( \mu \) with "square free conductor".

Now we explain contents of this paper. Let \( S \) be a finite set of finite places of \( F \). For \( v \in S \), \( k_v \) denotes the residual field of \( F_v \). Let \( r_v : K_v \to \text{Sp}^4(k_v) \) be the reduction homomorphism for \( v \in S \) and \( K_S = \{(k_v) \in K \mid k_v \in \text{Ker}(r_v) \text{ for } v \in S \} \) a normal subgroup of \( K \). We study the subspace \( L^2(B, K_S) \) consisting of \( K_S \)-invariant elements of \( L^2(B) \), which becomes naturally a representation space of the finite group \( K/K_S \). Thus we are interested in an irreducible decomposition of \( L^2(B, K_S) \).

First, it is not hard to show that there is a decomposition

\[
L^2(B, K_S) = \bigoplus_{\mu \in A_S(P)} L^2(B, K_S, X(\mu))
\]

(corollary to Proposition 1). Here, \( A_S(P) \) is the set of characters \( \mu \) of \( F^* \backslash A^* \) such that the corresponding \( \chi(\mu, \mu) \) are characters of \( T(F) \backslash T^1/(K_S \cap T(A)) \) of order at most 2 and \( L^2(B, K_S, X(\mu)) \) is the space generated by residues of \( E^1(g, \Phi, \Lambda) \) for \( K_S \)-invariant elements \( \Phi \in I(\chi(\mu, \mu))/A \).

Next, by further calculations of residues of Eisenstein series, it is shown that \( \mu = \mu_0 \) is the trivial character then \( L^2(B, K_S, X(\mu_0)) \) consists of constant functions (Theorem 2) and if \( \mu \) is nontrivial then one has an irreducible decomposition

\[
L^2(B, K_S, X(\mu)) = \bigoplus_{\lambda \in \Gamma(S, \mu)} L^2(B, K_S, X(\mu))_\lambda \cong \bigoplus_{\lambda \in \Gamma(S, \mu)} \lambda(\mu)
\]

(Theorem 1). Each \( \lambda(\mu) \) is a \( K/K_S \)-irreducible subspace in \( I(\chi(\mu, \mu))/A \) and \( \Gamma(S, \mu) \) is a certain subset of the set of all maps from \( S \) to the two points set \( \{0,1\} \). Isomorphisms \( L^2(B, K_S, X(\mu))_\lambda \cong \lambda(\mu) \) are derived from the constant term map. Irreducible representations of \( \text{Sp}_4(k_v), \forall \in S \) occurring in a tensor pro-
duct decomposition of $\lambda_{S}(\mu)$ are described by using the labelling of Enomoto ([7]) and Srinivasan ([20]) (Lemma 7).

Using these results, we can classify those automorphic representations realized in $L_{2}^{d}(B)$ which have (non-zero) vectors fixed by $K_{S}$ for some $S$. Let $A_{\omega}(F)$ be the union of $A_{S}(F)$ for all finite sets $S$ of finite places and $I^{1}(\mu, \beta_{i})$ for $\mu \in A_{\omega}(F)$ the $G(A)$ module generated by all vectors in $I(\mu, \beta_{i})$ fixed by $K_{S}$ for some $S$. Then Theorem 1 implies that there is a nontrivial intertwining operator $\delta_{I}$ from $P(\mu, \beta_{i})$ into $L_{2}^{d}(B)$ for each nontrivial $\mu \in A_{\omega}(F)$. Further, it is known by [3. Corollary 3.3.7] and [6. Proposition 3, 4° (b)] that $I(\mu, \beta_{i})$ coincides with $I(\mu, \beta_{i})$ if $F$ is totally imaginary. Unfortunately, we have been unable to prove the irreducibility of $\pi(\mu)$, the image of $\mathfrak{R}_{u}$. However, we can show that the number of irreducible constituents of $\pi(\mu)$ is at most $2^{|S_{r}(\mu)|}$ (Theorem 3), where $S_{r}(\mu)$ is the set of finite places $v$ such that $\mu_{v}$ is ramified, and each irreducible constituent of $\pi(\mu)$ is of multiplicity one in $L_{2}^{d}(B)$ (Theorem 4). The last assertion is derived from the fact that each $K$-type $\lambda_{S}(\mu)$ is of multiplicity one in $L_{2}^{d}(B, K_{S})$.

The major part of this paper will be devoted to calculations of residues of Eisenstein series. It will be carried out in Sections 2, 3, 5 and 7. Main results are stated in Sections 6, 7 and 8. In Section 4, we will recall the representation theory of the finite group $Sp_{4}(\mathbb{F}_{q})$, which we need for precise expressions of intertwining operators occuring in the constant terms of Eisenstein series.

1. Preliminaries

Let $F$ be an algebraic number field of finite degree over $\mathbb{Q}$. For each place $v$ of $F$, let $F_{v}$ be the completion of $F$ at $v$ and $| \cdot |_{v}$ the normalized absolute value on $F_{v}$. $V_{r}$ denotes the set of all finite places of $F$. For $v \in V_{r}$ let $\mathfrak{o}_{v}$ be the valuation ring of $F_{v}$, $\mathfrak{p}_{v} = \mathfrak{p}_{v}\mathfrak{o}_{v}$ the maximal ideal of $\mathfrak{o}_{v}$ and $k_{v} = \mathfrak{o}_{v}/\mathfrak{p}_{v}$ the residual field. Let $A$ be the adele ring of $F$, $\mathcal{A} = \prod_{v} | \cdot |_{v}$ the idele norm and $A^{1}$ the group consisting of ideles with idele norm one. The infinite part of $A$ is denoted by $A_{\infty}$.

We fix, once and for all, a finite subset $S$ of $V_{r}$. Let $U_{S}$ be the compact subgroup consisting of ideles $(a_{v}) \in A^{1}$ such that $|a_{v}|_{v} = 1$ for all places $v$ and $a_{v} \in 1 + \mathfrak{p}_{v}$ for any $v \in S$. $Q_{S}$ denotes the Pontrjagin dual of the compact group $F^{*}\backslash A^{1}/U_{S}$.

Take an element $\mu \in Q_{S}$. If we decompose $\mu$ to the product of characters $\mu_{v}$ of $F_{v}^{*}$, then, by definition, $\mu_{v}$ is trivial on $\mathfrak{o}_{v}^{*}$ or $1 + \mathfrak{p}_{v}$ according as $v \not\in S$ or $v \in S$. Hence, for each $v \in S$, the restriction of $\mu_{v}$ to $\mathfrak{o}_{v}^{*}$ induces the character $^{*}\mu_{v}$ of $k_{v}^{*}$. We denote by $\xi(z, \mu)$ the Hecke $L$-function of $\mu$ with the ordinary $\Gamma$ factor so
that it satisfies the functional equation \( \xi(z, \mu) = \varepsilon(\mu)\xi(1 - z, \mu^{-1}) \). If \( \mu \) is the trivial character \( \mu_0 \), then we write simply \( \xi(z) \) for \( \xi(z, \mu_0) \). The residue of \( \xi(z) \) at \( z = 1 \) is denoted by \( c(F) \).

For an algebraic group \( G \) defined over \( F \) and an \( F \)-algebra \( A \), \( G(A) \) denotes the group of \( A \)-rational points of \( G \).

Let \( G = Sp_4 \) be the symplectic group of degree two, that is

\[
G(F) = \{ g \in GL_4(F) \mid g \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right) g^{-1} = \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right) \}, \quad I = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

Let \( T \) and \( N \) be a maximal split torus and a maximal unipotent subgroup of \( G \), respectively, as follows:

\[
T(F) = \{ t(a, b) = \text{diag}(\beta, b, a^{-1}, a^2) \in G(F) \}
\]

\[
N(F) = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & A^{-1} \end{array} \right) \left( \begin{array}{cc} I & S \\ 0 & 1 \end{array} \right) \in G(F) \mid A = \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right), \quad S = 'S \right\}
\]

Then \( B = TN \) is a Borel subgroup in \( G \).

Let \( X(T) \) (resp. \( X^*(T) \)) be the character (resp. cocharacter) group of \( T \). There is a natural pairing \( \langle , \rangle : X(T) \times X^*(T) \rightarrow \mathbb{Z} \). We take \( \alpha_1, \alpha_2 \in X(T) \) such that \( \alpha_1(t(a, b)) = ab^{-1} \) and \( \alpha_2(t(a, b)) = b^2 \). Then \( \langle \alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = 2\alpha_1 + \alpha_2 \rangle \) (resp. \( \{ \alpha_1, \alpha_2 \} \)) is the set of positive roots (resp. simple roots) of \( G \) with respect to \( (B, T) \). Further, \( \beta_1 = \alpha_1 + \alpha_2/2 \) and \( \beta_2 = \alpha_1 + \alpha_2 \) are the fundamental weights of \( G \) with respect to \( (B, T) \). The coroot corresponding to \( \alpha_i \) is denoted by \( \alpha_i' \) for \( 1 \leq i \leq 4 \). Since \( G \) is simply connected, one has \( X(T) = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 \) and \( X^*(T) = \mathbb{Z}\alpha_1' + \mathbb{Z}\alpha_2' \).

Then the Weyl group \( W \) of \( G \) is generated by \( \sigma, \tau \). The homomorphism \( H \) from \( T(A) \) onto \( \alpha^* \) is defined to be \( H(t(a, b)) = \log |a|_{A} \alpha_1' + \log |ab|_A \alpha_2' \). We set \( T^1 = \text{Ker}(H) \) and \( T(R)_+ = \{ t(a, b) \mid a, b \in R_+ \} \). Then \( T(R)_+ \) is diagonally embedded in \( T(A_{oo}) \) and \( T(A) \) has a direct product decomposition \( T(A) = T(R)_+ T^1 \). Thus the map \( \alpha_C \rightarrow \text{Hom} \)
RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_4$

$\Omega(T(A)/T(C^*) : A \mapsto e^{A.H^*(\cdot)}$ becomes an isomorphism.

If we define the character $\chi(\mu, \nu)$ of $T(A)$ by $\chi(\mu, \nu)(\mu(a)\nu(b)) = \mu(a)\nu(b)$ for $\mu, \nu \in \Omega$, then the correspondence $(\mu, \nu) \mapsto \chi(\mu, \nu)$ gives a bijection from $\Omega \times \Omega$ to $\Omega(T) = \text{Hom}(T(F) \setminus T(\mathbb{A})/(K_\mathbb{A}) \cap T(A)), C^*)$. For $\chi \in \Omega(T)$ and a place $v$, $v$-component of $\chi$ is denoted by $\chi_v$. If $v \in S$, the restriction of $\chi_v$ to $T(\mathbb{F})$ induces the character $\chi_v$ of $T(\mathbb{F})$. Thus, $\chi_v$ is an isomorphism.

If we define the character $\chi(\mu, \nu)$ of $\Gamma(A)$ by $\chi(\mu, \nu)(t(a, b)) = \mu(a)v(b)$ for $\mu, v \in \Omega$, then the correspondence $(\mu, v) \mapsto \chi(\mu, v)$ gives a bijection from $\Omega \times \Omega$ to $\Omega(T(\mathbb{A})), C^*)$. For $\chi \in \Omega(T)$ and a place $v$, $v$-component of $\chi$ is denoted by $\chi_v$. If $v \in S$, the restriction of $\chi_v$ to $T(\mathbb{F})$ induces the character $\chi_v$ of $T(\mathbb{F})$. Thus, $\chi_v$ is an isomorphism.

The space of square integrable functions on $N(A)T(R)T(F) \setminus G(A)/K_\mathbb{A}$ has an orthogonal decomposition

$$L^2(N(A)T(R)T(F) \setminus G(A)/K_\mathbb{A}) = \bigoplus_{\chi \in \Omega(T)} I(\chi//A).$$

Here, for $\chi \in \Omega(T)$, let $I(\chi//A)$ be the space of all functions $\Phi$ on $G(A)$ satisfying $\Phi(ntg) = \chi(t)\Phi(g)$ for any $n \in N(A), t \in T(A)$ and $g \in G(A)$ invariant elements in $I(\chi//A)$.

For a moment, we fix a $\chi \in \Omega(T)$. Let $I(\chi_v//F_v)$ for each $v$ denote the space of functions $\Phi_v$ on $G(F_v)$ satisfying $\Phi_v(ngv) = \chi_v(t_v)\Phi_v(g_v)$ for any $n_v \in N(F_v), t_v \in T(F_v)$ and $g_v \in G(F_v)$. If $v \in S$, $I(\chi_v//F_v)$ contains a unique spherical function $e(\chi_v)$ such that $e(\chi_v)(1_v) = 1$. Then one has a restricted tensor product decomposition

$$I(\chi//A) = \otimes_v I(\chi_v//F_v)$$

with respect to $e(\chi_v), v \not\in S$.

