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RELATIVE NORMAL-CONVEXITY AND AMALGAMATIONS

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The notion of relative normal-convexity is introduced and its connection to equations over groups is studied. We study the effect on relative normal-convexity by free products, amalgamations, and HNN extensions.

0. INTRODUCTION

A subgroup S of a group G is said to be *normal-convex* if, given any subset $R \subset S$, the natural map

$$S/\langle\langle R\rangle\rangle_S \to G/\langle\langle R\rangle\rangle_G$$

is injective – where $\langle\langle A\rangle\rangle_B$ denotes the normal closure of A in B (we will also write $\langle\langle A\rangle\rangle$ as B is usually clear from the context). This concept was introduced by Stallings in [4].

In [1] and [2] we studied normal-convexity and its connection to solving equations over groups. We also proved the following three results about normal-convexity:

(1) if $S_1 \subset G_1$ and $S_2 \subset G_2$ are normal-convex then $S_1 * S_2 \subset G_1 * G_2$ is normal-convex,

(2) if $S_1 \subset G_1$ and $S_2 \subset G_2$ are normal-convex, and if $C \subset S_i$ for $i = 1,2$, then $S_1 *_C S_2 \subset G_1 *_C G_2$ is normal-convex,

(3) if $S \subset G$ is normal-convex and $C \cup \phi(C) \subset S$ then $S *_C \phi \subset G *_C \phi$ is normal-convex.

Or more succinctly, amalgamations and HNN extensions of group pairs preserve normal-convexity.

In this paper, we define a relative version of normal-convexity, which is connected to equations over subvarieties of groups — such as torsion-free groups or nilpotent groups, and obtain results analagous to the above.

Our results came about in an unsuccessful attempt to obtain a *Freiheitssatz* for torsion-free groups.

The layout of this paper is as follows. In Section 1 we give the definition of relative normal-convexity and show its connection to solving equations over groups. In Section 2 we prove that amalgamations and HNN extensions of group pairs preserve relative normal-convexity. In Section 3 we investigate more general amalgamations and describe briefly our approach to the Freiheitssatz for torsion-free groups.

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1. Definitions and preliminary results

Let P be a property of group elements — for instance "having finite order". We say that P is *functorial* if given a homomorphism $f : G \to H$ and an element $g \in G$ having property P, either $f(g)$ has property P or $f(g) = 1$. If we wish to stress the ambient group we will say g has property P in G. We will write

$$G_P = \{g \in G \mid g = 1 \text{ or } g \text{ has property } P\}.$$

Some examples of functorial properties are:

P_1: g has finite order
P_2: $\exists n \geqslant k$ with $g \in {}_pG_n$
P_3: $\exists n \geqslant k$ with $g \in {}_pG_{[n]}$

where p is either zero or a prime, $k \geqslant 1$ an integer, and ${}_pG_n$ and ${}_pG_{[n]}$ denote the n-th term of the p-central series and the p-derived series, respectively (when $p = 0$ it is the usual central or derived series; see [4]). For the rest of the paper let P be a fixed functorial property.

Define the class of groups determined by P by

$$\mathcal{C}_P = \{\, G \mid G_P = \{1\}\,\}.$$

For example, $\mathcal{C}_{P_1}$ is the class of torsion-free groups, while P_2 and P_3 yield p-nilpotent and p-solvable groups of class k (or $k-1$ depending on the indexing).

We now want to define a notion of normal-convexity that reduces to the standard concept in $\mathcal{C}_P$ and is preserved by amalgamations of group pairs. A subset W of a group H is said to be *chainable* or *chainable mod* P if we can write $W = \bigcup_{i \geqslant 1} W_i$ with each W_i finite and where, letting $H_0 = H$,

$$H_j = H/\left(\langle\langle \bigcup_{i=1}^{j} W_i \rangle\rangle\right)$$

for $j \geqslant 1$, and $p_j : H \to H_{j-1}$ be the natural map (for $j = 1$ it is the identity), one has $p_j(W_j) \subset (H_{j-1})_P$ for all j. We will call the expression for W in terms of the W_j's the *chaining* of W. Observe that W_1 is a subset of H consisting of elements having property P, and possibly the identity element. W_2 consists of elements of H that are either in the normal closure of W_1 or project to elements of $H/\langle\langle W_1\rangle\rangle$ having property P, and so forth. Thus if $H \in \mathcal{C}_P$ then there are no non-trivial chainable sets. A homomorphism $\alpha : S \to G$ is said to be *normal-convex mod* P if, given any $R \subset S$ the natural map

