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SOME COVERS AND ENVELOPES IN THE CHAIN COMPLEX CATEGORY OF *R*-MODULES

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Abstract

We study the existence of some covers and envelopes in the chain complex category of R-modules. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod and let $\mathcal{E}\mathcal{A}$ stand for the class of all exact complexes with each term in $\mathcal{A}$. We prove that $(\mathcal{E}\mathcal{A}, \mathcal{E}\mathcal{A}^{\perp})$ is a perfect cotorsion pair whenever $\mathcal{A}$ is closed under pure submodules, cokernels of pure monomorphisms and direct limits and so every complex has an $\mathcal{E}\mathcal{A}$-cover. As an application we show that every complex of R-modules over a right coherent ring R has an exact Gorenstein flat cover. In addition, the existence of $\overline{\mathcal{A}}$-covers and $\overline{\mathcal{B}}$-envelopes of special complexes is considered where $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ denote the classes of all complexes with each term in $\mathcal{A}$ and $\mathcal{B}$, respectively.

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1. Introduction

In this paper R denotes a ring with unity. We let $\mathscr{C}(R)$ denote the abelian category of complexes of left R-modules. A complex

$$\cdots \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^{0} \xrightarrow{\delta^{0}} C^{1} \xrightarrow{\delta^{1}} \cdots$$

of left R-modules will be denoted by (C, δ) or C. For a left R-module M we will use $\overline{M}$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the -1st and 0th positions in R-Mod. We denote by $\underline{M}$ and M^{+} the complex with M in the 0th place and 0 elsewhere, and the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ respectively. Given a complex C and an integer m, we denote by $C[m]$ the complex such that $C[m]^{n} = C^{m+n}$ and the boundary operators are $(-1)^{m}\delta^{m+n}$.

In this paper, we use both subscripts and superscripts. When we use superscripts for a complex, we use subscripts to distinguish positions within the complexes.

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For example, if $(K_i)_{i\in I}$ is a family of complexes, then K_i^n denotes the degree-n term of the complex K_i.

We denote by $\mathrm{Hom}(C, D)$ the abelian group of morphisms from C to D in $\mathscr{C}(R)$ and by $\mathrm{Ext}^i(C, D)$, where $i \geq 1$, the groups that we get from the right derived functor of Hom. We let $\mathcal{H}\mathrm{om}(C, D)$ denote the complex of abelian groups with

$$\mathcal{H}\mathrm{om}(C, D)^n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(C^i, D^{n+i})$$

and

$$\delta^n((f^i)_{i\in\mathbb{Z}}) = (\delta^{n+i} f^i - (-1)^n f^{i+1}\delta^i)_{i\in\mathbb{Z}}$$

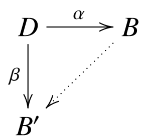
for $(f^i)_{i\in\mathbb{Z}} \in \mathcal{H}\mathrm{om}(C, D)^n$.

Let $Z(-)$, $B(-)$ and $H(-)$ denote the cycles, boundaries, and homology functors respectively. It is easy to see that

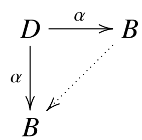
$$\mathrm{Hom}(C, D) = Z^0(\mathcal{H}\mathrm{om}(C, D)).$$

General background material can be found in [5–7, 11, 13].

Next we recall some known concepts and facts used in what follows. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of objects in an abelian category $\mathscr{D}$ which has enough projectives and enough injectives. Let D be an object of $\mathscr{D}$. We recall some definitions introduced in [4]. An object B in $\mathcal{B}$ is called a $\mathcal{B}$-preenvelope of D if there exists a homomorphism $\alpha\colon D \longrightarrow B$ such that the diagram



can be completed for each homomorphism $\beta : D \longrightarrow B'$ with B' in $\mathcal{B}$. Furthermore, if the triangle



can be completed only by automorphisms, then we say that $\alpha : D \longrightarrow B$ is a $\mathcal{B}$-envelope.

A monomorphism $\alpha : D \longrightarrow B$ with $B \in \mathcal{B}$ is said to be a special $\mathcal{B}$-preenvelope of D if $\mathrm{Coker}(\alpha) \in {}^{\perp}\mathcal{B}$. A class $\mathcal{B}$ is called (pre)enveloping if every object of $\mathscr{D}$ has a $\mathcal{B}$-(pre)envelope. We also have the dual concepts of a (special) $\mathcal{B}$-precover, $\mathcal{B}$-cover and (pre)covering class.

In [1, Theorem 2.10] the authors proved that every module has an $\mathcal{A}$-cover whenever it has an $\mathcal{A}$-precover and $\mathcal{A}$ is closed under direct limits. A pair of classes of objects $(\mathcal{A}, \mathcal{B})$ is called a cotorsion pair or cotorsion theory (see [17, 22]) if $\mathcal{A}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{A}$ where

$$\mathcal{A}^{\perp} = \{B \in \mathscr{D} \mid \mathrm{Ext}^1(A, B) = 0 \ \forall A \in \mathcal{A}\},$$

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and

$$^{\perp}\mathcal{B} = \{A \in \mathcal{D} \mid \mathrm{Ext}^1(A, B) = 0 \ \forall B \in \mathcal{B}\}.$$

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called hereditary if whenever

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is exact with $A, A'' \in \mathcal{A}$, then A' is also in $\mathcal{A}$. This is equivalent to the requirement that if whenever

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

is exact with B' and $B \in \mathcal{B}$, then B'' is also in $\mathcal{B}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called complete if every $D \in \mathcal{D}$ has a special $\mathcal{B}$-preenvelope and a special $\mathcal{A}$-precover. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called perfect if every $D \in \mathcal{D}$ has a $\mathcal{B}$-envelope and an $\mathcal{A}$-cover. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be cogenerated by a set X if $X^{\perp} = \mathcal{A}^{\perp}$.