For $\Phi = \sum_{\chi} \Phi_v \in C^*(\mathbb{A}) \otimes L^2(N(A)T(R)T(F) \setminus G(A)/K_\mathbb{A})$, the incomplete theta series

$$\varphi^\chi(g) = \sum_{\gamma \in B(F) \setminus G(F)} \sum_{\chi} \Phi_v(\gamma g)$$

For $\varphi = \sum f_i \otimes \Phi_i \in C^*(\mathbb{A}) \otimes L^2(N(A)T(R)T(F) \setminus G(A)/K_\mathbb{A})$, the incomplete theta series

$$\varphi^\chi(g) = \sum_{\gamma \in B(F) \setminus G(F)} \sum_{\chi} \Phi_v(\gamma g)$$

For $\varphi = \sum f_i \otimes \Phi_i \in C^*(\mathbb{A}) \otimes L^2(N(A)T(R)T(F) \setminus G(A)/K_\mathbb{A})$, the incomplete theta series

$$\varphi^\chi(g) = \sum_{\gamma \in B(F) \setminus G(F)} \sum_{\chi} \Phi_v(\gamma g)$$
converges and belongs to the Hilbert space $L^2(G(F) \backslash G(A)/K)$. Let $L^2(B, K)$ be the closure of $\{ \phi^+ | \phi \in C_c^*(a^*) \otimes L^2(N(A)T(R) \backslash G(A)/K) \}$ in $L^2(G(F) \backslash G(A)/K)$.

Take an orbit $X \in \Omega_s(T)/W$ and define the space $L^2(B, K, X)$ by the closed linear span of $\{ \phi^+ | \phi \in C_c^*(a^*) \otimes I(\chi//A), \chi \in X \}$. If $L^2(B, K)$ and $L^2(B, K, X)$ denote the space associated to discrete spectrums in $L^2(B, K)$ and $L^2(B, K, X)$, respectively, there is an orthogonal decomposition

$$L^2(B, K) = \bigoplus_{X \in \Omega_s(T)/W} L^2(B, K, X).$$

A theorem of Harish-Chandra says the space $L^2(B, K, X)$ is of finite dimension. Further, $K$ acts on this space by right translation. Since the action of $K$ is trivial, $L^2(B, K, X)$ becomes a representation space of the finite group $K/K_s$. In Section 6, we will give completely an irreducible decomposition of $L^2(B, K, X)$.

Let $X \in \Omega_s(T)/W$ and $\chi \in X$. For $\Phi(\chi//A)$ and $\Lambda \in a_c$, the Eisenstein series

$$E(g, \Phi, \Lambda) = \sum_{\tau \in B(F) \backslash G(F)} e^{(\Lambda + \delta, H(\tau g))} \Phi(\tau g)$$

defines an automorphic form on $G(F) \backslash G(A)$ provided $\text{Re} \Lambda \in \mathbb{C}^+ + \delta$, where let $\delta = \beta^1 + \beta^2$. The constant term of $E(g, \Phi, \Lambda)$ is given by

$$E_0(g, \Phi, \Lambda) = \sum_{w \in W} e^{(\omega + \delta, H(g))} M(w, \Lambda, \chi) \Phi(g).$$

Here, for any closed connected subgroup $N'$ of the unipotent group $N$, we use the Haar measure $dn$ on $N'(A)$ such that the volume of $N'(F) \backslash N'(A)$ equals one. This $M(w, \Lambda, \chi)$ defines a linear map from $I(\chi//A)_s$ to $I(w\chi//A)_s$. It is known by general theory that both $E(g, \Phi, \Lambda)$ and $M(w, \Lambda, \chi)$ have meromorphic continuation on $a_c$ as functions of $\Lambda$ and $M(w, \Lambda, \chi)$ satisfies the functional equation of the form

$$M(w_1 w_2, \Lambda, \chi) = M(w_1, w_2 \Lambda, w_2 \chi) M(w_2, \Lambda, \chi)$$

for any $w_1, w_2 \in W$. 
2. Calculation of $M(w, A, \chi)$

Throughout this section, we fix a $\chi \in \Omega_5(T)$ and $\Phi = \otimes_v \Phi_v \in I(\chi/A)_S$, $\Phi_v = \psi_v(\chi)$ for $v \not\in S$. First we calculate the integral (1.2) for the generator $\sigma, \tau$ of $W$. Let

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\]

be representatives in $K$ of $\sigma$ and $\tau$, respectively. Further, let

\[
\begin{pmatrix}
1 & x & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -x & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & x \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

be the one parameter subgroups associated to the simple roots $\alpha_1$ and $\alpha_2$, respectively.

In what follows we identify $\Phi$ (resp. $\Phi_v$) with $^*\Phi$ (resp. $^*\Phi_v$) by the isomorphism (1.1), hence we will often neglect the symbol $^*$. To mention a statement of results, we need a few more notations. Take an $r \in \mathbb{C}$ with absolute value one and an arbitrary $z \in \mathbb{C}$. Define, for each $v \in S$, two operators $\mathcal{A}_v(\sigma, *\chi_v)$ and $\mathcal{A}_v(\tau, z, r)$ on $I(*\chi_v/k_v)$ by

\[
\mathcal{A}_v(\sigma, *\chi_v) \Phi_v(k_v) = q_v^{-\frac{1}{2}} \sum_{x \in K_v} \Phi_v(w_v^{-1} n_v(x) r_v(k_v))
\]

\[
\mathcal{A}_v(\sigma, z, r) \Phi_v(k_v) = \left(1 - \frac{r q_v^{z}}{1 - r q_v^{z-1}} \right) q_v^{-1} \times \left\{ \sum_{x \in K_v} \Phi_v(w_v^{-1} n_v(x) r_v(k_v)) + \left( \frac{r q_v^{z}}{1 - r q_v^{z}} \right) (q_v - 1) \Phi_v(r_v(k_v)) \right\}
\]

for $k_v \in K_v$. Replacing $\sigma$ by $\tau$, $\mathcal{A}_v(\tau, *\chi_v)$ and $\mathcal{A}_v(\tau, z, r)$ are similarly defined.

Let $S_i(\chi)$ for $1 \leq i \leq 4$ be the set of $v \in S$ such that $*\chi_v \circ \alpha_i$ is trivial. The following is clear by definition.

**Lemma 1.** Let $v$ be in $S$ and $r$ a complex number with absolute value one.

1. Both $\mathcal{A}_v(\sigma, z, r)$ and $\mathcal{A}_v(\tau, z, r)$ are rational functions of $z$ and are holomorphic on $\{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \}$.
2. For $v \in S$, $\mathcal{A}_v(\sigma, *\chi_v)$ (resp. $\mathcal{A}_v(\tau, *\chi_v)$) is an intertwining operator from...
Now we can state an explicit formula of \( M(\sigma, \Lambda, \chi) \) and \( M(\tau, \Lambda, \chi) \).

**Lemma 2.** Let \( \chi \in \Omega_S(T) \) and \( \Phi = \otimes_v \Phi_v \in I(\chi//A) \). Then, one has

\[
M(\sigma, \Lambda, \chi) \Phi(k) = \frac{\xi(\langle \Lambda, \alpha_1^\vee \rangle, \chi \circ \alpha_1^\vee)}{\xi(\langle \Lambda, \alpha_1^\vee \rangle + 1, \chi \circ \alpha_1^\vee)} \prod_{v \in S_v(T)} \mathcal{A}_v(\sigma, \star \chi_v) \Phi_v(k_v)
\]

\[
\times \prod_{v \in S_v(T)} \mathcal{A}_v(\sigma, \langle \Lambda, \alpha_1^\vee \rangle, \chi \circ \alpha_1^\vee) \Phi_v(k_v)
\]

\[
M(\tau, \Lambda, \chi) \Phi(k) = \frac{\xi(\langle \Lambda, \alpha_2^\vee \rangle, \chi \circ \alpha_2^\vee)}{\xi(\langle \Lambda, \alpha_2^\vee \rangle + 1, \chi \circ \alpha_2^\vee)} \prod_{v \in S_v(T)} \mathcal{A}_v(\tau, \star \chi_v) \Phi_v(k_v)
\]

\[
\times \prod_{v \in S_v(T)} \mathcal{A}_v(\tau, \langle \Lambda, \alpha_2^\vee \rangle, \chi \circ \alpha_2^\vee) \Phi_v(k_v)
\]

for any \( \Lambda \in aC \) and \( k \in K \).

**Proof.** We prove only the first equality since the second is obtained by similar calculation. For each \( v \in V \) and \( x_v \in F_v \), elements \( c(x_v) \) and \( u(x_v) \) in \( \Theta_v \) are defined as follows:

\[
c(x_v) = \begin{cases} 
\rho_v^{-\sigma(x_v)} & \text{if } x_v \in F_v - \Theta_v \\
1 & \text{if } x_v \in \Theta_v
\end{cases}
\]

\[
u(x_v) = \begin{cases} 
(x_v c(x_v))^{-1} & \text{if } x_v \in F_v - \Theta_v \\
x_v & \text{if } x_v \in \Theta_v
\end{cases}
\]

Further, for each infinite place \( v \) and \( x_v \in F_v \), we set

\[
c(x_v) = \begin{cases} 
(x_v^2 + 1)^{-\frac{1}{2}} & \text{if } v \text{ is real} \\
(x_v \bar{x}_v + 1)^{-\frac{1}{2}} & \text{if } v \text{ is imaginary}'
\end{cases}
\]

\[
u(x_v) = \begin{cases} 
x_v c(x_v) & \text{if } v \text{ is real} \\
\bar{x}_v c(x_v) & \text{if } v \text{ is imaginary}'
\end{cases}
\]

where let \( x_v \mapsto \bar{x}_v \) be the complex conjugate for imaginary place \( v \). Obviously, for any \( x = (x_v) \in A \), both \( c(x) = (c(x_v)) \) and \( u(x) = (u(x_v)) \) are contained in \( A \).

For \( x = (x_v) \in A \), set \( k_v(x) = (k_v(x_v)) \), where
Let \( k_\sigma(x_v) \) be the following matrix:

\[
k_\sigma(x_v) = \begin{pmatrix}
    u(x_v) & u(x_v)x_v - c(x_v)^{-1} & 0 & 0 \\
    c(x_v) & c(x_v)x_v & 0 & 0 \\
    0 & 0 & c(x_v)x_v & -c(x_v) \\
    0 & 0 & -u(x_v)x_v + c(x_v)^{-1} & u(x_v)
\end{pmatrix}.
\]

Then \( k_\sigma(x) \) is contained in \( K \) and one has

\[
(2.1) \quad w_0^{-1}n_\sigma(x) = n_\sigma(-u(x)c(x))\alpha_\gamma(c(x))k_\sigma(x)
\]

for \( x \in A \).

By definition, for \( \Lambda \in (C^+ + \delta) + \sqrt{-1}a \) and \( k \in K \), \( M(\sigma, \Lambda, \chi)\Phi(k) \) equals

\[
\int_\Lambda e^{\langle A+\delta, H(w_0^{-1}n_\sigma(x)) \rangle} \Phi(w_0^{-1}n_\sigma(x)) k \, dx.
\]

The measure \( dx \) is normalized by \( \text{vol}(A/F) = 1 \). By \( (2.1) \), this equals

\[
\int_\Lambda | c(x) | \langle A + \delta, a_\gamma \rangle \chi(\alpha_\gamma(c(x))) \Phi(k_\sigma(x)) k \, dx = \prod_v I_v,
\]

where

\[
I_v = \int_{F_v} | c(x_v) | \langle A + \delta, a_\gamma \rangle \chi_v(\alpha_\gamma(c(x_v))) \Phi_v(k_\sigma(x_v)) k_v \, dx_v
\]

for each \( v \). If \( v \in S \), \( I_v \) is easily calculated because \( \chi_v \) is unramified and \( \Phi_v(k_\sigma(x_v) k_v) = 1 \). We obtain

\[
I_v = \begin{cases}
    \frac{\pi^{1/2}}{2\pi} \frac{\Gamma((z + d_v \sqrt{-1})/2)}{\Gamma((z + 1 + d_v \sqrt{-1})/2)} & \text{if } v \text{ is real} \\
    2\pi \frac{\Gamma(z + d_v \sqrt{-1})}{\Gamma(z + 1 + d_v \sqrt{-1})} & \text{if } v \text{ is imaginary}, \\
    \text{vol}(\mathcal{O}_v) \left\{ \frac{1 - \chi_v \circ \alpha_\gamma(p_v) q_v^{x_v}}{1 - \chi_v \circ \alpha_\gamma(p_v) q_v^{x_v}} \right\} & \text{if } v \in V_f - S
\end{cases}
\]

where let \( z = \langle \Lambda, \alpha_\gamma \rangle \) and \( d_v \) be the real number given by \( x_v \circ \alpha_\gamma = | \cdot | q_v^{x_v-1} \) for each infinite place \( v \).