$$\alpha_R : S/\langle\langle R\rangle\rangle_S \to G/\langle\langle R\rangle\rangle_G$$

has kernel being the normal closure of a chainable subset of $S/\langle\langle R\rangle\rangle$. If the property is P_1 then we will use the terminology *torsion-free normal-convex*. If P is understood from the context we will just say *relative normal-convex*.

We will need the following lemma on chainable subsets in Section 2:

Lemma 1.1.

(1) *If W and V are chainable subsets of G then $W \cup V$ is chainable.*
(2) *If $f : G \to H$ is a homomorphism and W is a chainable subset of G then $f(W)$ is a chainable subset of H.*

Proof: First, let $W = \bigcup_{i\geqslant 1} W_i$ and $V = \bigcup_{i\geqslant 1} V_i$ be chainings. We claim that the expression $\bigcup_{i\geqslant 1} (W_i \cup V_i)$ is a chaining for $W \cup V$. Clearly, $W_1 \cup V_1 \subset G_P$. Write $N = \langle\langle \bigcup_{i=1}^{j-1} (W_i \cup V_i)\rangle\rangle_G$ and $M = \langle\langle \bigcup_{i=1}^{j-1} W_i\rangle\rangle_G$. Further, let $G' = G/M$ The natural map

$$G \xrightarrow{\gamma} G/N$$

factors as a composition

$$G \xrightarrow{\alpha} G' \xrightarrow{\beta} G'/\left(\langle\langle \bigcup_{i=1}^{j} \alpha(V_i)\rangle\rangle\right).$$

By assumption $\alpha(W_j) \subset (G')_P$. By functorality $\gamma(W_j) = \beta(\alpha(W_j)) \subset (G/N)_P$. By symmetry $\gamma(V_j) \subset (G/N)_P$ as well.

Now suppose $f : G \to H$ is given. We claim that $\bigcup_{i\geqslant 1} f(W_i)$ is a chaining for $f(W)$. Consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ G_{j-1} & \xrightarrow{f_{j-1}} & H_{j-1} \end{array}$$

where f_{j-1} is induced by f, α and β are the natural maps, and G_{j-1} and H_{j-1} are the appropriate quotients. By assumption $\alpha(W_j) \subset (G_{j-1})_P$. Applying f_{j-1} and using functorality yields $f_{j-1}(\alpha(W_j)) \subset (H_{j-1})_P$. But $f_{j-1} \circ \alpha = \beta \circ f$ and so $\beta(f(W_j)) \subset (H_{j-1})_P$ as desired. $\square$

Just as there is a connection between normal-convexity and solving equations over groups, relative normal-convexity is related to solving equations over subclasses of groups.

Recall that an equation over a group G is of the form $w = 1$ (or $w(T_1, \ldots, T_n) = 1$) where $w \in G * F$ with F being free on $T_1, \ldots, T_n$. The T_j's are called the unknowns while the elements of G occurring in w are the coefficients. So for example the equation a^2TabT^{-1} over the group $\langle a, b \mid \ \rangle$ has T as its unknown and a^2 and ab as its coefficients.

The equation has a solution over G, or can be solved over G, if there is a group $\overline{G}$ containing G as a subgroup and possessing elements $t_1, \ldots, t_n$ with $w(t_1, \ldots, t_n) = 1$ in $\overline{G}$. Equivalently, the natural map

$$G \to G * F / \langle\langle w \rangle\rangle$$

is injective. Let $\exp_j(w)$ denote the exponent sum of T_j in w. The Kervaire conjecture speculates that any equation over any group with non-zero exponent sum for some unknown has a solution.

Note that sometimes an equation $w = 1$ over a group G may be considered an equation over a subgroup H of G. This happens when $w \in H * F \subset G * F$. And the equation can be solved over G if and only if it can be solved over H. One direction is immediate. Regarding the other, suppose a solution to $w = 1$ can be found in $\overline{H}$. Merely set $\overline{G} = G *_H \overline{H}$.