It is well known that a perfect cotorsion pair is complete, but the converse may be false in general. In [3] Eklof and Trlifaj proved that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod is complete when it is cogenerated by a set. This result actually holds in a Grothendieck category with enough projectives, as Hovey proved in [19]. For unexplained concepts and notation we refer the reader to [8, 13, 17, 24].

In [14] Gillespie introduced the following definition.

Definition 1.1 [14, Definition 3.3]. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair on an abelian category $\mathcal{C}$. Let X be a chain complex.

(1) X is called an $\mathcal{A}$ complex if it is exact and $Z^nX \in \mathcal{A}$ for all n.
(2) X is called a $\mathcal{B}$ complex if it is exact and $Z^nX \in \mathcal{B}$ for all n.
(3) X is called a dg-$\mathcal{A}$ complex if $X^n \in \mathcal{A}$ for each n and $\mathcal{H}\mathrm{om}(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.
(4) X is called a dg-$\mathcal{B}$ complex if $X^n \in \mathcal{B}$ for each n and $\mathcal{H}\mathrm{om}(A, X)$ is exact whenever A is an $\mathcal{A}$ complex.

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $\mathrm{dg}\,\widetilde{\mathcal{A}}$. Similarly, the class of $\mathcal{B}$ complexes is denoted by $\widetilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes is denoted by $\mathrm{dg}\,\widetilde{\mathcal{B}}$.

In [14] it was shown that $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ and $(\mathrm{dg}\,\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ are cotorsion pairs in $\mathcal{C}(R)$ if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod. It is also proved that $(\mathcal{A}, \mathcal{B})$ is hereditary if and only if $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ is hereditary or, equivalently, if $(\mathrm{dg}\,\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is hereditary. But the question of whether or not the induced cotorsion pairs are complete when the original cotorsion pair is complete is open (see [14]).

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod. In [15, Proposition 3.8] Gillespie proved that the induced cotorsion pair $(\mathrm{dg}\,\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is complete whenever $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set. In [23] it was proved that the induced cotorsion pairs $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ and $(\overline{\mathcal{A}}, \overline{\mathcal{A}}^{\perp})$ are complete whenever $\mathcal{A}$ is closed under pure submodules and cokernels of pure

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monomorphisms. Here $\overline{\mathcal{A}}$ stands for the class of all complexes with each term in $\mathcal{A}$. In [16] it was shown that $(\mathcal{E}\mathcal{A}, \mathcal{E}\mathcal{A}^{\perp})$ is a complete cotorsion pair whenever $\mathcal{A}$ is a Kaplansky class that is closed under direct limits.

In Section 2 of this paper we study complexes in the class $\mathcal{E}\mathcal{A}^{\perp}$ and the completeness of the cotorsion pair $(\mathcal{E}\mathcal{A}, \mathcal{E}\mathcal{A}^{\perp})$. Here $\mathcal{E}\mathcal{A}$ stands for the class of all exact complexes with each term in $\mathcal{A}$. It is shown that $(\mathcal{E}\mathcal{A}, \mathcal{E}\mathcal{A}^{\perp})$ is a complete cotorsion pair whenever $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod and $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms. This does not require $\mathcal{A}$ to be closed under direct limits. In addition, some applications are given.

Section 3 is devoted to studying the existence of $\overline{\mathcal{A}}$-covers and $\overline{\mathcal{B}}$-envelopes of special complexes. We prove that each complex of R-modules that is bounded above has an $\overline{\mathcal{A}}$-cover and each complex of R-modules that is bounded below has a $\overline{\mathcal{B}}$-envelope whenever $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class in R-Mod.

2. $\mathcal{E}\mathcal{A}$-covers of complexes

Let $\mathcal{E}\mathcal{A}$ denote the class of all exact complexes C with each term C^n in $\mathcal{A}$.

Proposition 2.1. *Let C be a complex. Then C is in $\mathcal{E}\mathcal{A}^{\perp}$ if and only if C^n is in $\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$ and $\mathcal{H}\mathrm{om}(G, C)$ is exact for each $G \in \mathcal{E}\mathcal{A}$.*

Proof. Suppose that (C, δ) is in $\mathcal{E}\mathcal{A}^{\perp}$ and let

$$0 \longrightarrow C^n \xrightarrow{\alpha} X \longrightarrow F \longrightarrow 0$$

be an extension in R-Mod with $F \in \mathcal{A}$. By the factor theorem (see [2, Theorem 3.6]) we have the following commutative diagram

$$\begin{array}{ccccc}
C^{n+1} & \xrightarrow{\eta} & \mathrm{Coker}(\delta^n) & \longrightarrow & 0 \\
{\scriptstyle \delta^{n+1}}\downarrow & \swarrow{\scriptstyle \theta} & & & \\
C^{n+2} & & & &
\end{array}$$

where $\eta : C^{n+1} \to \mathrm{Coker}(\delta^n)$ is the natural epimorphism. We form the pushout of $C^n \xrightarrow{\alpha} X$ and $C^n \xrightarrow{\delta^n} C^{n+1}$ and obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^n & \xrightarrow{\alpha} & X & \longrightarrow & F & \longrightarrow & 0 \\
 & & {\scriptstyle \delta^n}\downarrow & & {\scriptstyle \mu}\downarrow & & \| & & \\
0 & \longrightarrow & C^{n+1} & \xrightarrow{\nu} & P & \longrightarrow & F & \longrightarrow & 0 \\
 & & {\scriptstyle \eta}\downarrow & & {\scriptstyle g}\downarrow & & & & \\
 & & \mathrm{Coker}(\delta^n) & = & \mathrm{Coker}(\delta^n) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