If \( v \in S \), then one has

\[
I_v = \int_{\mathcal{O}_v} \Phi(k_\sigma(x_v) k_v) \, dx_v
\]

\[
\quad + \int_{F_v - \mathcal{O}_v} q_v^{\text{ord}(x_v)} \langle A, a_\gamma \rangle | x_v |^{-1} \chi_v \circ \alpha_\gamma(p_v^{-\text{ord}(x_v)}) \Phi_v(k_\sigma(x_v) k_v) \, dx_v
\]

\[
\quad = q_v^{-1} \text{vol}(\mathcal{O}_v) \sum_{x \in k_v} \Phi_v(w_0^{-1}n_\sigma(x) r_v(k_v))
\]

\[
\quad + \sum_{n=1}^m q_v^n \langle A, a_\gamma \rangle \chi_v \circ \alpha_\gamma(p_v)^n
\]
Therefore $I_v$ equals $v_0(\sigma, \chi_v) \Phi_v(k_v)$ times

$$q_v^{-\frac{1}{2}} \text{vol}(\mathcal{F}_v) \mathcal{A}_v(\sigma, \chi_v) \Phi_v(k_v)$$

if $v \notin S_1(\chi)$

and

$$q_v^{-1} \text{vol}(\mathcal{F}_v) \left( \frac{1 - \chi_v \circ \alpha^n_\sigma(p_\sigma) q_v^{(A, \alpha^n_\sigma)^{-1}}}{1 - \chi_v \circ \alpha^n_\sigma(p_\sigma) q_v^{(A, \alpha^n_\sigma)^{-1}}} \right) \mathcal{A}_v(\sigma, \langle A, \alpha^n_\sigma \rangle, \chi_v \circ \alpha^n_\sigma(p_\sigma) \Phi_v(k_v))$$

if $v \in S_1(\chi)$

Summing up, we obtain the desired equality for $A \in (C^+ + \delta) + \sqrt{-1} \alpha$. Then

$$\xi(\langle A, \alpha^n_\sigma \rangle + 1, \chi \circ \alpha^n_\sigma) \langle A, \alpha^n_\sigma \rangle (\langle A, \alpha^n_\sigma \rangle - 1)$$

$$\times \prod_{v \in S_1(\chi)} (1 - \chi \circ \alpha^n_\sigma(p_\sigma) q_v^{(A, \alpha^n_\sigma)^{-1}}) M(\sigma, \langle A, \alpha^n_\sigma \rangle \Phi(k))$$

must be holomorphic on $a_C$. Hence, the equality is valid for any $A \in a_C$.

Using the functional equation (1.3), one can describe $M(w, A, X) \Phi$ for any $w \in W$. If $w_1 \cdots w_m$, $w_1 \in \{\sigma, \tau\}$ for $1 \leq j \leq m$, is a reduced expression of an element of $W$, we define the operator $\mathcal{A}_v(w_1 \cdots w_m, \chi_v)$ on $I(\chi_v/k_v)$ by

$$\mathcal{A}_v(w_1, w_2 \cdots w_m, \chi_v) = \mathcal{A}_v(w_2, w_3 \cdots w_m, \chi_v) \cdots \mathcal{A}_v(w_m, \chi_v).$$

This definition is independent of reduced expressions, actually one sees $\mathcal{A}_v((\sigma \tau)^2, \chi_v) = \mathcal{A}_v((\tau \sigma)^2, \chi_v)$ by $(w_1 w_2)^2 = (w_2 w_1)^2$. We set

$$\mathcal{E}_w(A, \chi) = \prod_{1 \leq i < j \leq \delta} \frac{\xi(\langle A, \alpha^n_\sigma \rangle, \chi \circ \alpha^n_\sigma)}{\xi(\langle A, \alpha^n_\sigma \rangle + 1, \chi \circ \alpha^n_\sigma)}$$

for each $w \in W$ and let $S_v(\chi)$ be the set consisting of $\nu \in S$ such that $\chi_v$ is unramified. Then, for any $A \in a_C$ one has the following:

$$M(\sigma \tau, A, \chi) \Phi$$

$$= \mathcal{E}_{\sigma \tau}(A, \chi) \left[ \otimes_{\nu \in S - S_v(\chi) - S_v(\chi)} \mathcal{A}_v(\sigma \tau, \chi_v) \Phi_v \right]$$
\[ M(\tau\sigma, \Lambda, \chi) \Phi = \mathbb{E}_\tau(\Lambda, \chi) \left[ \bigotimes_{v \in S(2)} \mathcal{A}_v(\tau\sigma, \sigma^* \chi_v^*) \Phi_v \right] \]

\[ M(\sigma\tau\sigma, \Lambda, \chi) \Phi = \mathbb{E}_{\sigma\tau\sigma}(\Lambda, \chi) \left[ \bigotimes_{v \in S(2)} \mathcal{A}_v(\sigma\tau\sigma, \sigma^* \chi_v^*) \Phi_v \right] \]
\[ M(\tau \sigma, \Lambda, \chi) \Phi \]
\[ = \mathcal{E}_{\tau \sigma}(\Lambda, \chi) \left\{ \bigotimes_{\nu \in S} A_\nu(\tau, \langle \Lambda, \alpha_\nu^x \rangle, \chi_\nu \circ \alpha_\nu^x(p_\nu)) \circ A_\nu(\tau, \sigma, \chi_\nu) \Phi_\nu \right\} \]
\[ \bigotimes \left\{ \bigotimes_{\nu \in S} A_\nu(\tau, \langle \Lambda, \alpha_\nu^x \rangle, \chi_\nu \circ \alpha_\nu^x(p_\nu)) \circ A_\nu(\tau, \sigma, \chi_\nu) \Phi_\nu \right\} \]
\[ \bigotimes \left\{ \bigotimes_{\nu \in S} A_\nu(\tau, \langle \Lambda, \alpha_\nu^x \rangle, \chi_\nu \circ \alpha_\nu^x(p_\nu)) \circ A_\nu(\tau, \sigma, \chi_\nu) \Phi_\nu \right\} \]
\[ M((\sigma \tau)^2, \Lambda, \chi) \Phi \]
\[ = \mathcal{E}_{(\sigma \tau)^2}(\Lambda, \chi) \left\{ \bigotimes_{\nu \in S} A_\nu((\sigma \tau)^2, \chi_\nu) \Phi_\nu \right\} \]
\[ \bigotimes \left\{ \bigotimes_{\nu \in S} A_\nu((\sigma \tau)^2, \chi_\nu) \Phi_\nu \right\} \]
In this paper, we are interested in the singular hyperplanes of $M(w, \Lambda, \chi)$ contributing to the spectral decomposition of $L^2(B, K)$. Such a singular hyperplane $S$ has to satisfy the following condition (cf. [14, Theorem 7.1], [17, Theorem 5.12]).

$(2.2)$ $S$ is defined by a linear equation of the form $\langle \Lambda, \alpha \rangle = c$ with positive real constant $c$.

From this and Lemma 1 (1), we obtain:

**Lemma 3.** Let $S_i = \{ \Lambda \in \mathcal{A} \mid \langle \Lambda, \alpha_i \rangle = 1 \}$ for $1 \leq i \leq 4$. Then $S_i$, $1 \leq i \leq 4$, exhaust the singular hyperplanes of $M(w, \Lambda, \chi)$, $w \in W$, $\chi \in \Omega(T)$, contributing to the spectral decomposition of $L^2(B, K_s)$.

### 3. Residues of $M(w, \Lambda, \chi)$ (1)

Each $S_i$ is rewritten as $S_i = Cu_i + v_i$, $1 \leq i \leq 4$, where $u_1 = \beta_2$, $u_2 = \beta_1$, $u_3 = \alpha_1$, $u_4 = \alpha_2$ and $v_i = \alpha_i/2$ for $1 \leq i \leq 4$. Then we take a coordinate $z_i(\Lambda)$ on $S_i$ as $\Lambda = z_i(\Lambda)u_i + v_i$ for $\Lambda \in S_i$, $1 \leq i \leq 4$.

Next, for $1 \leq i, j \leq 4$, let $W_{ij}$ be the set of elements $w \in W$ such that $-wS_i = S_j$. Obviously $W_{ij}$ is empty unless $i \equiv j$ modulo 2. For $1 \leq i \leq 4$, let $W_i$ be the union of $W_{ij}$, $1 \leq j \leq 4$.

For $\chi \in \Omega(T)$, $\Phi \in I(\chi//A)_s$ and $w \in W$, we set

$$M^i(w, \Lambda, \chi) = \frac{\xi(2)}{2\pi c(F)} \int_C M(w, \Lambda + zv_i, \chi) \, dz$$

$$E^i(g, \Phi, \Lambda) = \frac{\xi(2)}{2\pi c(F)} \int_C E(g, \Phi, \Lambda + zv_i) \, dz$$

for $\Lambda \in S_i$, $1 \leq i \leq 4$, where $C$ is a small contour around the origin in the complex plane. Let $\Lambda_{ij}$ denote the intersection of $S_i$ and $S_j$ for $i \neq j$. One sees that the order of pole of $E^i(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{ij}$ is at most one for any $\Phi \in I(\chi//A)_s$, $\chi \in \Omega(T)$, $1 \leq j \leq 4$, $j \neq i$ (cf. Lemma 11). Then the main theorem of [11] deduces that, for $X \in \Omega(T)/W$, the space $L^2(B, K_s, X)$ is spanned by

$$\text{Res}_{\Lambda = \Lambda_{ij}} E^i(g, \Phi, \Lambda), \quad \Phi \in I(\chi//A)_s, \chi \in X, \quad i = 1, 2, 1 \leq j \leq 4, \, j \neq i$$

belonging to $L^2(G(F) \setminus G(A))$. Here, notice that, since principal singular hyperplanes (in the sense of [17]) are only $S_1$ and $S_2$, it is enough for our purpose to consider only residues of $E^i(g, \Phi, \Lambda)$ for $i = 1, 2$ (cf. [17, Chapter 6]). Since the
residue \( \text{Res}_{\Lambda=A} E^t(g, \Phi, \Lambda) \) is completely determined by its constant term

\[
\int_{N(F) \setminus N(A)} \text{Res}_{\Lambda=A} E^t(ng, \Phi, \Lambda) \, dn
\]

\[
= \text{Res}_{\Lambda=A} \left\{ \sum_{w \in W_i} e^{\omega A + \delta H(g)} M^t(w, \Lambda, \chi) \Phi(g) \right\},
\]

what we should do is to calculate residues of \( M^t(w, \Lambda, \chi) \) at \( \Lambda = \Lambda_i \) for \( i = 1, 2 \), \( 1 \leq j \leq 4 \), \( j \neq i \).

Let \( A_S(F) \) be the set of characters \( \mu \in \Omega_S \) such that \( \mu^2 = \mu_0 \), where \( \mu_0 \) is the trivial character. For each \( \mu \in A_S(F) \), the \( W \) orbit of \( \chi(\mu, \mu) \) is denoted by \( X(\mu) \), that is, \( X(\mu) = \{ \chi(\mu, \mu) \} \). Then the next proposition follows from direct calculations.

**PROPOSITION 1.** Let \( X \in \Omega_S(T)/W \) be a \( W \) orbit. Assume \( X \notin \{ X(\mu) \mid \mu \in A_S(F) \} \). Then, for any \( \chi \in X \), \( w \in W \) and \( 1 \leq i \neq j \leq 4 \), \( \text{Res}_{\Lambda=A} M^t(w, \Lambda, \chi) = 0 \)

By that mentioned above, we obtain

**COROLLARY.** \( L^2(B, K_S) = \bigoplus_{\mu \in A_S(F)} L^2(B, K_S, X(\mu)) \).

The major part of remains of this paper will be devoted to a detailed calculation of residues of \( M^t(w, \Lambda, \chi) \) for \( \chi = \chi(\mu, \mu), \mu \in A_S(F) \). At first, we give an explicit form of \( M^t(w, \Lambda, \chi) \). For each \( \mu \in A_S(F) \), the subset \( S_u(\mu) \) of \( S \) is defined to be \( S_u(\mu) = \{ v \in S \mid \mu_v \text{ is unramified} \} \).