Similarily we could speak of a system or set of equations over a group, $\mathcal{S} = \{w_i = 1\}$. If the set is finite then the Kervaire conjecture says that $\mathcal{S}$ has a solution if the matrix of exponent sums $[\exp_j(w_i)]$ has maximal rank, where, by adding superfluous unknowns, we assume that the number of unknowns is $\geqslant$ the number of equations. As for solving an infinite system of equations, observe that it suffices to solve each finite subsystem.

Given $\mathcal{S} = \{w_i = 1\}$, a system of equations over a group G, let F be the free group on the unknowns. Any homomorphism $\phi : G \to H$ extends to a map $\phi_F : G * F \to H * F$ by mapping each $T_i \mapsto T_i$. Thus $\mathcal{S}$ and ϕ give rise to a set of equations $\phi(\mathcal{S}) = \{\phi_F(w_i) = 1\}$ over H (in fact $\phi(\mathcal{S})$ is a set of equations over the subgroup $\phi(G) \subset H$). Using this notation we have the following:

Proposition 1.2. *Suppose $\mathcal{S} = \{w_i = 1\}$ is a system of equations over a group G. Let F be the free group on the unknowns. Suppose $\phi : G \to H$ is a homomorphism and $H_P = \{1\}$. Assume the natural map*

$$\alpha : G \to G * F / \langle\langle \{w_i\} \rangle\rangle$$

is normal-convex modulo P. Then the system $\phi(\mathcal{S})$ can be solved over H.

Proof: As noted above, it suffices to show that $\phi(\mathcal{S})$ can be solved over $G_1 = \phi(G) \subset H$. let $K = \text{kernel}(\phi)$. The natural map

$$\alpha_1 : G_1 \to G_1 * F / \langle\langle \{\phi_F(w_i)\} \rangle\rangle$$

is equivalent to the map

$$G/K \to \Big((G/K) * F\Big)/\langle\langle\{\phi_F(w_i)\}\rangle\rangle$$

which is equivalent to the map

$$G/K \to \Big((G * F)/\langle\langle\{w_i\}\rangle\rangle\Big)/\langle\langle K\rangle\rangle.$$

By relative normal-convexity this map has kernel the normal closure of a chainable mod P subset of G/K. But $G/K = G_1 = \phi(G) \subset H$ and as P is functorial $(G/K)_P \subset H_P = \{1\}$. Hence there are no non-trivial chainable subsets of G/K. Thus α is injective and $\phi(\mathcal{S})$ can be solved. ∎

We will use the preceding result to reduce solving equations over classes of groups to showing that a map of a free group is relatively normal-convex.

Let $\mathcal{S} = \{w_i = 1\}$ be a system of equations over a group G. We will say that $\mathcal{S}$ is *universal* if G is free, each coefficient appears only once in the equations, and the set of coefficients forms a basis for G. A system of equations $\mathcal{S}'$ over a group H is said to be *modelled on* $\mathcal{S}$ if there is a homomorphism $\phi : G \to H$ with $\mathcal{S}' = \phi(\mathcal{S})$. Clearly any system of equations over any group is modelled on some universal system. With this in mind, we have:

COROLLARY 1.3. *Assume $\mathcal{S} = \{w_i = 1\}$ is a universal system of equations over a free group G. Let F be the free group on the unknowns. Assume the natural map*

$$G \to G * F/\langle\langle\{w_i\}\rangle\rangle$$

is normal-convex modulo P. Then any system modelled on $\mathcal{S}$ over a group in $\mathcal{C}_P$ can be solved.

We close this section with an example. Consider the equation $aTbT^{-1}$ over the free group $\langle\, a, b \mid \,\rangle$. As an HNN extension of a group contains that group as a subgroup, it is a simple exercise to see that the inclusion

$$\langle a, b\rangle \hookrightarrow \langle\, a, b, T \mid aTbT^{-1}\,\rangle$$

is torsion-free normal-convex.