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So we have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{n-2} & \xrightarrow{\mathrm{id}} & C^{n-2} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \delta^{n-2}} & & \downarrow{\scriptstyle \delta^{n-2}} & & \downarrow & & \\
0 & \longrightarrow & C^{n-1} & \xrightarrow{\mathrm{id}} & C^{n-1} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \delta^{n-1}} & & \downarrow{\scriptstyle \alpha\delta^{n-1}} & & \downarrow & & \\
0 & \longrightarrow & C^{n} & \xrightarrow{\alpha} & X & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta^{n}} & & \downarrow{\scriptstyle \mu} & & \| & & \\
0 & \longrightarrow & C^{n+1} & \xrightarrow{\nu} & P & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta^{n+1}} & & \downarrow{\scriptstyle \theta g} & & \downarrow & & \\
0 & \longrightarrow & C^{n+2} & \xrightarrow{\mathrm{id}} & C^{n+2} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \delta^{n+2}} & & \downarrow{\scriptstyle \delta^{n+2}} & & \downarrow & & \\
0 & \longrightarrow & C^{n+3} & \xrightarrow{\mathrm{id}} & C^{n+3} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

and can form the complex

$$W = \cdots \longrightarrow C^{n-2} \longrightarrow C^{n-1} \longrightarrow X \longrightarrow P \longrightarrow C^{n+2} \longrightarrow \cdots .$$

Thus we have an exact sequence of complexes

$$0 \longrightarrow C \longrightarrow W \longrightarrow \overline{F}[-n-1] \longrightarrow 0.$$

By our hypothesis, the sequence splits in $\mathscr{C}(R)$ and so the sequence

$$0 \longrightarrow C^n \longrightarrow X \longrightarrow F \longrightarrow 0$$

splits in R-Mod. Therefore, C^n is in $\mathcal{A}^\perp$.

For each $G \in \mathcal{E}\mathcal{A}$ we have that $\mathcal{H}\mathrm{om}(G, C)$ is exact if and only if for each n each map of complexes $f : G \to C[n]$ is homotopic to 0. This is equivalent to the requirement that for each n and each map of complexes $f : G \to C[n]$ the sequence

$$0 \longrightarrow C[n] \longrightarrow M(f) \longrightarrow G[1] \longrightarrow 0$$

splits or, equivalently, that for each n and each map of complexes $f : G \to C[n]$ the sequence

$$0 \longrightarrow C \longrightarrow M(f)[-n] \longrightarrow G[1-n] \longrightarrow 0$$

splits where $M(f)$ denotes the mapping cone of f.

Since G is in $\mathcal{E}\mathcal{A}$ we also have $G[1-n]$ in $G \in \mathcal{E}\mathcal{A}$. By our hypothesis we have $\operatorname{Ext}^1(G[1-n], C) = 0$. So the sequence

$$0 \longrightarrow C \longrightarrow M(f)[-n] \longrightarrow G[1-n] \longrightarrow 0$$

splits and $\mathcal{H}\mathrm{om}(G, C)$ is an exact complex.

Suppose that C^n is in $\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$ and that $\mathcal{H}\mathrm{om}(G, C)$ is exact for each $G \in \mathcal{E}\mathcal{A}$. Each exact sequence

$$0 \longrightarrow C \longrightarrow W \longrightarrow G \longrightarrow 0$$

of complexes with $G \in \mathcal{E}\mathcal{A}$ splits at the module level. So this sequence is isomorphic to

$$0 \longrightarrow C \longrightarrow M(f) \longrightarrow G \longrightarrow 0$$

where $f : G[-1] \to C$ is a map of complexes.

Since $\mathcal{H}\mathrm{om}(G[-1], C)$ is exact the sequence

$$0 \longrightarrow C \longrightarrow M(f) \longrightarrow G \longrightarrow 0$$

splits in $\mathscr{C}(R)$ by [13, Lemma 2.3.2]. Therefore

$$0 \longrightarrow C \longrightarrow W \longrightarrow G \longrightarrow 0$$

also splits and our result is established. □

Lemma 2.2. *If G is in $\mathcal{A}^{\perp}$, then $\overline{G}[-n]$ is in $\mathcal{E}\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$.*

Proof. It is enough to prove that $\operatorname{Ext}^1(F, \overline{G}[-n]) = 0$ for each $F \in \mathcal{E}\mathcal{A}$. Let

$$0 \longrightarrow \overline{G}[-n] \xrightarrow{\alpha} X \longrightarrow F \longrightarrow 0$$

be an extension in $\mathscr{C}(R)$ and consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & X^{n-2} & \longrightarrow & F^{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & X^{n-1} & \longrightarrow & F^{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \xrightarrow{\alpha^n} & X^{n} & \longrightarrow & F^{n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & X^{n+1} & \longrightarrow & F^{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

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Since F^n is in $\mathcal{A}$ and G is in $\mathcal{A}^{\perp}$ we have $\mathrm{Ext}^1(F^n, G) = 0$. That is, the sequence

$$0 \longrightarrow G \xrightarrow{\alpha^n} X^n \longrightarrow F^n \longrightarrow 0$$

splits in R-Mod. So there exists $h^n : X^n \to G$ such that $h^n \alpha^n = 1$.