**LEMMA 4.** Let \( \mu \in A_S(F) \) be a nontrivial character, \( \chi = \chi(\mu, \mu) \in \Omega_S(T) \) and \( \Phi \in I(\chi//A)_S \) an arbitrary element. Then, for any \( w \in W_2 \), \( M^2(w, \Lambda, \chi) \Phi \) is identically zero. Further one has

\[
M^1(\sigma, \Lambda, \chi) \Phi = \bigotimes_{v \in S} A_v(\sigma, 1, 1) \Phi_v
\]

\[
M^1(\tau \sigma, \Lambda, \chi) \Phi = \frac{\xi(z + 2, \mu)}{\xi(z + 3, \mu)} \left( \bigotimes_{v \in S} A_v(\tau, \chi_v) \right) \Phi_v
\]

\[
\bigotimes \left( \bigotimes_{v \in S} A_v(\tau, z + 1, \mu_v) \right) \Phi_v
\]

\[
M^1(\sigma \tau \sigma, \Lambda, \chi) \Phi = \frac{\xi(2z)\xi(z + 1, \mu)}{\xi(2z + 1)\xi(z + 3, \mu)} \times
\]

[Note: The rest of the text contains mathematical expressions and formulas that are not visible in the provided image.]
RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_4$

\[
\left\{ \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\sigma, 2z, 1) \otimes A_\nu(\tau, *\chi_\nu) \otimes A_\nu(\sigma, 1, 1) \Phi_v \right\}
\]

\[
\otimes \left\{ \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\sigma, 2z, 1) \otimes A_\nu(\tau, z + \frac{1}{2}, \mu_v(\nu)) \otimes A_\nu(\sigma, 1, 1) \Phi_v \right\}
\]

\[
M^1((\sigma \tau)^2, \Lambda, \chi) \Phi = \frac{\xi(2z)\xi(z - \frac{1}{2}, \mu)}{\xi(2z + 1)\xi(z + \frac{3}{2}, \mu)} \times \]

\[
\left\{ \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\sigma, 1, 1) \otimes A_\nu(\tau, *\chi_\nu) \otimes A_\nu(\sigma, 2z, 1) \otimes A_\nu(\tau, *\chi_v) \Phi_v \right\}
\]

\[
\otimes \left\{ \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\sigma, 1, 1) \otimes A_\nu(\tau, z + \frac{1}{2}, \mu_v(\nu)) \right. \]

\[
\left. \otimes A_\nu(\sigma, 2z, 1) \otimes A_\nu(\tau, z - \frac{1}{2}, \mu_v(\nu)) \Phi_v \right\}
\]

for any $\Lambda = zu_1 + v_1 \in S_1$.

**Lemma 5.** Let $\chi = \chi(\mu_0, \mu_v) \in \Omega_3(T)$ be the trivial character and $\Phi \in I(\chi//A)_S$ an arbitrary element. Then one has the following.

For $\Lambda = zu_1 + v_1 \in S_1$,

\[
M^1(\sigma, \Lambda, \chi) \Phi = \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\sigma, 1, 1) \Phi_v
\]

\[
M^1(\tau \sigma, \Lambda, \chi) \Phi = \frac{\xi(z + \frac{1}{2})}{\xi(z + \frac{3}{2})} \left\{ \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\tau, z + \frac{1}{2}, 1) \otimes A_\nu(\sigma, 1, 1) \Phi_v \right\}
\]

\[
M^1((\tau \sigma)^2, \Lambda, \chi) \Phi = \frac{\xi(2z + 1)\xi(z + \frac{3}{2})}{\xi(2z)\xi(z + \frac{1}{2})} \times \]

\[
\left\{ \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\sigma, 2z, 1) \otimes A_\nu(\tau, z + \frac{1}{2}, 1) \otimes A_\nu(\sigma, 1, 1) \Phi_v \right\}
\]

\[
M^1((\sigma \tau)^2, \Lambda, \chi) \Phi = \frac{\xi(2z + 1)\xi(z + \frac{3}{2})}{\xi(2z)\xi(z - \frac{1}{2})} \left\{ \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\sigma, 1, 1) \right. \]

\[
\left. \otimes A_\nu(\tau, z + \frac{1}{2}, 1) \otimes A_\nu(\sigma, 2z, 1) \otimes A_\nu(\tau, z - \frac{1}{2}, 1) \Phi_v \right\}
\]

For $\Lambda = zu_2 + v_2 \in S_2$,

\[
M^2(\tau, \Lambda, \chi) \Phi = \otimes_{\nu \in S_{\nu}(\nu)} A_\nu(\tau, 1, 1) \Phi_v
\]
\[ M^2(\sigma \tau, \Lambda, \chi) \Phi = \frac{\xi(z + 1)}{\xi(z + 2)} \left\{ \otimes_{v \in S} A_v(\sigma, z + 1, 1) \cdot A_v(\tau, 1, 1) \Phi \right\} \]

\[ M^2(\tau \sigma \tau, \Lambda, \chi) \Phi = \frac{\xi(z)}{\xi(z + 2)} \left\{ \otimes_{v \in S} A_v(\tau, z, 1) \cdot A_v(\sigma, z + 1, 1) \right\} \]

\[ M^2((\sigma \tau)^2, \Lambda, \chi) \Phi = \frac{\xi(z - 1)}{\xi(z + 2)} \left\{ \otimes_{v \in S} A_v(\sigma, z - 1, 1) \cdot A_v(\tau, z, 1) \right\} \]

These lemmas are easily proved. Consequently, we obtain the following:

**Proposition 2.** Let \( \mu \in A_5(F) \) be a nontrivial character. Then \( L^2(B, K_s, X(\mu)) \) is spanned by \( \text{Res}_{A=\beta_1} E^1(g, \Phi, \Lambda), \Phi \in I(\chi(\mu) \Phi) \), where \( \beta_1 = \Lambda_{13} = \Lambda_{14} \) is a fundamental weight of \( G \).

**Proof.** From Lemma 4, it follows

\[ \text{Res}_{A=\Lambda_12} E^1(g, \Phi, \Lambda) = 0 \]

\[ \text{Res}_{A=\Lambda_1} E^2(g, \Phi, \Lambda) = 0, \quad 1 \leq j \leq 4, \ j \neq 2 \]

for all \( \Phi \in I(\chi(\mu) \Phi) \). On the other hand, by Lemma 4 again, it is possible that \( E^1(g, \Phi, \Lambda) \) has a simple pole at \( \Lambda = \beta_1 \). If so, the constant term of \( \text{Res}_{A=\beta_1} E^1(g, \Phi, \Lambda) \) equals

\[ e^{(-\beta_1 + \delta \Phi(g))} \{ \text{Res}_{A=\beta_1} M^1(\sigma \tau \sigma, \Lambda, \chi) \Phi + \text{Res}_{A=\beta_1} M^1((\sigma \tau)^2, \Lambda, \chi) \Phi \}. \]

Thus, Langlands’ \( L^2 \)-ness criterion deduces \( \text{Res}_{A=\beta_1} E^1(g, \Phi, \Lambda) \) is square integrable for any \( \Phi \in I(\chi(\mu) \Phi) \). This implies the assertion.

Residues of \( M^1(\sigma \tau \sigma, \Lambda, \chi(\mu, \mu)) \) and \( M^1((\sigma \tau)^2, \Lambda, \chi(\mu, \mu)) \) at \( \Lambda = \beta_1 \) for nontrivial \( \mu \in A_5(F) \) are given as follows.

\[ \text{Res}_{A=\beta_1} M^1(\sigma \tau \sigma, \Lambda, \chi(\mu, \mu)) \Phi \]

\[ = \frac{c_F(\mu)}{2} \left\{ \otimes_{v \in S - S_{\mu}(D)} A_v(\sigma, 1, 1) \cdot A_v(\tau, 1, 1) \cdot A_v(\sigma, 1, 1) \Phi \right\} \]

\[ \otimes \left\{ \otimes_{v \in S_{\mu}(D)} A_v(\sigma, 1, 1) \cdot A_v(\tau, 1, 1, \mu_v(\Phi_v)) \cdot A_v(\sigma, 1, 1) \Phi \right\} \]

\[ \text{Res}_{A=\beta_1} M^1((\sigma \tau)^2, \Lambda, \chi(\mu, \mu)) \Phi \]
RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $\text{Sp}_4$

$= \frac{\epsilon(\mu) c_F(\mu)}{2} \left\{ \bigotimes_{v \in S - S_w(\mu)} \mathcal{A}_v(\sigma, 1, 1) \ast \mathcal{A}_v(\tau, *\chi_v) \right. \\
\left. \ast \mathcal{A}_v(\sigma, 1, 1) \ast \mathcal{A}_v(\tau, *\chi_v) \Phi_v \right\} \\
\bigotimes \left\{ \bigotimes_{v \in S_w(\mu)} \mathcal{A}_v(\sigma, 1, 1) \ast \mathcal{A}_v(\tau, 1, \mu_v(p_v)) \ast \mathcal{A}_v(\sigma, 1, 1) \right. \\
\left. \ast \mathcal{A}_v(\tau, 0, \mu_v(p_v)) \Phi_v \right\},$

where $c_F(\mu) = c(F) \xi(1, \mu) \xi(2)^{-1} \xi(2, \mu)^{-1}$. Hence, in order to describe $L^2(B, K_S X(\beta))$ as a representation space of $K/K_S$ in more detail, we have to investigate the intertwining operators $\mathcal{A}_v(\sigma, 1, 1), \mathcal{A}_v(\tau, *\chi_v), \ldots$. To do so, we use a result of Gurtis and Fossum. Thus, in next section, we recall the representation theory of $G(k)$. 

4. Principal series of $G(F_q)$

Throughout this section, $k$ denotes the finite field of cardinality $q$. For a character $*\chi$ of $T(k)$, let $I(*\chi//k)$ be the representation of $G(k)$ induced by the trivial extension to $B(k)$ of $*\chi$ and $H(G, B; *\chi)$ denote the centralizer ring of $I(*\chi//k)$. Dropping the lower index $v$, operators $\mathcal{A}(w, *\chi)$ and $\mathcal{A}(w, z, r)$ for $w \in \{\sigma, \tau\}$ on $I(*\chi//k)$ are similarly defined as in Section 2. In this section, we explain an irreducible decomposition of $I(*\chi//k)$ for some particular $*\chi$ and then represent the operators $\mathcal{A}(w, *\chi)$ and $\mathcal{A}(w, z, r)$ by linear combinations of those projections to irreducible subspaces of $I(*\chi//k)$ which are constructed by using a theorem of Curtis and Fossum.

For a character $\mu$ of $k^*$, the character $*\chi(\mu)$ of $T(k)$ is defined to be $*\chi(\mu)(t(a, b)) = \mu(ab)$ for $t(a, b) \in T(k)$. Then we consider the following three cases.

$(\#-1)$ $\mu$ is the trivial character. 

$(\#-2)$ $q \equiv 1 \text{ mod } 4$ and $\mu$ is the quadratic character. 

$(\#-3)$ $q \equiv 3 \text{ mod } 4$ and $\mu$ is the quadratic character.

Before going to case by case consideration, we state a result deduced from general theory (cf. [5]).

**Lemma 6.** Let $*\chi$ be an arbitrary character of $T(k)$. Then there is a bijection $\eta \mapsto \theta(\eta)$ from the set of equivalence classes of irreducible representations of $H(G, B; *\chi)$ to the set of equivalence classes of irreducible constituents of $I(*\chi//k)$ such that the character of $\eta$ is equal to the restriction to $H(G, B; *\chi)$ of the character
of \( \theta(\eta) \). Here, notice that \( H(G, B; \ast \chi) \) is considered as a subalgebra of the group ring of \( G(k) \). Furthermore, the multiplicity of \( \theta(\eta) \) in \( I(\ast \chi//k) \) is equal to the degree of \( \eta \).

Now we start with the case (\#-1).

(\#-1) \( \ast \chi = \ast \chi(\mu) \), \( \mu \) is the trivial character.

The self-intertwining operators \( \alpha_\sigma \) and \( \alpha_\tau \) of \( I(\ast \chi//k) \) are defined to be

\[
\alpha_\sigma(\Phi)(g) = \sum_{x \in k} \Phi(w_\sigma^{-1}n_\sigma(x)g), \quad \alpha_\tau(\Phi)(g) = \sum_{x \in k} \Phi(w_\tau^{-1}n_\tau(x)g)
\]

for \( \Phi \in I(\ast \chi//k) \). Further, if \( w = w_1 \cdots w_m, w_j \in \langle \sigma, \tau \rangle, 1 \leq j \leq m \) is a reduced expression, we define \( \alpha_w \) by \( \alpha_{w_1} \cdots \alpha_{w_m} \), which does not depend on reduced expressions. Then \( H(G, B; \ast \chi) \) is generated by \( \alpha_\sigma \) and \( \alpha_\tau \) together with the relations

\[
\alpha_\sigma^2 = q\alpha_\sigma + (q-1)\alpha_\sigma, \quad \alpha_\tau^2 = q\alpha_\tau + (q-1)\alpha_\tau, \quad (\alpha_\sigma \alpha_\tau)^2 = (\alpha_\tau \alpha_\sigma)^2,
\]

where \( \alpha_e \) is the identity map of \( I(\ast \chi//k) \). Irreducible representations of \( H(G, B; \ast \chi) \) are exhausted by the following ones up to equivalence.

\[
\eta_1: \begin{cases} \alpha_\sigma \mapsto q, \\ \alpha_\tau \mapsto q \end{cases}, \quad \eta_2: \begin{cases} \alpha_\sigma \mapsto q, \\ \alpha_\tau \mapsto -1 \end{cases}, \quad \eta_3: \begin{cases} \alpha_\sigma \mapsto q, \\ \alpha_\tau \mapsto -1 \end{cases}, \quad \eta_4: \begin{cases} \alpha_\sigma \mapsto -1, \\ \alpha_\tau \mapsto -1 \end{cases}, \quad \eta_5: \begin{cases} \alpha_\sigma \mapsto \frac{1}{q+1} \left( \frac{q-1}{2q(q^2+1)} \frac{1}{q(q-1)} \right), \\ \alpha_\tau \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}
\]

Let \( \theta(\eta_i) \) be the irreducible representation of \( G(k) \) corresponding to \( \eta_i \) by Lemma 6 and \( V_i \) the \( \theta(\eta_i) \)-isotypic subspace of \( I(\ast \chi//k) \) for \( 1 \leq i \leq 5 \). Then one has

\[
I(\ast \chi//k) = \bigoplus_{i=1}^5 V_i, \quad \begin{cases} V_i \cong \theta(\eta_i) \text{ for } 1 \leq i \leq 4, \\ V_5 \cong \theta(\eta_3) \oplus \theta(\eta_3). \end{cases}
\]

If \( P_i \) denotes the projection from \( I(\ast \chi//k) \) onto \( V_i \) for \( 1 \leq i \leq 5 \), then \( [5, \text{ Theorem (2.4)}] \) allows us to represent \( P_i \) by linear combinations of \( \alpha_w, w \in W \).