2. Amalgamations and HNN extensions of group pairs

We need to recall and adapt the topological formulation of normal-convexity from [1] (also see [5, Section 7] for similar constructions). Let $f : F \to X$ be a map of a disk with holes into a topological space (we will also consider maps of pairs). We may choose a component of ∂F and call it the *outer* or *outermost boundary component*. When such a choice has been made we will call f a *based map*. The other components of ∂F are called the *inner boundary components*.

We need a few topological definitions. Let Y be a topological space. If ω is a loop in Y then ω determines a conjugacy class, $[\omega]$, of elements of $\pi_1(Y,y)$ for any basepoint y in the path component containing the image of ω. We will say that ω has property P in Y if that conjugacy class contains an element having property P. Similarly, given a set of loops, $\Omega = \{\omega_r\}$, each mapping to the same path component, then we say that $\{\omega_r\}$ is *chainable* (or *chainable mod* P) if, for any basepoint y, there is some choice $w_r \in [\omega_r]$ with $\{w_r\}$ chainable in $\pi_1(Y,y)$. And if the chaining of $\{w_r\}$ is $W_j = \{w_{j_r}\}$ with $w_{j_r} \in [\omega_{j_r}]$ then we will say that Ω has the chaining $\Omega_j = \{\omega_{j_r}\}$. If we wish to stress the space, we will use the phrase "chainable in Y".

Now fix a based map $f : (F, \partial F) \to (X, A)$. Enumerate the components of ∂F as $a_0, a_1, \ldots, a_n$, where a_0 is outermost. Construct a space A_f by attaching n disks to A using the maps $f \restriction a_i$, for $i \geqslant 1$ as attaching maps. If $\overline{f} : (\overline{F}, \partial\overline{F}) \to (X, A)$ is a based map of another disk with holes, with outer boundary component $\overline{a}_0$ and inner boundary components $\overline{a}_1, \ldots, \overline{a}_m$, then we say that $\overline{f}$ is *derived from* f *mod* P if there is a $k \geqslant 0$ such that the following holds:

(1) the map $f \restriction a_0$ is homotopic in A to $\overline{f} \restriction \overline{a}_0$,
(2) if $1 \leqslant i \leqslant k$ then the map $\overline{f} \restriction \overline{a}_i$ is homotopic in A to a map $f \restriction a_t$ for some $t \geqslant 1$ (so if $k = 0$ then there are no such index i),
(3) the set of loops $\{\overline{f} \restriction \overline{a}_i \mid i > k\}$ is chainable in A_f.

We will call the set of loops, $\{\overline{f} \restriction \overline{a}_i \mid i > k\}$, the *$P$-loops*. If we wish to stress the maps involved we will call them the P-loops of $(f, \overline{f})$. Observe that the loops in (2) could also be considered P-loops as they are all null-homotopic in A_f. However it turns out to be useful to focus on the loops that satisfy condition (3) but not condition (2). So we view the P-loops as being those that do not satisfy condition (2). Note that in [1] we define a notion of "derived from", but there we do not allow any P-loops.

Here is a lemma that will useful in Section 3. It is proven by applying the definition of "derived".

LEMMA 2.1. *Suppose $f_i : (F_i, \partial F_i) \to (X, A)$ are based maps of disks with holes for $i = 1, 2, 3$ with f_3 derived from f_2 and f_2 derived from f_1. Assume at least one of the pairs (f_2, f_3) and (f_1, f_2) has no P-loops. Then f_3 is derived from f_1. Moreover, if both of the pairs has no P-loops then neither does (f_1, f_3).*

We will say that a pair of spaces (X, A) is *topologically normal-convex mod* P if given any based map $f : (F, \partial F) \to (X, A)$ of a disk with holes there is a map, $\overline{f}$, of another disk with holes, $\overline{F}$, that is derived from f and with $\overline{f}(\overline{F}) \subset A$. It is then immediate that we have the following:

Proposition 2.2. *Let (X, A) be a pair of connected topological spaces. Then (X, A) is topologically normal-convex mod P if and only if given any $x \in A$ the inclusion map induces a map $\pi_1(A, x) \to \pi_1(X, x)$ that is normal-convex mod P.*

When the topological spaces are not connected relative normal-convexity can be viewed as a statement about systems of subgroups.