We define $h^{n-1} : X^{n-1} \to G$ by $h^{n-1} = h^n \delta_X^{n-1}$ and $h^i = 0$ for $i \neq n, n-1$. Thus we obtain a map of complexes $h : X \to \overline{G}[-n]$ such that $h\alpha = 1$. So the sequence

$$0 \longrightarrow \overline{G}[-n] \xrightarrow{\alpha} X \longrightarrow F \longrightarrow 0$$

splits in $\mathscr{C}(R)$ and our result is established. □

Lemma 2.3. *If an injective module I is in $\mathcal{A}$, then $\underline{I}[-n]$ is in $\mathcal{E}\mathcal{A}^{\perp}$ for all $n \in \mathbb{Z}$.*

Proof. It is enough to prove that each map $f : F \to \underline{I}[-n]$ is homotopic to zero for each $F \in \mathcal{E}\mathcal{A}$. Since $f^n d^{n-1} = 0$ we obtain $Z^n(F) = B^n(F) \subseteq \mathrm{Ker}(f^n)$ and so the following diagram

$$\begin{array}{ccccc} F^n & \xrightarrow{d^n} & B^{n+1}(F) & \longrightarrow & 0 \\ {\scriptstyle f^n}\downarrow & \swarrow{\scriptstyle \theta^n} & & & \\ I & & & & \end{array}$$

commutes.

Again, since I is injective there exists $S^{n+1} : F^{n+1} \to I$ such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & B^{n+1}(F) & \longrightarrow & F^{n+1} \\ & & {\scriptstyle \theta^n}\downarrow & \swarrow{\scriptstyle S^{n+1}} & \\ & & I & & \end{array}$$

is commutative. Thus $S^{n+1} d^n = f^n$. That is, the map f is null homotopic and our result follows. □

Theorem 2.4. *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod, then $(\mathcal{E}\mathcal{A}, \mathcal{E}\mathcal{A}^{\perp})$ is a cotorsion pair in $\mathscr{C}(R)$.*

Proof. It suffices to prove that ${}^{\perp}(\mathcal{E}\mathcal{A}^{\perp}) \subseteq \mathcal{E}\mathcal{A}$. If $F \in {}^{\perp}(\mathcal{E}\mathcal{A}^{\perp})$, then $\mathrm{Ext}^1(F, C) = 0$ for all $C \in \mathcal{E}\mathcal{A}^{\perp}$. For each $n \in \mathbb{Z}$ and each $G \in \mathcal{B} = \mathcal{A}^{\perp}$ let

$$0 \longrightarrow G \xrightarrow{\alpha} X \xrightarrow{\beta} F^n \longrightarrow 0$$

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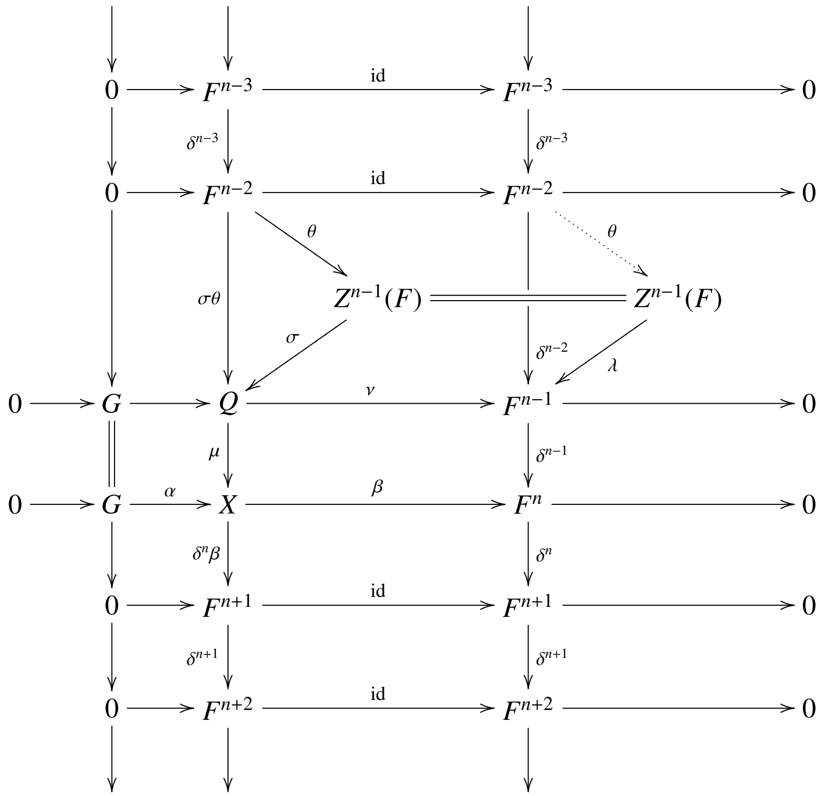
be an extension in R-Mod. We consider the following commutative diagram



where $\lambda : Z^{n-1}(F) \to F^{n-1}$ is the natural inclusion and Q is the pullback of β and δ^{n-1}.

We get a complex

$$W = \cdots \longrightarrow F^{n-2} \longrightarrow Q \longrightarrow X \longrightarrow F^{n+1} \longrightarrow \cdots$$

and an exact sequence

$$0 \longrightarrow \overline{G}[-n] \longrightarrow W \longrightarrow F \longrightarrow 0 \tag{2.1}$$

in $\mathscr{C}(R)$.

Since G is in $\mathcal{A}^{\perp}$ we have that $\overline{G}[-n]$ is in $\mathcal{E}\mathcal{A}^{\perp}$ by Lemma 2.2. By the hypothesis $\mathrm{Ext}^1(F, \overline{G}[-n]) = 0$. So the sequence (2.1) splits and the sequence

$$0 \longrightarrow G \xrightarrow{\alpha} X \xrightarrow{\beta} F^n \longrightarrow 0$$

in R-Mod splits. Thus F^n is in $\mathcal{A}$ for all $n \in \mathbb{Z}$.