Actually one has

\[
\begin{align*}
P_1 &= \frac{1}{(q^2+1)} \sum_{w \in W} \alpha_w \\
P_2 &= \frac{1}{2(q+1)^2} \{ q\alpha_e - \alpha_\sigma + q\alpha_\tau - \alpha_\sigma \alpha_\tau - \alpha_\tau \alpha_\sigma + q^{-1}\alpha_\sigma \alpha_\tau + q^{-1}\alpha_\sigma^2 \alpha_\tau \} \\
P_3 &= \frac{1}{2(q+1)^2} \{ q\alpha_e + q\alpha_\sigma - \alpha_\tau - \alpha_\sigma \alpha_\tau + \alpha_\tau \alpha_\sigma + q^{-1}\alpha_\tau \alpha_\sigma + q^{-1}\alpha_\sigma \alpha_\tau \} \\
P_4 &= \frac{1}{(q^2+1)} \{ q^4\alpha_e - q^3\alpha_\sigma - q^3\alpha_\tau + q^2\alpha_\sigma \alpha_\tau \}
\end{align*}
\]
RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_{33}$

$$P_s = \frac{1}{2(q^2 + 1)}(2q\alpha_e + (q - 1)\alpha_\sigma + (q - 1)\alpha_\tau + q^2\alpha_\sigma\sigma - q\alpha_\sigma\tau - q\alpha_\tau\sigma + \alpha_{(\sigma\tau)}{\tau e})$$

$$+ q^{-1}(q - 1)\alpha_{(\sigma)\alpha} + q^{-1}(q - 1)\alpha_{(\tau)\alpha} - 2q^{-1}\alpha_{(\sigma\tau)\alpha}.$$ 

We express $A(\sigma, 1, 1)$ and $A(\tau, 1, 1)$ by linear combinations of projections to irreducible subspaces. By definition,

$$A(\sigma, 1, 1) = \frac{1}{q + 1} (\alpha_e + \alpha_\sigma), \quad A(\tau, 1, 1) = \frac{1}{q + 1} (\alpha_e + \alpha_\tau).$$

Thus, it follows from easy calculation

$$A(\sigma, 1, 1) = P_1 + P_3 + Q_\sigma, \quad Q_\sigma = P_s A(\sigma, 1, 1)$$
$$A(\tau, 1, 1) = P_1 + P_3 + Q_\tau, \quad Q_\tau = P_s A(\tau, 1, 1).$$

Then $Q_\sigma$ and $Q_\tau$ are considered as elements in $End_{G(k)}(V_\delta)$ and satisfy the following:

$$Q_\sigma Q_\tau Q_\sigma = \frac{2q}{(q + 1)^2} Q_\sigma, \quad Q_\tau Q_\sigma Q_\tau = \frac{2q}{(q + 1)^2} Q_\tau$$

We define four elements of $End_{G(k)}(V_\delta)$ by

$$P_\sigma = \frac{1}{(q^2 + 1)} (-2qQ_\tau (q - 1)^2 Q_\sigma),$$
$$P_\tau = \frac{1}{(q^2 + 1)} (-2qQ_\sigma (q + 1)^2 Q_\tau)$$
$$R_+ = \sqrt{2q} (q + 1) (q - 1), \quad R_- = \sqrt{2q} q + 1 (Q_\sigma Q_\tau Q_\tau),$$

where $\sqrt{2q}$ is the positive square root of $2q$. Then these elements satisfy

$$P_\sigma^2 = R_- R_+ = P_\sigma, \quad P_\tau^2 = R_+ R_- = P_\tau,$$
$$P_\sigma P_\tau = P_\tau P_\sigma = (R_+)^2 = (R_-)^2 = 0.$$ 

Therefore $P_\sigma, P_\tau, R_, R_-$ becomes a base of the four dimensional space $End_{G(k)}(V_\delta)$ and both $P_\sigma$ and $P_\tau$ are projections to irreducible subspaces in $V_\delta$.

Representing $Q_\sigma$ and $Q_\tau$ by these we obtain

$$A(\sigma, 1, 1) = P_1 + P_3 + P_\sigma + \sqrt{2q} \frac{q + 1}{R_-} R_-$$
$$A(\tau, 1, 1) = P_1 + P_2 + P_\tau + \sqrt{2q} \frac{q + 1}{R_+} R_+. $$

https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0027763000004086

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 14 May 2019 at 12:13:20, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0027763000004086
We set particularly
\[ P^*_5 = Q_\sigma = P_\sigma + \frac{\sqrt{2q}}{q + 1} R_. \]
Then \( P^*_5 I (\chi//k) \cong \theta (\eta_5) \) and one has
\[ (4.3) \quad \mathcal{A} (\sigma, 1, 1) \circ \mathcal{A} (\tau, 1, 1) \circ \mathcal{A} (\sigma, 1, 1) = P_1 + \frac{2q}{(q + 1)^2} P^*_5 \]
Since \( \mathcal{A} (w, z, r), w \in \{ \sigma, \tau \}, \) equals
\[ \frac{1}{1 - rq^{-1}} ((1 - rq^{-1} )(1 + q^{-1}) \mathcal{A} (w, 1, 1) + q^{-1}(rq^{-1} - 1) \alpha_\varsigma), \]
one has
\[ (4.4) \quad \mathcal{A} (\sigma, 1, 1) \circ \mathcal{A} (\tau, 1, - 1) \circ \mathcal{A} (\sigma, 1, 1) = P_1 - \frac{2q}{q^2 + 1} P_3 \]
\[ \mathcal{A} (\sigma, 1, 1) \circ \mathcal{A} (\tau, 1, - 1) \circ \mathcal{A} (\sigma, 1, 1) \circ \mathcal{A} (\tau, 0, - 1) = P_1 + \frac{2q}{q^2 + 1} P_3. \]
\((\# - 2)\) \( q \equiv 1 \text{ mod } 4, \) *\( \chi = \chi (\mu), \) \( \mu \) is the quadratic character.

We define the self-intertwining operators \( \alpha_w, w \in W \) of \( I (\chi//k) \) as in (4.1). Then the centralizer ring \( H (G, B; \chi) \) is generated by \( \alpha_\sigma \) and \( \alpha_\tau \) together with the relations
\[ \alpha_\tau^2 = q \alpha_\tau + (q - 1) \alpha_\sigma, \quad \alpha_\sigma^2 = q \alpha_\sigma, \quad (\alpha_\sigma \alpha_\tau)^2 = (\alpha_\tau \alpha_\sigma)^2 \]
Irreducible representations of \( H (G, B; \chi) \) are exhausted by the following ones up to equivalence.
\[ \eta'_1: \{ \alpha_\sigma \mapsto q, \quad \alpha_\tau \mapsto q^{\frac{1}{2}}, \quad \eta'_2: \{ \alpha_\sigma \mapsto q^{\frac{1}{2}}, \quad \alpha_\tau \mapsto q^{\frac{1}{2}}, \quad \eta'_3: \{ \alpha_\sigma \mapsto - q^{\frac{1}{2}}, \quad \alpha_\tau \mapsto - q^{\frac{1}{2}}, \quad \eta'_5: \{ \alpha_\sigma \mapsto q^{\frac{1}{2}}, \quad \alpha_\tau \mapsto 0, \quad \eta'_6: \{ \alpha_\sigma \mapsto q^{\frac{1}{2}}, \quad \alpha_\tau \mapsto q^{\frac{1}{2}}, \quad \eta'_7: \{ \alpha_\sigma \mapsto 0, \quad \alpha_\tau \mapsto q^{\frac{1}{2}} \}, \]
where \( q^{\frac{1}{2}} \) is the positive square root of \( q. \) Let \( \theta (\eta'_i) \) be the irreducible representation of \( G(k) \) corresponding to \( \eta'_i \) and \( V'_i \) the \( \theta (\eta'_i) \)-isotypic subspace of \( I (\chi//k) \) for \( 1 \leq i \leq 5 \). Then it follows from Lemma 6
\[ I (\chi//k) = \bigoplus_{i=1}^5 V'_i, \quad \begin{cases} V'_i \cong \theta (\eta'_i), & \text{for } 1 \leq i \leq 4 \smallskip V'_5 \cong \theta (\eta'_5), & \text{for } 5 \end{cases} \]
If \( P'_i \) denote projections from \( I (\chi//k) \) to \( V'_i \) for \( 1 \leq i \leq 5 \), then by [5], one has
\[ P'_i = \frac{1}{2q(q + 1)^2} \{ q \alpha_\sigma + q \alpha_\tau + q^{\frac{1}{2}} \alpha_\sigma + q^{\frac{1}{2}} \alpha_\tau + q^{\frac{1}{2}} \alpha_{\sigma \tau} + q^{\frac{1}{2}} \alpha_{\tau \sigma} + \alpha_{\sigma \tau} + \alpha_{\tau \sigma} \}\]
\[
P_2' = \frac{1}{2(q + 1)^2} \left( q^2 \alpha_e - q \alpha_\sigma + q^2 \alpha_\tau - q^2 \alpha_{\sigma \tau} - q^2 \alpha_{\sigma \tau} + q^{-1} \alpha_{(\sigma \tau)z} \right)
\]
\[
P_3' = \frac{1}{2q(q + 1)^2} \left( q \alpha_e + q \alpha_\sigma - q^3 \alpha_\tau - q^3 \alpha_{\sigma \tau} - q^3 \alpha_{\sigma \tau} + q^{-1} \alpha_{(\sigma \tau)z} \right)
\]
\[
P_4' = \frac{1}{2(q + 1)^2} \left( q^2 \alpha_e - q \alpha_\sigma - q^2 \alpha_\tau + q^2 \alpha_{\sigma \tau} + q^2 \alpha_{\sigma \tau} - q^{-1} \alpha_{(\sigma \tau)z} \right)
\]
\[
P_5' = \frac{1}{(q + 1)^2} \left( 2q \alpha_e + (q - 1) \alpha_\sigma + q^{-1} (q - 1) \alpha_{\sigma \tau} - 2q^{-1} \alpha_{(\sigma \tau)z} \right).
\]

We set
\[
Q_+ = \frac{1}{2} (\alpha_e + \mathcal{A}(\tau, *_\chi)) \cdot P_5', \quad Q_- = \frac{1}{2} (\alpha_e - \mathcal{A}(\tau, *_\chi)) \cdot P_5',
\]
\[
R_+ = 2\mathcal{A}(\sigma, 1, 1) Q_+ - Q_+, \quad R_- = 2\mathcal{A}(\sigma, 1, 1) Q_- - Q_-.
\]

Since \( \mathcal{A}(\sigma, 1, 1) = \frac{1}{q + 1} (\alpha_e + \alpha_\sigma) \), \( \mathcal{A}(\tau, *_\chi) = q^{-\frac{1}{2}} \alpha_\tau \) and \( \mathcal{A}(\tau, *_\chi)^2 = \alpha_e \)
The centralizer ring \( H(G, B; *_\chi) \) is generated by \( \alpha_e \) and \( \alpha_\tau \) together with the relations

Hence we obtain

\[
\mathcal{A}(\sigma, 1, 1) \cdot \mathcal{A}(\tau, *_\chi) \cdot \mathcal{A}(\sigma, 1, 1) = P_1' - P_3',
\]
\[
\mathcal{A}(\sigma, 1, 1) \cdot \mathcal{A}(\tau, *_\chi) \cdot \mathcal{A}(\sigma, 1, 1) \cdot \mathcal{A}(\tau, *_\chi) = P_1' + P_3'.
\]