Now we turn to our main result. Recall that a (tame) subcomplex T of a cell complex X is said to be *two-sided* or *bicollared* in X if a regular neighbourhood N of T in X is PL-equivalent to $T \times [-1, +1]$ with T corresponding to $T \times \{0\}$. If T is two-sided in X then the complex $X_T =$ the closure of $(X \setminus N)$ is said to be obtained by *cutting X along T*. Note that the interior of each component of X_T is equivalent to the corresponding component of $X \setminus T$.

Suppose $T \subset A \subset X$ are cell complexes with T two-sided in both A and X. By choosing an appropriate regular neighbourhood we may cut A and X along T simultaneously, yielding $A_T \subset X_T$.

Proposition 2.3. *Assume $T \subset A \subset X$ are cell complexes with T two-sided in both A and X. If (X_T, A_T) is normal-convex mod P then (X, A) is normal-convex mod P.*

Proof: Fix $f : (F, \partial F) \to (X, A)$ a map of a disk with holes. We need to show that there is a map of a disk with holes $\overline{f} : \overline{F} \to A$ that is derived from f. We will do this in steps, each step resulting in a map into X of a disk with holes derived from f. Additionally, each such map will be transverse to T.

We start with a few definitions. Assume we are given $f' : (F', \partial F') \to (X, A)$, a map of a disk with holes, that is derived from f and transverse to T. Transversality implies that $(f')^{-1}(T)$ is a properly and tamely embedded, though possibly disconnected, one-manifold in F' (see Figure 2.1). By a *T-component* of f' let us mean the closure of a component of $F' \setminus (f')^{-1}(T)$. Each T-component K is a disk with holes, and $f' \restriction K$ can be viewed as a map of the pair $(K, \partial K)$ into the pair (X_T, A_T). If K and K' are two T-components we will say that K' is *interior to* K if K separates K' from the outer boundary component of F'. Let $I(K)$ be the union of K together with all T-components that are interior to K (see Figure 2.2). We will say that f' is *I-derived from* f if

(1) f' is derived from f, and
(2) if the T-component K contains a P-loop then $f(I(K)) \subset A$.

Define the *complexity* of the map f' to be the number of T-components that are not mapped entirely into A. By Proposition 2.2, we need only find a map f' dervived from f of complexity zero.

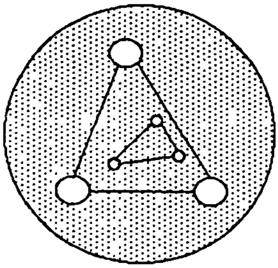

Figure 2.1

Figure 2.2

We start by homotoping f to make it transverse to T. The resulting map, which we call f_0, is I-derived from f because there are no P-loops and the definition of derived allows homotopies.

Now suppose we have constructed maps $f_0, f_1, \ldots, f_\ell$ of disks with holes $F_0, F_1, \ldots, F_\ell$, each transverse to T and with decreasing complexities such that each map is I-derived from f.

If the complexity of f_ℓ is non-zero we will show how to construct a map $f_{\ell+1}$ of a disk with holes $F_{\ell+1}$ transverse to T that is I-derived from f and has strictly smaller complexity than f_ℓ. Clearly this suffices.

Take a T-component K that is innermost with respect to "not mapping entirely to A". Consider the map $g = f_\ell \restriction K : (K, \partial K) \to (X_T, A_T)$. This map is based since K inherits from F_ℓ a notion of outermost. Enumerate the the boundary components of K as $b_0, b_1, \ldots, b_m$ with b_0 the outermost component. Given an index $t \geqslant 1$, let L_t be the subcomplex of F_ℓ for which b_t is the outer boundary component (note that L_t may be empty). Since (X_T, A_T) is normal-convex mod P there is a based map of a disk with holes $\overline{g} : \overline{K} \to A_T$ derived from g (see Figure 2.3). We can assume that $\overline{g}$ is transverse to T. Write $\overline{b}_0, \overline{b}_1, \ldots, \overline{b}_s$ for the boundary components of $\overline{K}$ where, as usual, $\overline{b}_0$ is the outermost one. We will glue together the closure of $F_\ell \backslash K$, the new disk with holes $\overline{K}$, various copies of the L_t's, and assorted annuli to get another disk with holes $F_{\ell+1}$ and a map $f_{\ell+1}$ which is still I-derived from f and with lesser complexity.