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Next we prove that F is exact. Let $f^n : F^n/B^n(F) \to I$ be an injection with I injective. Then f^n induces a map $f : F \to \underline{I}[-n]$ as follows

$$\begin{array}{ccccccccc} F = \cdots & \longrightarrow & F^{n-1} & \xrightarrow{d^{n-1}} & F^n & \xrightarrow{d^n} & F^{n+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f^n\eta} & & \downarrow & & \\ \underline{I}[-n] = \cdots & \longrightarrow & 0 & \longrightarrow & I & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

where $\eta : F^n \to F^n/B^n(F)$ is the natural surjection. By Lemma 2.3, f is homotopic to zero. Let $\{S^n\}$ be the homotopy. Then $S^{n+1}d^n = f^n\eta$ and so $Z^n(F) \subseteq B^n(F)$. Thus F is in $\mathcal{E}\mathcal{A}$. Therefore, we may deduce that $(\mathcal{E}\mathcal{A}, \mathcal{E}\mathcal{A}^\perp)$ is a cotorsion pair and our result is established. □

Remark 2.5. Proposition 2.1 and Theorem 2.4 are similar to [16, Proposition 3.3] by Gillespie, but our proofs are more direct.

Lemma 2.6. *Suppose that S, T and M are modules such that $S \subseteq T \subseteq M$. If S is pure in M and T/S is pure in M/S, then T is pure in M.*

We define the cardinality of a complex C to be $|\coprod_{n\in\mathbb{Z}} C^n|$.

Lemma 2.7 [1, Proposition 4.1]. *Let $|R| \leq \aleph$ where $\aleph$ is some infinite cardinal. Then for each $C \in \mathscr{C}(R)$ and each element $x \in C$ (that is, $x \in C^n$ for some n) there exists an exact subcomplex $L \leq X$ such that $x \in L^k$, $|L| \leq \aleph$ and $L^j \leq C^j$ is a pure submodule for all $j \in \mathbb{Z}$.*

Theorem 2.8. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod. If $\mathcal{A}$ is closed under taking pure submodules and cokernels of pure monomorphisms, then the cotorsion pair $(\mathcal{E}\mathcal{A}, \mathcal{E}\mathcal{A}^\perp)$ is complete.*

Proof. Suppose that G is in $\mathcal{E}\mathcal{A}$ and $|R| \leq \aleph$ for some infinite cardinal $\aleph$. We will show that G is equal to the union of a continuous chain $(P_\alpha)_{\alpha<\lambda}$ of exact subcomplexes of G where $|P_0| \leq \aleph$, $|P_{\alpha+1}/P_\alpha| \leq \aleph$ and P^i_α is pure G^i for all α and all $i \in \mathbb{Z}$.

Set $T = \coprod_{n\in\mathbb{Z}} G^n$. We may well-order the set T so that for some ordinal λ

$$T = \{x_0, x_1, x_2, \ldots, x_\alpha, \ldots\}_{\alpha<\lambda}.$$

For x_0 we use Lemma 2.7 to find an exact subcomplex $P_1 \subseteq G$ containing x_0 such that $|P_1| \leq \aleph$ and P^i_1 is pure in G^i for all $i \in \mathbb{Z}$. Then G/P_1 is in $\mathcal{E}\mathcal{A}$.

Now $\overline{x_1} \in G/P_1$. Therefore we can find an exact subcomplex $P_2/P_1 \subseteq G/P_1$ containing $\overline{x_1}$ such that $|P_2/P_1| \leq \aleph$ and $(P_2/P_1)^i$ is pure in $(G/P_1)^i$ for all $i \in \mathbb{Z}$. Then $(G/P_1)/(P_2/P_1) \cong G/P_2$ is in $\mathcal{E}\mathcal{A}$, P_2 is exact and P^i_2 is pure in G^i by Lemma 2.6. Note that $P_1 \subseteq P_2$ and $x_0,\ x_1 \in P_2$.

In general, given an ordinal α and having constructed exact subcomplexes $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\alpha$ where $x_\gamma \in P_\alpha$ for all $\gamma < \alpha$, we find an exact subcomplex $P_{\alpha+1} \subseteq G$ as follows. We have $\overline{x_\alpha} \in G/P_\alpha$ and so by Lemma 2.7 we can find an exact subcomplex $P_{\alpha+1}/P_\alpha \subseteq G/P_\alpha$ containing $\overline{x_\alpha}$ such that $|P_{\alpha+1}/P_\alpha| \leq \aleph$ and $(P_{\alpha+1}/P_\alpha)^i$ is pure in $(G/P_\alpha)^i$ for all $i \in \mathbb{Z}$. Thus $(G/P_\alpha)/(P_{\alpha+1}/P_\alpha) \cong G/P_{\alpha+1}$ is in $\mathcal{EA}$, whence $P_{\alpha+1}$ is exact and $P^i_{\alpha+1}$ is pure in G^i.

We now have

$$P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\alpha \subseteq P_{\alpha+1}$$

and

$$x_0, x_1, \ldots, x_\alpha \in P_{\alpha+1}.$$

In the case where α is a limit ordinal we just define $P_\alpha = \bigcup_{\gamma<\alpha} P_\gamma$. Then, as we noted above, P_α is exact, $x_\gamma \in P_\alpha$ and P^i_α is pure in G^i for all $i \in \mathbb{Z}$ and all $\gamma < \alpha$. This construction gives us the directed continuous chain $(P_\alpha)_{\alpha<\lambda}$.

If C is a complex such that $\mathrm{Ext}^1(P_0, C) = 0$ and $\mathrm{Ext}^1(P_{\alpha+1}/P_\alpha, C) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^1(G, C) = 0$ by [14, Lemma 4.5]. Let X be a set of representatives of all complexes $G \in \mathcal{EA}$ with $|G| \leq \aleph$. Then $\mathcal{EA}^\perp = X^\perp$. That is, $(\mathcal{EA}, \mathcal{EA}^\perp)$ is cogenerated by X. Thus $(\mathcal{EA}, \mathcal{EA}^\perp)$ is a complete cotorsion pair. □

Remark 2.9. In [16] it was shown that $(\mathcal{EA}, \mathcal{EA}^\perp)$ is a complete cotorsion pair whenever $\mathcal{A}$ is a Kaplansky class that is closed under direct limits. In Theorem 2.8 we assume that $\mathcal{A}$ is closed under pure submodules and cokernels of pure monomorphisms. Such a class is automatically a Kaplansky class, but need not be closed under direct limits.