This case is similar to the case (\#-2), hence we will omit the details. Define \( \alpha_w, w \in W \), as in (4.1). Then the centralizer ring \( H(G, B; *_\chi) \) is generated by \( \alpha_\sigma \) and \( \alpha_\tau \) together with the relations

\[
(\#-3) \quad q \equiv 3 \mod 4, \quad *_\chi = *_\chi(\mu), \quad \mu \text{ is the quadratic character.}
\]

Define \( \alpha_w, w \in W \), as in (4.1). Then the centralizer ring \( H(G, B; *_\chi) \) is generated by \( \alpha_\sigma \) and \( \alpha_\tau \) together with the relations
\[ \alpha^2 = q\alpha + (q - 1)\alpha, \quad \alpha^2 = -q\alpha, \quad (\alpha\alpha\alpha)^2 = (\alpha\alpha\alpha)^2. \]

Irreducible representations of \( H(G, B; *\chi) \) are exhausted by the following ones up to equivalence.

\[ \eta''_i : \begin{cases} \alpha \mapsto q, \\ \alpha \mapsto -(-q)^{\frac{1}{2}} \end{cases}, \quad \eta''_2 : \begin{cases} \alpha \mapsto -1, \\ \alpha \mapsto -(-q)^{\frac{1}{2}} \end{cases}, \quad \eta''_3 : \begin{cases} \alpha \mapsto q, \\ \alpha \mapsto (-q)^{\frac{1}{2}} \end{cases}, \quad \eta''_4 : \begin{cases} \alpha \mapsto -1, \\ \alpha \mapsto -(-q)^{\frac{1}{2}} \end{cases}, \quad \eta''_5 : \begin{cases} \alpha \mapsto 0, \\ \alpha \mapsto -(-q)^{\frac{1}{2}} \end{cases}, \]

where \((-q)^{\frac{1}{2}}\) lies in the upper half plane of the complex plane. Let \( \theta(\eta'') \) be the irreducible representation of \( G(k) \) corresponding to \( \eta'' \) and \( V''_i \) the \( \theta(\eta'') \)-isotypic subspace of \( I(*\chi//k) \) for \( 1 \leq i \leq 5 \). Then it follows from Lemma 6

\[ I(*\chi//k) = \bigoplus_{i=1}^{5} V''_i, \quad \begin{cases} V''_i \cong \theta(\eta''_i) & \text{for } 1 \leq i \leq 4 \\ V''_5 \cong \theta(\eta''_5) \oplus \theta(\eta''_5) \end{cases}. \]

Let \( P''_i \) be the projection from \( I(*\chi//k) \) to \( V''_i \) for each \( 1 \leq i \leq 5 \). Then, from the argument similar to the case \((#-2)\), it follows

\[ (4.6) \quad \mathcal{A}(\sigma, 1, 1) \cdot \mathcal{A}(\tau, *\chi) \cdot \mathcal{A}(\sigma, 1, 1) = -\sqrt{-1}(P''_1 - P''_5) \]

\[ \mathcal{A}(\sigma, 1, 1) \cdot \mathcal{A}(\tau, *\chi) \cdot \mathcal{A}(\sigma, 1, 1) \cdot \mathcal{A}(\tau, *\chi) = -(P''_1 + P''_5). \]

5. Residues of \( M(w, \Lambda, \chi) \) (2)

We return to calculations of residues of \( M(w, \Lambda, \chi) \). In this section, we fix a nontrivial character \( \mu \in A_\delta(F) \) and put \( \chi = \chi(\mu, \mu) \).

Define four subsets of \( S \) as follows:

\[ S^+_\mu = \{ v \in S \mid *\mu_v \text{ is the quadratic character and } *\mu_v(-1) = 1 \} \]

\[ S^-_\mu = \{ v \in S \mid *\mu_v \text{ is the quadratic character and } *\mu_v(-1) = -1 \} \]

\[ S^\mu = \{ v \in S \mid \mu_v \text{ is trivial} \} \]

\[ S^-_\mu = \{ v \in S \mid \mu_v \text{ is unramified and } \mu_v(q_v) = -1 \} \]

Then \( S \) is the disjoint union of these subsets. Notice that if \( v \in S \) lies above 2 then \( v \) is contained in \( S^+_\mu = S^+_\mu \cup S^-_\mu \). We apply results in Section 4 to \( I(*\chi_v//k_v) \) for each \( v \in S \). \( I(*\chi_v//k_v) \) takes the case \((#-1), (#-2)\) or \((#-3)\) according as \( v \in S^+_\mu, S^-_\mu \) or \( S^\mu \). Then, using the notations of Section 4 with respect to \( k = k_v \) and \( *\chi = *\chi_v \), we define irreducible subspaces \( Y^v \) and \( Y^v_v \) of \( I(*\chi_v//k_v) \) as follows:
RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_3$

Let $R^*_{v}(v)$ be the projection of $I(v\chi_v//k_v)$ to $Y_{i}(v)$ for $i = 0, 1$.

**Lemma 7.** Using the labelling of Srinivasan ([20]) for $v \neq 2$ and Enomoto ([7]) for $v | 2$,

$$
Y_{i}(v) \equiv \begin{cases} 
\text{the trivial representation} & \text{if } v \in S_u(\mu) \\
\theta_3 & \text{if } v \in S_r(\mu) \\
\theta_9 & \text{if } v \in S^*_u(\mu) \text{ and } v \not| 2 \\
\theta_1 & \text{if } v \in S^*_u(\mu) \text{ and } v | 2 \\
\theta_{11} & \text{if } v \in S_w(\mu) \text{ and } v \not| 2, \\
\theta_2 & \text{if } v \in S_w(\mu) \text{ and } v | 2 \\
\theta_4 & \text{if } v \in S_r(\mu) 
\end{cases}
$$

where $S_r(\mu)$ is the union of $S^*_r(\mu)$ and $S_r(\mu)$.

We remark that Enomoto's character table contains some misprints. The degrees of $\theta_1$ and $\theta_2$ are correctly \(\frac{1}{2} q_v(q_v + 1)^2\) and \(\frac{1}{2} q_v(q_v^2 + 1)\), respectively. This is easily checked from the defining equations of $\theta_1$ and $\theta_2$ in [7, p. 83].

**Proof.** First we assume $v \in S_u(\mu)$, hence one has the case ($\# - 1$). When $q = q_v$ is odd, the correspondence $\eta_i \mapsto \theta(\eta_i)$ is well known (cf. [23]). When $q = q_v$ is even, one can compute explicitly $\theta_3(Bw_sB)$ and $\theta_3(Bw_tB)$ by using the tables of conjugacy classes and characters in [7]. Then one has $\theta_3(Bw_sB) = \eta_2(\alpha_v)$ and $\theta_3(Bw_tB) = \eta_2(\alpha_v)$, hence $V_3 \cong \theta(\eta_3) = \theta_3$. Further, from the formula in [5], it follows $\dim \theta(\eta_2) = \dim \theta(\eta_3) = \frac{1}{2} q(q^2 + 1)$ and $\dim \theta(\eta_5) = \frac{1}{2} q(q + 1)^2$. Thus the character table concludes $V_3 \cong \theta(\eta_3) = \theta_2$ and $P^*_5 V_5 \cong \theta(\eta_5) = \theta_1$.

Next, assume $v \in S^*_u(\mu)$, hence one has the case ($\# - 2$). The formula in [5] deduces that $\dim \theta(\eta'_i) = \dim \theta(\eta'_3) = \frac{1}{2} (q_v^2 + 1)$, hence $\{\theta(\eta'_i), \theta(\eta'_3)\} = \{\theta_3, \theta_4\}$. Furthermore, Littlewood's formula (cf. [12]) deduces $\theta(\eta'_i)(g) \geq \theta(\eta'_3)(g)$ for any $g \in G(k_v)$. Then, by the character table in [20], it is known that $V'_3 \cong \theta(\eta'_1) = \theta_4$ and $V'_5 \cong \theta(\eta'_3) = \theta_5$.

Finally, assume $v \in S_r(\mu)$. From the similar arguments to the second case, it follows $\{\theta(\eta'_i), \theta(\eta'_3)\} = \{\theta_3, \theta_4\}$ and Imaginary$(\theta(\eta'_i)(g)) \geq$ Imaginary$(\theta(\eta'_3))$. 
(g)) for any \( g \in G(k) \). The character table in [20] allows us to conclude \( V_1'' \cong \theta(\eta_1'') = \theta_3, \ V_3'' \cong \theta(\eta_3'') = \theta_4 \).

The following lemma is an immediate consequence of (4.3), (4.4), (4.5) and (4.6). Here, note that the cardinality of \( S_\gamma(\mu) \) is even since \( \mu \) is an even character.

**Lemma 8.** Let \( \mu \in A_S(F) \) be a nontrivial character, \( \chi = \chi(\mu, \mu) \) and \( \Phi \in I(\chi//A)_s \) an arbitrary element. Then

\[
\text{Res}_{A=B} \ M^1(\sigma \tau \sigma, \Lambda, \chi) \Phi
\]

\[
= c_{\gamma}(\mu) \left( -\frac{1}{2} \right) \left( R_1^\mu(v) + \frac{2q_v}{(q_v + 1)^2} R_0^\mu(v) \right) \Phi_v
\]

\[
\otimes \left( \otimes_{\tau \in S_\gamma(\mu)} \left( R_1^\mu(v) - \frac{2q_v}{q_v^2 + 1} R_0^\mu(v) \right) \Phi_v \right) \otimes \left( \otimes_{\tau \in S_\gamma(\mu)} \left( R_1^\mu(v) - R_0^\mu(v) \right) \Phi_v \right)
\]

6. **Decomposition of** \( L_2^s(B, K_s, X(\mu)) \) **for nontrivial** \( \mu \)

We take a nontrivial \( \mu \in A_S(F) \) and put \( \chi = \chi(\mu, \mu) \). Let \( \Gamma(S) \) be the set of all maps from \( S \) to \( \{0,1\} \). For each \( \lambda \in \Gamma(S) \), \( \lambda_S(\mu) = \bigotimes_{\tau \in S} Y_{\lambda(\tau)}^\mu(v) \) is an irreducible subspace in \( \bigotimes_{\tau \in S} I(*)_{\lambda(\tau)/k_0} \) and \( R_1^\mu = \bigotimes_{\tau \in S} R_{\lambda(\tau)}^\mu(v) \) the projection of \( \bigotimes_{\tau \in S} I(*)_{\lambda(\tau)/k_0} \) to \( \lambda_S(\mu) \). By the isomorphism (1.1), \( \lambda_S(\mu) \) is identified with a subspace in \( I(\chi//A)_s \). The subset \( \Gamma(S, \mu) \) of \( \Gamma(S) \) is defined to be

\[
\Gamma(S, \mu) = \{ \lambda \in \Gamma(S) | \varepsilon(\mu) = (-1)^{|S_{\gamma(\mu)}| + \frac{1}{2}|S_{\gamma(\mu)|} + |2^{-1}(0) \cap S_{\gamma(\mu)}| + |2^{-1}(0) \cap S_{\gamma(\mu)}^0|} \}
\]

where \( \lambda^{-1}(0) \) is the inverse image of 0 by \( \lambda \). By Lemma 8, we obtain

**Proposition 3.** Let \( \mu \in A_S(F) \) be a nontrivial character, \( \lambda \in \Gamma(S) \) and \( \Phi \in \lambda_S(\mu) \) an arbitrary element. Then the constant term of \( \text{Res}_{A=B} E^1(g, \Phi, \Lambda) \) is equal to the following.
RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_4$

\[
\begin{cases}
\varepsilon(\mu)c_F(\mu)q_\mu(\lambda)e^{-\beta_1+\delta_H(\phi)}\Phi(g) & \text{if } \lambda \in \Gamma(S, \mu) \\
0 & \text{if } \lambda \notin \Gamma(S, \mu)
\end{cases}
\]

where

\[
c_F(\mu) = \frac{e(F)_2(1, \mu)}{\xi(2)(\mu)}, \quad q_\mu(\lambda) = \frac{2q_v}{\prod_{v \in \Omega \cap S(\mu)}(q_v + 1)^2} \times \frac{2q_\lambda}{\prod_{v \in \Omega \cap S(\mu)} q_v^2 + 1}
\]

**Corollary.** For any $\Phi \in I(\chi(\mu//A)_S$, one has

\[
\text{Res}_{\lambda = \beta_1} E^1(g, \Phi, \Lambda) = \sum_{\lambda \in \Gamma(S, \mu)} \text{Res}_{\lambda = \beta_1} E^1(g, R^\Phi, \Lambda).
\]

**Proof.** The constant term of the left hand side is equal to that of the right hand side. Hence, Langlands' lemma implies the assertion.