Start by letting k be the index that satisfies condition (2) of the definition of derived for the maps g and $\overline{g}$. For each index i with $1 \leqslant i \leqslant k$, let $t_i \geqslant 1$ be chosen so that the map $\overline{g} \restriction \overline{b}_i$ is homotopic in A to $g \restriction b_{t_i}$. Now $\overline{g} \restriction \overline{b}_0$ is homotopic in A to $g \restriction b_0$. We can find an annulus and a map into A, that is transverse to T, representing this homotopy. Identify one boundary component of the annulus with b_0 in the closure of $F_\ell \backslash K$ and the other boundary component with $\overline{b}_0$ in $\overline{K}$. We can then use f_ℓ, $\overline{g}$, and the homotopy to define a map on the resulting complex. Similarly, find annuli α_i and transverse maps into A that represent the homotopies between $\overline{g} \restriction b_i$ and $g \restriction b_{t_i}$ for $1 \leqslant i \leqslant k$. Glue these annuli to $\overline{b}_i$ in $\overline{K}$ and ∂L_{t_i} and use the homotopies together

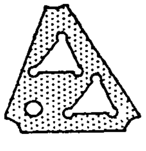

Figure 2.3

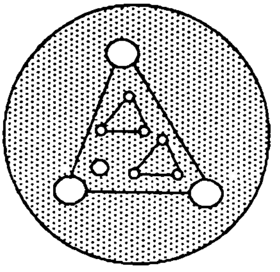

Figure 2.4

with the maps f_ℓ and $\overline{g}$ to get a map of a disk with holes (see Figure 2.4). Call it $f_{\ell+1}$. Clearly it is transverse to T and the complexity has been decreased. We only need see that $f_{\ell+1}$ is I-derived from f.

Let K' be a T-component of $F_{\ell+1}$. We consider three cases.

First, assume that $\overline{K} \not\subset I(K')$. Then we can identify $I(K')$ with some $I(K'')$ where K'' is a T-component of F_ℓ. In fact, if K' is interior to $\overline{K}$ in $F_{\ell+1}$, there may be other T-components of $F_{\ell+1}$ that are identified with the same K'' in F_ℓ. In any case, under this identification $f_{\ell+1} \restriction I(K') = f_\ell \restriction I(K'')$. Hence if K' contains some P-loop then $f_{\ell+1}(I(K')) \subset A$ (condition (2) of the definition of I-derived).

Second, consider the T-component $K' = \overline{K}$. By assumption $\overline{g} = f_{\ell+1} \restriction \overline{K}$ is derived from $g = f_\ell \restriction K$. So the set, Ω_0, of P-loops of $(g, \overline{g})$ in $\overline{K}$ is chainable in $(A_T)_g$. The inner boundary loops of $I(\overline{K})$ are of three types– those in Ω_0, those lying in $\overline{K} \cap \partial F_{\ell+1}$ but not in Ω_0, and those not lying in $\overline{K}$. We begin with the first type. The inclusion map $A_T \hookrightarrow A$ induces, for any choice of basepoint, a homomorphism $\beta : \pi_1\Big((A_T)_g\Big) \to \pi_1(A_f)$ as $g \restriction b_i$, for $i \geqslant 1$, is null-homotopic in A_f. By Lemma 1.1, Ω_0 is chainable in A_f. We turn to loops of the second type. If c is such a loop then c must correspond to some inner boundary component of $I(K)$ lying in K. So $f_{\ell+1} \restriction c$ is homotopic in A to some $f \restriction a$, where a is an inner boundary loop of F. Now we handle the loops of the third type. By the initial case, the set, Ω_1 of P-loops of $I(\overline{K})$ that are not in $\overline{K}$ is chainable in A_f. Taking the union of Ω_0 and Ω_1 yields a chainable set in A_f by Lemma 1.1. Note further that $f_{\ell+1}\big(I(\overline{K})\big) \subset A$, so condition (2) of the definition of I-derived holds.

Third, suppose $\overline{K}$ is interior to K'. Since K did not map entirely into A under f_ℓ, and since $f_{\ell+1}$ agrees with f_ℓ on K', it follows that any inner boundary component of $F_{\ell+1}$ lying in K' is not a P-loop, that is it must be homotopic in A to some $f \restriction a$, where a is an inner boundary component of F. Thus the set of P-loops of $I(K')$ is the same as the set of P-loops of $I(\overline{K})$, which is chainable in A_f by the above. Also condition (3) holds trivially.