Corollary 2.10. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-*Mod*. If $\mathcal{A}$ is closed under pure submodules, cokernels of pure monomorphisms and direct limits, then the cotorsion pair $(\mathcal{EA}, \mathcal{EA}^\perp)$ is perfect.*

According to [10] a module M is called Gorenstein flat if there exists an exact sequence

$$\cdots \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \cdots$$

in R-Mod of flat R-modules such that $M = \mathrm{Ker}(F_0 \to F_1)$ and the sequence remains exact whenever $E \otimes -$ is applied, where E is an injective right R-module.

Let $\mathcal{GF}$ denote the class of all Gorenstein flat left R-modules. In [12, Theorem 3.1.9] (see also [9]) it was proved that over a right coherent ring, $(\mathcal{GF}, \mathcal{GF}^\perp)$ is a perfect and hereditary cotorsion pair.

Corollary 2.11. *Every complex over a right coherent ring has an $\mathcal{EGF}$-cover.*

Proof. By [12, Corollary 2.1.9] the class $\mathcal{GF}$ is closed under direct limits. Thus it is enough to prove that $\mathcal{GF}$ is closed under pure submodules and cokernels of pure monomorphisms.

Suppose that

$$0 \to P \to M \to M/P \to 0$$

is pure exact in R-Mod, where $M \in \mathcal{GF}$. Then

$$0 \to (M/P)^+ \to M^+ \to P^+ \to 0$$

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is split and $M^+ \in \mathcal{GI}$ by [18, Theorem 3.6]. Here $\mathcal{GI}$ denotes the class of Gorenstein injective modules. Thus $(M/P)^+$ and P^+ are in $\mathcal{GI}$ by [18, Theorem 2.6], which implies that M/P and P are in $\mathcal{GF}$. □

We use the symbol $\mathcal{F}_n$ to denote the class of all left R-modules with flat dimension less than or equal to a fixed nonnegative integer n. In [21, Theorem 3.4] it was proved that $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ is a perfect and hereditary cotorsion pair. Note that $\mathcal{F}_n$ is closed under pure submodules, cokernels of pure monomorphisms and direct limits. Thus we have the following result.

Corollary 2.12. *Every complex has an $\mathcal{EF}_n$-cover.*

A left R-module M is called min-flat (see [20]) if $\mathrm{Tor}_1(R/I, M) = 0$ for each simple right ideal I. Let $\mathcal{MF}$ denote the class of all min-flat left R-modules. In [20, Theorem 3.4] it was proved that $(\mathcal{MF}, \mathcal{MF}^{\perp})$ is a perfect cotorsion pair. Note that $\mathcal{MF}$ is closed under pure submodules, cokernels of pure monomorphisms and direct limits.

Corollary 2.13. *Every complex has an $\mathcal{EMF}$-cover.*

3. Covers and envelopes of special complexes

Let $\mathcal{A}$ and $\mathcal{B}$ be classes of R-modules. In this section we consider the existence of a $\overline{\mathcal{A}}$-cover of a complex that is bounded above and a $\overline{\mathcal{B}}$-envelope of a complex that is bounded below. Here $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ stand for the classes of all complexes with each term in $\mathcal{A}$ and $\mathcal{B}$, respectively.

Lemma 3.1. *Let C be a complex.*

(1) *If $\varphi : G \to C$ is an $\overline{\mathcal{A}}$-precover in $\mathscr{C}(R)$, then $\varphi^n : G^n \to C^n$ is an $\mathcal{A}$-precover in* R-Mod *for all $n \in \mathbb{Z}$.*
(2) *If $\varphi : C \to G$ is a $\overline{\mathcal{B}}$-preenvelope in $\mathscr{C}(R)$, then $\varphi^n : C^n \to G^n$ is a $\mathcal{B}$-preenvelope in* R-Mod *for all $n \in \mathbb{Z}$.*

Proof. (1) Let D be in $\mathcal{A}$ and let $f : D \to C^n$ be an R-homomorphism. We define a map of complexes $\overline{f} : \overline{D}[-n-1] \to C$ as follows:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & D & \xrightarrow{\ \mathrm{id}\ } & D & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \delta^n f} & & \downarrow & & \\ \cdots & \longrightarrow & C^{n-1} & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow & C^{n+2} & \longrightarrow & \cdots \end{array}$$

Since $\overline{D}[-n-1]$ is in $\overline{\mathcal{A}}$ there is a map $h : \overline{D}[-n-1] \to G$ such that $\varphi h = \overline{f}$. So we have a commutative diagram

$$\begin{array}{ccc} & & D \\ & \overset{h^n}{\swarrow} & \downarrow{\scriptstyle f} \\ G^n & \xrightarrow[\varphi^n]{} & C^n \end{array}$$

This means that $\varphi^n : G^n \to C^n$ is an $\mathcal{A}$-precover of C^n.