Let $L^2(B, K_S, X(\mu))_1$ be the space spanned by $\text{Res}_{\lambda = \beta_1} E^1(g, \Phi, \Lambda), \Phi \in \lambda_S(\mu)$ for each $\lambda \in \Gamma(S, \mu)$. Combining with Proposition 2, one has

**Theorem 1.** Let $\mu \in A_5(F)$ be a nontrivial character. Then one has a $K/K_S$-irreducible decomposition

\[
L^2(B, K_S, X(\mu)) = \bigoplus_{\lambda \in \Gamma(S, \mu)} L^2(B, K_S, X(\mu))_1.
\]

For each $\lambda \in \Gamma(S, \mu)$, the constant term map gives rise to a $K/K_S$-isomorphism from $L^2(B, K_S, X(\mu))$ onto $\lambda_S(\mu)$.

7. Decomposition of $L^2(B, K_S, X(\mu))$ for trivial $\mu$

Throughout this section, $\mu_0$ and $\chi = \chi(\mu_0, \mu_0)$ denote the trivial characters. We prove the following:

**Theorem 2.** $L^2(B, K_S, X(\mu_0))$ consists of constant functions.

We must calculate residues of $E^1(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{12}, \Lambda_{13}$ and of $E^2(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{12}, \Lambda_{23}, \Lambda_{24}$. In what follows, $E_i(g, \Phi, \Lambda)$ denote the constant terms of $E^i(g, \Phi, \Lambda)$ for $i = 1, 2$.

**Lemma 9.** $E^2(g, \Phi, \Lambda)$ is holomorphic at $\Lambda = \Lambda_{23} = \alpha_2/2$ and $\Lambda = \Lambda_{24} = \beta_2$ for any $\Phi \in I(\chi//A)_S$. 
Proof. It is sufficient to show that $E^0_0(g, \Phi, \Lambda)$ is holomorphic at $\Lambda = \alpha_2/2$ and $\beta_2$. By Lemma 5, one has

$$\text{Res}_{\Lambda = \alpha_2/2} E^0_0(bk, \Phi, \Lambda)$$

$$= c(F) \zeta(2) e^{-\beta_1 + \rho(kv)} \{ \otimes_{v \in S} \mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 1, 1) \Phi_v(k_v)$$

$$- \otimes_{v \in S} \mathcal{A}_v(\sigma, 0, 1) \cdot \mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 1, 1) \Phi_v(k_v) \}$$

$$= c(F) \zeta(2) e^{-\beta_1 + \rho(kv)} \{ \otimes_{v \in S} \mathcal{A}_v(\tau, 0, 1) \cdot \mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 1, 1) \Phi_v(k_v)$$

for any $b \in B(A)$ and $k \in K$. Since both $\mathcal{A}_v(\tau, 0, 1)$ and $\mathcal{A}_v(\sigma, 0, 1)$ are the identity map of $I(\chi_v//k_v)$ for any $v \in S$, the residues are identically zero.

**Lemma 10.** The residues of $E^i_0(g, \Phi, \Lambda)$, $i = 1, 2$ at $\Lambda = \Lambda_{12} = \delta$ are constant functions for any $\Phi \in I(\chi//k)_S$.

**Proof.** $\text{Res}_{\Lambda = \delta} E^0_0(bk, \Phi, \Lambda)$ equals

$$c(F) \zeta(2) \{ \otimes_{v \in S} \mathcal{A}_v(\tau, 0, 1) \cdot \mathcal{A}_v(\tau, 2, 1) \cdot \mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 1, 1) \Phi_v(k_v) \}$$

for any $b \in B(A)$, $k \in K$ and $i = 1, 2$. Since

$$\mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 2, 1) \cdot \mathcal{A}_v(\sigma, 3, 1) \cdot \mathcal{A}_v(\tau, 1, 1)$$

is the projection to the space of constant functions for any $v \in S$, we obtain the assertion.

**Lemma 11.** The order of pole of $E^1_0(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{13} = \beta_1$ is at most one for any $\Phi \in I(\chi//A)_S$.

**Proof.** By Lemma 5, one has

$$\lim_{z_1(\Lambda) \to \frac{1}{2}} (z_1(\Lambda) - \frac{1}{2})^2 E^1_0(k, \Phi, \Lambda)$$

$$= c(F)^2 \zeta(2)^2 \{ \otimes_{v \in S} \mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 1, 1) \cdot \mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 1, 1) \Phi_v(k_v)$$

$$- \otimes_{v \in S} \mathcal{A}_v(\sigma, 0, 1) \cdot \mathcal{A}_v(\tau, 1, 1) \cdot \mathcal{A}_v(\sigma, 1, 1) \cdot \mathcal{A}_v(\tau, 0, 1) \Phi_v(k_v) \}.$$
This vanishes since $\mathcal{A}_v(\tau, 0, 1)$ is the identity map for any $v \in S$.

By Lemmas 9, 10, 11 and the main theorem of [14], it is known that the space $L_2^2(B, K_\nu, X(\mu_0))$ is spanned by the constant functions and those residues of $E^1(g, \Phi, \Lambda), \Phi \in I(\chi//A)_S$ at $\Lambda = j$ which are square integrable. Hence, in order to finish the proof of Theorem 2, we must show the following.

PROPOSITION 4. Let $\Phi \in I(\chi//A)_S$. If $\text{Res}_{A = j}E^1(g, \Phi, \Lambda)$ is square integrable on $G(F) \backslash G(A)$, then it is identically zero.

Proof. Let $z = z_1(\Lambda)$ be the coordinate of $\Lambda$ on $S_i$. We note that $M^1(\sigma \tau, \Lambda, \chi)$ and $M^1((\sigma \tau)^2, \Lambda, \chi)$ may have double pole at $\Lambda = j$. By Lemma 5, the residue of $E_0^1(bk, \Phi, \Lambda)$ at $\Lambda = j$ equals

$$
\sum_{w \in W_1} \text{Res}_{A = j} \left\{ e^{\langle w_1 + \beta, H(b) \rangle} M^1(w, \Lambda, \chi) \Phi(k) \right\} 
$$

$$
= e^{\langle -\alpha/2 + \beta, H(b) \rangle} \text{Res}_{A = j} M^1(\tau \sigma, \Lambda, \chi) \Phi(k)
$$

$$
+ \lim_{z \to j} \frac{d}{dz} \left( z - \frac{1}{2} \right)^2 e^{\langle \alpha_2 \Lambda + \beta, H(b) \rangle} M^1((\sigma \tau)^2, \Lambda, \chi) \Phi(k)
$$

$$
+ \lim_{z \to j} \frac{d}{dz} \left( z - \frac{1}{2} \right)^2 e^{\langle (\sigma \tau)^2 \Lambda + \beta, H(b) \rangle} M^1((\sigma \tau)^2, \Lambda, \chi) \Phi(k)
$$

$$
= e^{\langle -\alpha/2 + \beta, H(b) \rangle} \text{Res}_{A = j} M^1(\tau \sigma, \Lambda, \chi) \Phi(k)
$$

$$
+ \left\{ \lim_{z \to j} (z - \frac{1}{2})^2 M^1((\sigma \tau)^2, \Lambda, \chi) \Phi(k) \right\} \frac{d}{dz} e^{\langle (\sigma \tau)^2 \Lambda + \beta, H(b) \rangle} \bigg|_{\Lambda = j}
$$

$$
+ e^{\langle -j + \beta, H(b) \rangle} \text{Res}_{A = j} M^1((\sigma \tau)^2, \Lambda, \chi) \Phi(k)
$$

Since the second and fourth terms are cancelled out each other, one has

$$
\text{Res}_{A = j} E_0^1(bk, \Phi, \Lambda)
$$

$$
= e^{\langle -\alpha/2 + \beta, H(b) \rangle} \text{Res}_{A = j} M^1(\tau \sigma, \Lambda, \chi) \Phi(k)
$$

$$
+ e^{\langle -j + \beta, H(b) \rangle} \text{Res}_{A = j} \left( M^1(\sigma \tau, \Lambda, \chi) \Phi(k) + M^1((\sigma \tau)^2, \Lambda, \chi) \Phi(k) \right)
$$

for $b \in B(A), k \in K$. Then it follows from Langlands $L^2$-ness criterion that the residue of $E^1(g, \Phi, \Lambda)$ at $\Lambda = j$ is square integrable on $G(F) \backslash G(A)$ if and only if the first term of the right hand side vanishes. Therefore, the next lemma completes the proof of Proposition 4.
Lemma 12. Let $\Phi \in I(\chi//A)_S$. If $\text{Res}_{A=\overline{A}}(M^{1}(\sigma\tau, \Lambda, \chi) \Phi)$ is identically zero, then so is $\text{Res}_{A=\overline{A}}(M^{1}(\sigma\tau, \Lambda, \chi) \Phi + M^{1}(\langle \sigma \rangle^2, \Lambda, \chi) \Phi)$. 

Proof. We assume $\text{Res}_{A=\overline{A}}(M^{1}(\sigma\tau, \Lambda, \chi) \Phi)$ is identically zero. Then, by Lemma 5, one has $\otimes_{v \in S} \mathcal{A}_v(\tau, 1, 1) \cdot \mathcal{A}_v(\sigma, 1, 1) \Phi_v = 0$. Hence, there exists at least one place $u \in S$ such that $\mathcal{A}_u(\tau, 1, 1) \cdot \mathcal{A}_u(\sigma, 1, 1) \Phi_u = 0$. We fix such a place $u$. For $k = k_u$ and $\ast \chi = \ast \chi_u$, we use the same notations as in Section 4 case $(\# - 1)$. Then, by (4.2),

$$\mathcal{A}_u(\tau, 1, 1) \cdot \mathcal{A}_u(\sigma, 1, 1) = P_1 + \left( P_1 + \frac{\sqrt{2} q_u}{q_u + 1} R_u \right) P_5^*.$$ 

Therefore, $\Phi_u$ must belong to the space $(P_2 + P_3 + P_4 + P_5^*) I(\ast \chi_u//k_u), \text{ where}$

$$P_5^* = P_1 - \frac{\sqrt{2} q_u}{q_u + 1} R_u$$

is a projection satisfying $P_5^* P_5^- = P_5^- P_5^* = 0$.

Assume $\Phi_u \in (P_2 + P_3) I(\ast \chi//k_u)$. Then one has

$$\mathcal{A}_u(\sigma, 2z, 1) \cdot \mathcal{A}_u(\tau, z + \frac{1}{2}, 1) \cdot \mathcal{A}_u(\sigma, 1, 1) \Phi_u = 0$$

and hence $M^1(\sigma\tau, \Lambda, \chi) \Phi = M^1((\sigma\tau)^2, \Lambda, \chi) \Phi = 0$. This implies the assertion.

Assume $\Phi_u \in P_2 I(\ast \chi_u//k_u)$. Then one has

$$\mathcal{A}_u(\sigma, 2z, 1) \cdot \mathcal{A}_u(\tau, z + \frac{1}{2}, 1) \cdot \mathcal{A}_u(\sigma, 1, 1) \Phi_u$$

and

$$\mathcal{A}_u(\sigma, 1, 1) \cdot \mathcal{A}_u(\tau, z + \frac{1}{2}, 1) \cdot \mathcal{A}_u(\sigma, 2z, 1) \cdot \mathcal{A}_u(\tau, z - \frac{1}{2}, 1) \Phi_u$$

and

$$\text{Res}_{A=\overline{A}} M^1(\sigma\tau, \Lambda, \chi) \Phi$$

$$= \text{Res}_{z=\frac{1}{2}} \frac{\xi(2z) \xi(z + \frac{1}{2})}{\xi(2z + 1) \xi(z + \frac{3}{2})} \left\{ - \frac{(q_u^{-\frac{1}{2}} - 1) q_u}{q_u^{-\frac{3}{2}} - 1} \Phi_u \right\} \otimes \ldots$$
\[
\left\{ \otimes_{v \in S-(u)} A_v(\sigma, 2z, 1) \ast A_v(\tau, z + \frac{1}{2}, 1) \ast A_v(\sigma, 1, 1) \Phi_v \right\} \\
= - \frac{c(F)^2 q_u \log q_u}{2 \xi(2)q_u^2 - 1} \Phi_v \otimes \\
\left\{ \otimes_{v \in S-(u)} A_v(\sigma, 1, 1) \ast A_v(\tau, 1, 1) \ast A_v(\sigma, 1, 1) \Phi_v \right\} \\
\text{Res}_{\lambda = \beta_1} M^1((\sigma \tau)^2, \lambda, \chi) \Phi \\
= \text{Res}_{\tau = \frac{1}{2}} \frac{\xi(2z)\xi(z - \frac{1}{2})}{\xi(2z + 1)\xi(z + \frac{3}{2})} \left\{ \frac{(q_u^{z-\frac{3}{2}} - 1)(q_u^{z-\frac{1}{2}} - 1)q_u^2 \Phi_v}{(q_u^{z+\frac{1}{2}} - 1)(q_u^{z+\frac{3}{2}} - 1)} \right\} \otimes \\
\left\{ \otimes_{v \in S-(u)} A_v(\sigma, 1, 1) \ast A_v(\tau, 1, 1) \ast A_v(\sigma, 1, 1) \ast A_v(\tau, 0, 1) \Phi_v \right\} \\
\text{Res}_{\lambda = \beta_1} M^1((\sigma \tau)^2, \lambda, \chi) \Phi \\
= \frac{c(F)^2 q_u \log q_u}{2 \xi(2)q_u^2 - 1} \Phi_v \otimes \\
\left\{ \otimes_{v \in S-(u)} A_v(\sigma, 1, 1) \ast A_v(\tau, 1, 1) \ast A_v(\sigma, 1, 1) \ast A_v(\tau, 0, 1) \Phi_v \right\}.
\]

Since \( A_v(\tau, 0, 1) \) is the identity map for any \( v \in S-(u) \), \( \text{Res}_{\lambda = \beta_1} \left( M^1(\sigma \tau, \lambda, \chi) \Phi + M^1((\sigma \tau)^2, \lambda, \chi) \Phi \right) \) vanishes.