It only remains to see that $f_{\ell+1}$ is derived from f. We can write

$$F_{\ell+1} = \bigcup_{i=1}^{d} K_i$$

where each K_i is a T-component and $K_s \cap K_t$, for $s \neq t$, is either empty or a subset of both their boundaries. By the above, the set of P-loops W_s in K_s is chainable in A_f. Taking the union of the sets of P-loops, and using Lemma 1.1, proves the result. $\square$

Taking for X and Y appropriate complexes yields the following result on amalgamations and HNN extensions:

Corollary 2.4.

(1) *If $S_1 \subset G_1$ and $S_2 \subset G_2$ are normal-convex mod P then $S_1 * S_2 \subset G_1 * G_2$ is normal-convex mod P.*

(2) *If $S_1 \subset G_1$ and $S_2 \subset G_2$ are normal-convex mod P, and if $C \subset S_i$ for $i = 1, 2$, then $S_1 *_C S_2 \subset G_1 *_C G_2$ is normal-convex mod P.*

(3) *If $S \subset G$ is normal-convex mod P and $C \cup \phi(C) \subset S$ then $S *_C \phi \subset G *_C \phi$ is normal-convex mod P.*

We close this section with an application of our amalgamation result. We answer a question of Mark Rinker's, who asked us whether a one-relator group can split as a free product with amalgamation along $\mathbb{Z} \oplus \mathbb{Z}$. Use the notation $F(x_1, x_2)$ for the free group on x_1, x_2. Now it is a well known fact (see, for instance, [1]) that $\langle ece^{-1}, cec^{-1} \rangle \subset F(e, c)$ is normal-convex (this is absolute, not relative, normal-convexity). From this, it easily follows that $\langle c, dcdc^{-1}d^{-1} \rangle$ is normal-convex in $F(c, d)$. Also $\langle a^2, b^2 \rangle$ is normal-convex in $F(a, b)$. Let G be the amalgamated product $F(a, b) *_S F(c, d)$ where $S = \langle a^2, b^2 \rangle$ in $F(a, b)$ is identified with $\langle c, dcdc^{-1}d^{-1} \rangle$ in $F(c, d)$. Then the subgroup S is normal-convex in G by our amalgamation result. But G is isomorphic to $F(x, y)$. To see this, map $F(x, y)$ to G by $x \mapsto a$ and $y \mapsto c^{-1}d^{-1}bdc$. The inverse, call it α, is defined as follows:

$$\begin{aligned} a &\mapsto x \\ b &\mapsto y^2 x^2 y x^{-2} y^{-2} \\ c &\mapsto x^2 \\ d &\mapsto y^2. \end{aligned}$$

So $\alpha(S) = \langle x^2, y^2 x^2 y^2 x^{-2} y^{-2} \rangle$ is normal-convex in $F(x, y)$. Now G splits as a free product with amalgamation over S. Let $w = [a^2, b^2]$. By normal-convexity $S/\langle\langle w \rangle\rangle_S \to G/\langle\langle w \rangle\rangle_G$ is injective. Hence $G/\langle\langle w \rangle\rangle_G$ splits as a free product with amalgamation over $S/\langle\langle w \rangle\rangle_S = \mathbb{Z} \oplus \mathbb{Z}$. Applying α yields the fact that the one-relator group

$$\langle\, x, y \mid [x^2, y^2 x^2 y^2 x^{-2} y^{-2}] \,\rangle$$

splits as a free product with amalgamation over $\mathbb{Z} \oplus \mathbb{Z}$.

splits as a free product with amalgamation over $\mathbb{Z} \oplus \mathbb{Z}$.

3. Amalgamations

As mentioned in the introduction, this paper is a result of an attempt to prove a Freiheitssatz for torsion-free groups — that is to prove that any equation $w = 1$ over any torsion-free group H has a solution as long as w is not conjugate to an element of H.