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(2) Let F be in $\mathcal{B}$ and let $f : C^n \to F$ be an R-homomorphism. We define a map of complexes $\alpha : C \to \overline{F}[-n]$ as follows:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C^{n-2} & \longrightarrow & C^{n-1} & \longrightarrow & C^{n} & \longrightarrow & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f\delta^{n-1}} & & \downarrow{\scriptstyle f} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F & \xrightarrow{\text{id}} & F & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Since $\overline{F}[-n]$ is in $\overline{\mathcal{B}}$ there is a map $\beta : G \to \overline{F}[-n]$ such that $\varphi\beta = \alpha$. So we have a commutative diagram

$$\begin{array}{ccc}
C^n & \xrightarrow{\varphi^n} & G^n \\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle \beta^n} & \\
F & &
\end{array}$$

That is, $\varphi^n : C^n \to G^n$ is a $\mathcal{B}$-preenvelope of C^n. □

Theorem 3.2. *Let $\mathcal{A}$ be a covering class in R-*Mod *and let the complex*

$$C = \cdots \to C^{-2} \to C^{-1} \to C^0 \to 0 \cdots$$

be bounded above. Then:

(1) *C has an $\overline{\mathcal{A}}$-cover;*
(2) *if $\varphi : G \to C$ is an $\overline{\mathcal{A}}$-cover in $\mathscr{C}(R)$, then $\varphi^0 : G^0 \to C^0$ is an $\mathcal{A}$-cover.*

Proof. Part (1) follows from some ideas in the proof of [13, Theorem 3.3.10].

(2) We begin by proving that the complex G is bounded. The complex

$$G^* = \cdots \to 0 \to G^0 \to G^1 \to \cdots$$

is in $\overline{\mathcal{A}}$ and the obvious induced map $G^* \to C$ is an $\overline{\mathcal{A}}$-precover. So G is a direct summand of G^* and hence G is bounded above.

Next we prove that $\varphi^0 : G^0 \to C^0$ is an $\mathcal{A}$-cover of C^0. By Lemma 3.1 we know that $\varphi^0 : G^0 \to C^0$ is an $\mathcal{A}$-precover of C^0. Let $\varphi^0 : G(C^0) \to C^0$ be the $\mathcal{A}$-cover of C^0 in R-Mod. We consider the splitting epimorphism $\beta : G(C^0) \to C^0$ such that $\alpha^0\beta = \varphi^0$. We take the complex

$$G^* = \cdots \longrightarrow G^{-2} \xrightarrow{\delta_G^{-2}} G^{-1} \xrightarrow{\beta\delta_G^{-1}} G(C^0) \longrightarrow 0 \longrightarrow \cdots .$$

We also consider the map of complexes given by

$$\begin{array}{ccccccccc}
G^* = \cdots & \longrightarrow & G^{-2} & \longrightarrow & G^{-1} & \longrightarrow & G(C^0) & \longrightarrow & 0 \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle \varphi^{-2}} & & \downarrow{\scriptstyle \varphi^{-1}} & & \downarrow{\scriptstyle \alpha^0} & & \\
C = \cdots & \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & C^0 & \longrightarrow & 0 \longrightarrow \cdots
\end{array}$$

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It is easy to check that the above map, which we call $\alpha : G^* \to C$, is an $\overline{\mathcal{A}}$-precover. Thus there exists a splitting epimorphism $g : G^* \to G$ such that $\varphi g = \alpha$. That is, the diagram

$$\begin{array}{ccc} G & & \\ {\scriptstyle f}\,\vdots & \searrow^{\varphi} & \\ G^* & \xrightarrow{\ \alpha\ } & C \\ {\scriptstyle g}\,\vdots & \nearrow_{\varphi} & \\ G & & \end{array}$$

commutes and gf is an automorphism. Hence $\alpha^0\beta g^0 = \varphi^0 g^0 = \alpha^0$ and so βg^0 is an automorphism, which means that $\varphi^0 : G^0 \to C^0$ is an $\mathcal{A}$-cover of C^0. □

THEOREM 3.3. *Let $\mathcal{B}$ be an enveloping class in R-*Mod *and let the complex*

$$(C, \delta) = \cdots \to 0 \to C^0 \to C^1 \to C^2 \to \cdots$$

be bounded below. Then:

(1) *C has a $\overline{\mathcal{B}}$-envelope;*
(2) *if $\varphi : C \to G$ is a $\overline{\mathcal{B}}$-envelope, then $\varphi^0 : C^0 \to G^0$ is a $\mathcal{B}$-envelope.*

PROOF. (1) By the hypothesis we may choose a $\mathcal{B}$-envelope $\varphi^0 : C^0 \to G^0$. By analogy with the proof of [13, Theorem 3.3.10] we are going to construct a complex

$$G = \cdots \longrightarrow 0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

with each term in $\mathcal{B}$ and a map of complexes $\varphi : C \to G$ in the following way. For $i < 0$ we take $G^i = 0$ and $\varphi^i = 0$. For $i = 0$ we take the above envelope. Now for $i > 0$ we proceed inductively. Suppose that we have constructed

$$\begin{array}{ccccc} C^{i-1} & \xrightarrow{\delta^{i-1}} & C^i & \xrightarrow{\delta^i} & C^{i+1} \\ {\scriptstyle \varphi^{i-1}}\downarrow & & \downarrow{\scriptstyle \varphi^i} & & \\ G^{i-1} & \xrightarrow{\alpha^{i-1}} & G^i & & \end{array}$$

We consider the pushout diagram

$$\begin{array}{ccc} C^i & \xrightarrow{\qquad \delta^i \qquad} & C^{i+1} \\ {\scriptstyle \varphi^i}\downarrow & & \\ G^i & & \downarrow{\scriptstyle \nu^{i+1}} \\ {\scriptstyle \eta^i}\downarrow & & \\ \operatorname{Coker}(\alpha^{i-1}) & \xrightarrow[\mu^{i+1}]{\qquad} & P^{i+1} \end{array}$$

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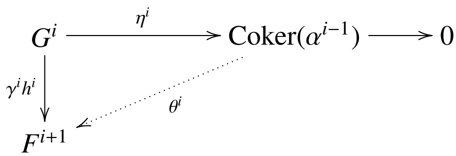
where $\eta^i : G^i \to \mathrm{Coker}(\alpha^{i-1})$ is the natural epimorphism. Then we take a $\mathcal{B}$-envelope of $P^{i+1}, \beta^{i+1} : P^{i+1} \to G^{i+1}$. We define $\alpha^i : G^i \to G^{i+1}$ to be the composition $\alpha^i = \beta^{i+1}\mu^{i+1}\eta^i$ and define $\varphi^{i+1} : C^{i+1} \to G^{i+1}$ by $\varphi^{i+1} = \beta^{i+1}\nu^{i+1}$. It is not hard to check that this construction gives a complex G with terms in $\mathcal{B}$ and a map of complexes $\varphi : C \to G$.