Assume \( \Phi_v \in P_S \left( \{ \chi_u \} / k_u \right) \). From \( P_S^* \Phi_v = 0 \), it follows \( M^1(\sigma \tau, \lambda, \chi) \Phi = 0 \). On the other hand, one has

\[
\text{Res}_{\lambda = \beta_1} M^1((\sigma \tau)^2, \lambda, \chi) \Phi \\
= \frac{\xi(2z)\xi(z - \frac{1}{2})}{\xi(2z + 1)\xi(z + \frac{3}{2})} \left\{ \frac{(q_u^{z-\frac{3}{2}} - 1)(q_u^{z-\frac{1}{2}} - 1)q_u^2 \Phi_v}{(q_u^{z+\frac{1}{2}} - 1)(q_u^{z+\frac{3}{2}} - 1)} \right\} \\
\times \frac{(q_u^2 + 1)^{\sqrt{2} q_u}}{(q_u + 1)^3} R \Phi_v \\
\otimes \left\{ \otimes_{v \in S-(u)} A_v(\sigma, 1, 1) \ast A_v(\tau, z + \frac{1}{2}, 1) \ast A_v(\sigma, 2z, 1) \ast A_v(\tau, z - \frac{1}{2}, 1) \Phi_v \right\} \\
= - \frac{c(F)^2}{2 \xi(2)q_u^2 - 1} \left\{ \frac{2 q_u \log q_u}{q_u^2 - 1} + \frac{2 q_u \log q_u}{q_u^2 - 1} \right\} \frac{(q_u^2 + 1)^{\sqrt{2} q_u}}{(q_u + 1)^3} R \Phi_v \\
\otimes \left\{ \otimes_{v \in S-(u)} A_v(\sigma, 1, 1) \ast A_v(\tau, 1, 1) \ast A_v(\sigma, 1, 1) \ast A_v(\tau, 0, 1) \Phi_v \right\} \\
= 0.
\]

This completes the proof of Lemma and hence Theorem 2.
8. Residual automorphic representations

In this section, we give another representation theoretic interpretation of Theorem 1. For this, it is convenient to consider various $S$ simultaneously. Hence, we set $A_{\infty}(F) = \cup_{S} A_{S}(F)$, where $S$ runs over finite subsets of $V_{f}$. For each $\mu \in A_{\infty}(F)$, $S_{r}(\mu)$ also denotes the set of $v \in V_{f}$ such that $\mu_{v}$ is ramified.

We fix a nontrivial $\mu \in A_{\infty}(F)$. Let

$$I(\mu, \beta_{i}) = \text{Ind}(B(A) \uparrow G(A) : e^{(g_{1}H^{-1})}\chi(\mu, \mu))$$

be the normalized induced representation of $G(A)$. This $I(\mu, \beta_{i})$ has a restricted tensor product decomposition: $I(\mu, \beta_{i}) = \bigotimes_{s} I_{s}(\mu, \beta_{i})$ (cf. [15]). For any finite set $S \subset V_{f}$ containing $S_{r}(\mu)$, we denote by $I(\mu, \beta_{i})_{S}$ the subspace spanned by right $K_{S}$-invariant elements of $I(\mu, \beta_{i})$. Further, $I^{1}(\mu, \beta_{i})$ denotes the $G(A)$-submodule of $I(\mu, \beta_{i})$ generated by $I(\mu, \beta_{i})_{S}$ for all finite sets $S \subset V_{f}$ containing $S_{r}(\mu)$. Then the residue map

$$I(\chi(\mu, \mu)\big/ A)_{S} \rightarrow L^{2}_{d}(B, K_{S}) : \Phi \mapsto \text{Res}_{A=\beta_{i}, E^{1}(g, \Phi, \Lambda)}$$

induces an intertwining operator from $I^{1}(\mu, \beta_{i})$ into the space associated with residual spectrums $L^{2}_{d}(B)$. We write $\pi(\mu)$ for the image of this intertwining operator. Namely, $\pi(\mu)$ is an automorphic representation generated by $L^{2}_{d}(B, K_{S}, X(\mu))$ for all finite sets $S \subset V_{f}$ containing $S_{r}(\mu)$.

**Proposition 5.** Let $\mu \in A_{\infty}(F)$ be a nontrivial character and $\bigotimes_{s} \pi_{s}(\mu)$ a restricted tensor product decomposition of $\pi(\mu)$. If $u \notin S_{r}(\mu)$, then $\pi_{u}(\mu)$ is a spherical irreducible representation of $G(F_{u})$.

*Proof.* From the above construction, $\pi_{u}(\mu)$ is clearly spherical. We show the irreducibility of it. In the following, we denote by $\pi_{u}(\mu)_{L}$ for an open subgroup $L \subset K_{u}$ the subspace consisting of $L$-invariant elements of $\pi_{u}(\mu)$.

First, assume $u$ is a finite place. Then it follows from [3. Corollary 3.3.7] that the $u$-component of a restricted tensor product of $I^{1}(\mu, \beta_{i})$ coincides with $I_{u}(\mu, \beta_{i})$, so that $\pi_{u}(\mu)$ is isomorphic to a quotient representation of $I_{u}(\mu, \beta_{i})$. We take a finite set $S \subset V_{f}$ such that $S_{r}(\mu) \cup \{u\} \subset S$ and $|S_{r}(\mu)| \geq 2$. Then, by Theorem 1,

$$\pi(\mu)_{K_{S}} = L^{2}_{d}(B, K_{S}, X(\mu)) \cong \bigoplus_{\lambda \in \Gamma(S,\mu)} \lambda_{S}(\mu)$$

and

$$\pi_{u}(\mu)_{\text{Ker}(\nu)} = Y_{\nu}(u) \oplus Y_{\nu}(u).$$
Let $L_u = r^{-1}_u(B(K_u))$ be an Iwahori subgroup of $K_u$. By the Frobenius reciprocity law and Lemma 7, it is seen

\[ \dim \pi_u(\mu)_{L_u} = \begin{cases} 3 & \text{if } u \in S_u^+(\mu) \\ 2 & \text{if } u \in S_u^-\text{(}\mu) \end{cases}. \]

On the other hand, from [2, Lemma 4.7] and [18, Chapters 3 and 6], it follows that $I_u(\mu, \beta_1)$ has a composition series of the form

\[ (0) = J_4 \subset J_3 \subset J_2 \subset J_1 \subset J_0 = I_u(\mu, \beta_1) \]

$J_0/J_1$ is spherical

and

\[ \dim (J_0/J_1)_{L_u} = \begin{cases} 3 & \text{if } u \in S_u^+(\mu) \\ 2 & \text{if } u \in S_u^-\text{(}\mu) \end{cases}. \]

This implies $\pi_u(\mu) \cong J_0/J_1$.

Next let $\pi_\omega(\mu)$ (resp. $\pi_f(\mu)$) be the infinite part (resp. finite part) of $\pi(\mu)$. We show $\pi_\omega(\mu)$ is irreducible. Since $L_\lambda^H(B)$ decomposes to the sum of irreducible subspaces, so is $\pi(\mu)$. Hence, if $\pi_\omega(\mu)$ is reducible, it decomposes to the sum of proper subspaces $\pi^\omega(\mu)$ and $\pi^\omega(\mu)$. Then, by Theorem 1, one has

\[ (\pi^\omega(\mu)_{K_S} \oplus \pi^\omega(\mu)_{K_u}) \otimes \pi_f(\mu)_{K_{S,f}} \cong \sum_{\lambda \in \Gamma(\omega)} \lambda_S(\mu) \]

for any finite set $S \subset V_f$ containing $S_t(\mu)$, where $K_{S,f}$ is the finite part of $K_S$. Since $\pi_f(\mu)_{K_{S,f}} \cong \bigoplus_{\lambda \in \Gamma(S,\omega)} \lambda_S(\mu)$ and the multiplicity of $\lambda_S(\mu)$ in $\pi(\mu)_{K_S}$ is one for any $\lambda \in \Gamma(S,\mu)$, either $\pi^\omega(\mu)_{K_u}$ or $\pi^\omega(\mu)_{K_u}$ must be trivial. We assume $\pi^\omega(\mu)_{K_u}$ is trivial. Then $\pi(\mu)_{K_S} = \pi^\omega(\mu)_{K_u} \otimes \pi_f(\mu)_{K_{S,f}}$. Since $\pi(\mu)$ is generated by $\pi(\mu)_{K_S}$ for finite sets $S \subset V_f$ one has $\pi(\mu) = \pi^\omega(\mu) \otimes \pi_f(\mu)$, and hence $\pi_\omega(\mu) = \pi^\omega(\mu)$.

It seems that $\pi(\mu)$ is irreducible. However, at present, we know only the upper bounds of the number of irreducible components of $\pi(\mu)$.

**Theorem 3.** Let $\mu \in A_\omega(F)$ be a nontrivial element. The number of irreducible components of $\pi(\mu)$ is less than or equal to $2^{\lvert S_\omega(\mu) \rvert}$.

**Proof.** By Proposition 5, it is sufficient to show that the number of irreducible components of $\pi_v(\mu)$ is at most 2 for any $v \in S_v(\mu)$. Thus we fix a $v \in S_v(\mu)$. Since the $v$-component of a restricted tensor product of $I^1(\mu, \beta_1)$ equals $I_v(\mu, \beta_1)$ by [3, Corollary 3.3.7], $\pi_v(\mu)$ is a quotient representation of $I_v(\mu, \beta_1)$. Further, by [3, Theorem 7.2.4], each constituent $J$ of $\pi_v(\mu)$ has a non-zero vector...
fixed by \( \text{Ker}(r_v) \). In other words, the subspace of \( \text{Ker}(r_v) \)-invariant elements of \( J \) contains necessarily at least one nontrivial representation of \( G(k_v) \). On the other hand, if we take a finite set \( S \subset V_f \) such that \( S_r(\mu) \subset S \) and \( |S_r(\mu)| \geq 2 \), then one has \( \pi(\mu)_{K_S} \cong \bigoplus_{\lambda \in \Gamma(S, \mu)} \lambda_S(\mu) \) and hence \( \pi_v(\mu)_{Ker(r_v)} \cong Y_{\psi}(v) \bigoplus Y_{\psi}(v) \) by Theorem 1. This implies that the number of irreducible constituents of \( \pi_v(\mu) \) is at most 2.

**Theorem 4.** Let \( \mu \in A_{\omega}(F) \) be a nontrivial character. Then each irreducible constituent of \( \pi(\mu) \) is of multiplicity one in \( L^2(B) \).

**Proof.** By Proposition 5 and the proof of Theorem 3, it is known that each irreducible constituent of \( \pi(\mu) \) has a non-zero vector fixed under \( K_S \) for sufficiently large \( S \). Then the assertion follows from the fact that \( \lambda_S(\mu) \) is of multiplicity one in \( L^2(B, K_S) \) for any \( \lambda \in \Gamma(S, \mu) \).

Finally, we remark about an \( L \)-function of \( \pi(\mu) \). Proposition 5 implies that the set of irreducible constituents of \( \pi(\mu) \) becomes an \( L \)-packet of automorphic representations. We write \( L(s, \pi(\mu)) \) for the standard (degree 5) \( L \)-function attached to this \( L \)-packet. If we define the factor \( L_v(s, \pi(\mu)) \) attached to a ramified place \( v \in S_r(\mu) \) by \( (1 - q_v^{-s})^{-1} \), then the simple calculation gives

\[
L(s, \pi(\mu)) = \zeta_F(s) L(s, \mu)^2 L(s - 1, \mu) L(s + 1, \mu),
\]

where \( \zeta_F(s) \) is the Dedekind zeta function of \( F \).

**References**

7. H. Enomoto, The characters of the finite symplectic group \( Sp(4, q) \), \( q = 2' \), Osaka J. Math. 9 (1972), 75–94.
RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_4$


Department of Mathematics
College of General Education
Tohoku University
Kawauchi Sendai 980
Japan