Let $G = \langle x_1, \dots, x_n, t \mid r \rangle$ where r is cyclically reduced and involves t. Set $F = \langle x_1, \dots, x_n \rangle$. Our hope was to show that $F \subset G$ is torsion-free normal-convex. Our plan was to use HNN extension approach to one-relator groups as described in [**3**]. There are three cases:

(1) r involves only t,
(2) $\exp_{x_i}(r) = 0$ for some x_i appearing in r, and
(3) $\exp_{x_i}(r) \neq 0$ for all x_i appearing in r.

The first case is handled easily by our amalgamation result. The third case is approached by adjoining a root of one of the generators and can be tackled by using Proposition 3.1 below. However the second case requires HNN extensions. And the way that an HNN extension affects relative normal-convexity is quite complicated.

We end with the following result about amalgamations:

Proposition 3.1. *Suppose $S \subset H$ is normal-convex mod P If $G = H *_C K$ and $C \subset K$ is normal-convex then $S \subset G$ is normal-convex mod P.*

Proof: Construct complexes A, X, B, Y, and Z where, for some basepoint x_0,

$$x_0 \in A \subset X,\ x_0 \in X \cap Y = B,$$
$$Z = X \cup Y,\ B \text{ is two-sided in } Z,\ Z \setminus B = (X \setminus B) \cup (Y \setminus B),$$
$$\pi_1(X, x_0) = H,\ \pi_1(A, x_0) = S,\ \pi_1(B, x_0) = C, \text{ and } \pi_1(Y, x_0) = K.$$

Observe that $\pi_1(Z, x_0) = G$. See Figure 3.1.

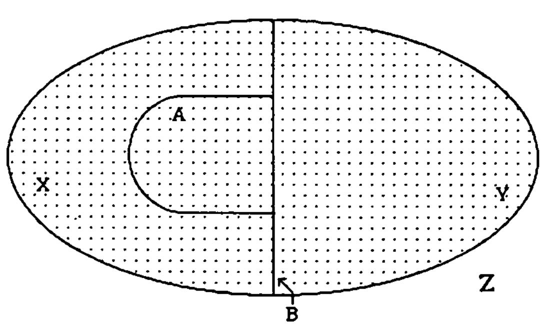

Figure 3.1

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By Proposition 2.2, the pair (X, A) is topologically normal-convex mod P and the pair (Y, B) is topologically normal-convex (not relative). It suffices to show that the pair (Z, A) is topologically normal-convex mod P.

Let $f : (F, \partial F) \to (Z, A)$ be a based map of a disk with holes. We will prove that there is a map $f' : (F', \partial F') \to (X, A)$ of another disk with holes that is derived from f, and involving no P-loops. Then, by topological relative normal-convexity, there is a map f'' of a disk with holes derived from f' whose image lies in A. Since there are no P-loops for (f, f'), we can conclude, using lemma 2.1, that f'' is derived from f. Thus we need only construct f'.

As in the proof of Proposition 2.3, we proceed in steps, each step resulting in a map into Z of a disk with holes derived from f and transverse to B. Instead of T-components we use B-components. And here, by complexity of a map $\overline{f}$ of a disk with holes, we mean the number of B-components with image intersecting $Y \setminus B$.

We start by homotoping f to make it transverse to B. The resulting map, f_0, is clearly derived from f and there are no P-loops.

Now suppose $f_i : F_i \to Z$ is transverse to B, and derived from f without P-loops. Let K be some B-component whose image meets $Y \setminus B$. The map $g = f_i \restriction K$ is a based map of the pair $(K, \partial K)$ into the pair (Y, B). By assumption the pair (Y, B) is topologically normal-convex (this is absolute – no P loops). We can thus find another based map of another disk with holes $\overline{g} : \overline{K} \to B$ that is derived from g. A cut-and-paste technique similar to that in the proof of Proposition 2.3 yields a based map of a disk with holes, $f_{i+1} : F_{i+1} \to Z$ of lesser complexity. Pushing the image of $\overline{K}$ into X using the collar on B results in a map transverse to B. Also this map is derived from f_i without P-loops. And, as there are no P-loops for (f, f_i), we may conclude, by lemma 2.1, that f_{i+1} is derived from f with no P-loops.

Since the complexity of f_0 was finite, this process has to terminate with the desired map f'. $\square$

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