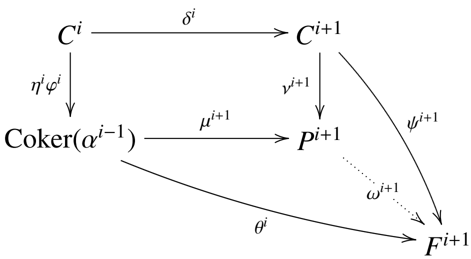
Let

$$F = \cdots \longrightarrow F^i \xrightarrow{\gamma^i} F^{i+1} \xrightarrow{\gamma^{i+1}} F^{i+2} \longrightarrow \cdots$$

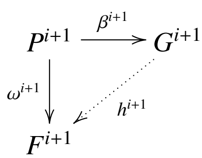
be in $\overline{\mathcal{B}}$ and let $\psi : C \to F$ be a map of complexes. We are going to construct a morphism of complexes $h : G \to F$ such that $h\varphi = \psi$. For $i < 0$ we take $h^i = 0$. For $i = 0$, since $\varphi^0 : C^0 \to G^0$ is a $\mathcal{B}$-envelope, there exists $h^0 : G^0 \to F^0$ such that $h^0\varphi^0 = \psi^0$. We proceed by induction. Suppose that $h^i : G^i \to F^i$ is defined such that $h^i\varphi^i = \psi^i$ and $h^i\alpha^{i-1} = \gamma^{i-1}h^{i-1}$. By the factor theorem (see [2, Theorem 3.6]) we have the commutative diagram



We consider the commutative diagram induced by the pushout



Since G^{i+1} is a $\mathcal{B}$-envelope of P^{i+1} there exists $h^{i+1} : G^{i+1} \to F^{i+1}$ such that the diagram



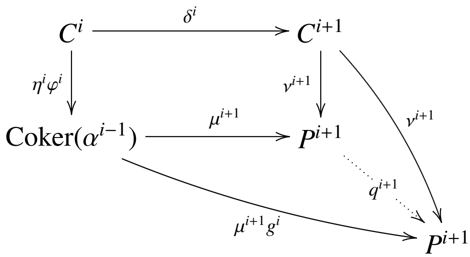
can be completed commutatively. It is easy to see that in this way we obtain a map of complexes $h : G \to F$ such that $h\varphi = \psi$.

Now let $f : G \to G$ be a map of complexes such that $f\varphi = \varphi$. For $i < 0$ we have $f^i = 0$. For $i = 0$ we know that f^0 is an automorphism because $\varphi^0 : C^0 \to G^0$ is a $\mathcal{B}$-envelope. For $i > 0$ we proceed inductively. Suppose that f^{i-1} and f^i are automorphisms.

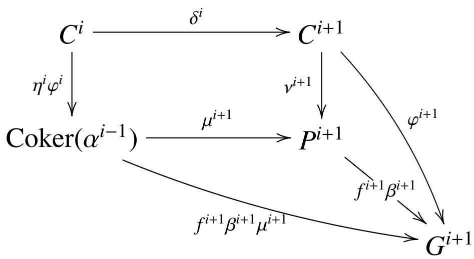
We show that $f^{i+1}: G^{i+1} \to G^{i+1}$ is also an automorphism. We consider the commutative diagram

$$\begin{array}{ccccccc} G^{i-1} & \longrightarrow & G^{i} & \longrightarrow & \mathrm{Coker}(\alpha^{i-1}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^{i}} & & \vdots{\scriptstyle g^{i}} & & \\ G^{i-1} & \longrightarrow & G^{i} & \longrightarrow & \mathrm{Coker}(\alpha^{i-1}) & \longrightarrow & 0 \end{array}$$

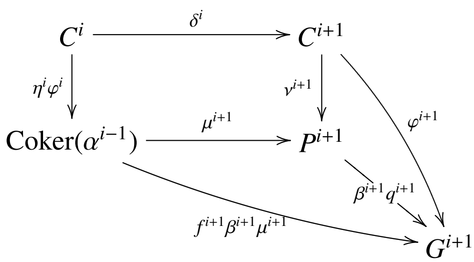
and get that g^i is an automorphism. By the properties of a pushout diagram, we get that the diagram



commutes and $q^{i+1}: P^{i+1} \to P^{i+1}$ is an automorphism. Since $f^{i+1}\alpha^i = \alpha^i f^i$, we get $f^{i+1}\beta^{i+1}\mu^{i+1} = \beta^{i+1}q^{i+1}\mu^{i+1}$ and so we obtain the commutative diagrams



and



By the properties of pushout diagrams, $f^{i+1}\beta^{i+1} = \beta^{i+1}q^{i+1}$. That is, the diagram

$$\begin{array}{ccc} P^{i+1} & \xrightarrow{\beta^{i+1}} & G^{i+1} \\ {\scriptstyle q^{i+1}}\downarrow & & \downarrow{\scriptstyle f^{i+1}} \\ P^{i+1} & \xrightarrow[\beta^{i+1}]{} & G^{i+1} \end{array}$$

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is commutative. Since $\beta^{i+1} : P^{i+1} \to G^{i+1}$ is a $\mathcal{B}$-envelope and q^{i+1} is an automorphism, it follows that f^{i+1} is an automorphism.

Part (2) follows by an argument like that to prove [13, Proposition 3.2.14]. □

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