AN ANALOGUE OF A RESULT OF JACOBSTHAL

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Dedicated to Professor Jacobsthal on his eightieth birthday

1. Introduction

Jacobsthal (**4**) has proved that the $n \times n$ matrix

$$A = ([r/s]),\ r,\ s = 1, 2, \ldots, n, \quad \ldots\ldots\ldots\ldots(1)$$

is invertible with the inverse,

$$A^{-1} = (\mu(r/s) - \mu(r/(s+1))),\ r,\ s = 1, 2, \ldots, n. \quad \ldots\ldots\ldots\ldots(2)$$

Here $\mu(x)$ denotes the Möbius function for positive integral x and is assumed to be 0 for other values; $[x]$ has its usual meaning as the number of positive integers $\leqq x$.

Carlitz (**1**) has given a particularly simple proof of this result, starting from the identity,

$$\sum_{d \mid n} \mu(d) = \epsilon(n) \overset{\text{def}}{=} \begin{cases} 1 (n = 1) \\ 0 (n > 1). \end{cases} \quad \ldots\ldots\ldots\ldots(3)$$

In this note we prove a " unitary " analogue of Jacobsthal's result, contained in (7) below. The proof is based upon the analogue (5) of (3) for unitary divisors. We indicate in § **3** a generalisation from which several special results are then drawn (§ **4**).

2. An Analogue of (2)

We say that d is a unitary divisor of n, written $d \| n$ or $d^* \delta = n$, provided $n = d\delta$, $(d, \delta) = 1$; in such a case, we also call n a unitary multiple of d. Let us place $\mu^*(n) = \mu(\nu(n))$ where $\nu(n)$ is the maximal square-free divisor of n and define

$$\mu^*(r; s) = \begin{cases} \mu^*(r/s) & \text{if } s \| r, \\ 0 & \text{otherwise.} \end{cases} \quad \ldots\ldots\ldots\ldots(4)$$

We recall from (**2**, (2.5)) that

$$\sum_{d \| n} \mu^*(d) = \epsilon(n). \quad \ldots\ldots\ldots\ldots(5)$$

In addition, let us define $[x, s]$ to be the number of unitary multiples of s not exceeding x.

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We prove now that the matrix

$$A^* = ([r, s]), \quad r, s = 1, 2, \ldots, n, \qquad \text{......(6)}$$

has the inverse

$$(\delta_{rs})/A^* = (\mu^*(r; s) - \mu^*(r; s+1)), \quad r, s = 1, \ldots, n, \qquad \text{......(7)}$$

δ_{rs} denoting of course the Kronecker delta function.

Letting $\rho(n)$ denote the function with the value 1 for all n, we place

$$\rho^*(r; s) = \begin{cases} \rho(r/s) = 1 & \text{if } s \| r, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{......(8)}$$

One obtains from (4) and (8), if $s \| r$,

$$\sum \equiv \sum_{k=1}^{n} \mu^*(r; k)\rho^*(k; s) = \sum_{\substack{k \| r \\ s \| k}} \mu^*(r/k) \qquad \text{......(9)}$$

$$= \sum_{\substack{s * s' = k \\ k * d = r}} \mu^*(d) = \sum_{\substack{d \| r \\ d * s' = r/s}} \mu^*(d) = \sum_{d \| r/s} \mu^*(d);$$

since Σ is 0 in case $s \not\| r$, it follows from (5) that

$$\sum = \sum_{k=1}^{n} \mu^*(r; k)\rho^*(k; s) = \delta_{r,s}(r, s = 1, 2, \ldots, n). \qquad \text{......(10)}$$

Noting that

$$\sum_{k=1}^{r} \rho(k; s) = [r, s], \qquad \text{......(11)}$$

one finds on applying partial summation that

$$\sum = \sum_{k=1}^{n} (\mu^*(r; k) - \mu^*(r; k+1))[k; s] + [n, s]\mu^*(r; n+1),$$

so that (10) yields

$$\sum_{k=1}^{n} (\mu^*(r; k) - \mu^*(r; k+1))[k, s] = \delta_{r,s}(r, s = 1, 2, \ldots, n). \qquad \text{......(12)}$$

This is equivalent to the desired result.

From (11) we have

$$\rho(r; s) = [r, s] - [r-1, s]; \qquad \text{......(13)}$$

hence (10) may be put into the matric form,

$$(\mu^*(r; s))([r, s] - [r-1, s]) = (\delta_{rs}), \; r, s = 1, 2, \ldots, n. \qquad \text{......(14)}$$

3. A Generalisation

For arbitrary arithmetical functions $f(n)$, we write

$$f(r; s) = \begin{cases} f(r/s) & \text{if } s \| r, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{......(15)}$$

Let now f, g, h be arithmetical functions related by

$$\sum_{d*\delta=n} f(d)g(\delta) = h(n). \qquad \text{.....(16)}$$

In the same way that (5) implied (10), (16) implies that

$$\sum_{k=1}^{n} f(r;\, k)g(k;\, s) = h(r;\, s), \quad r, s = 1, 2, \ldots, n. \qquad \text{.....(17)}$$

Placing

$$G(r, s) = \sum_{k=1}^{r} g(k;\, s), \quad G(0, s) = 0, \qquad \text{.....(18)}$$

and letting Σ denote the left member of (17), one obtains by partial summation,

$$\sum = \sum_{k=1}^{n} f(r;\, k)(G(k, s) - G(k-1, s))$$

$$= \sum_{k=1}^{n} (f(r;\, k) - f(r;\, k+1))G(k, s) + f(r;\, n+1)G(n, s).$$

That is,

$$\sum_{k=1}^{n} f(r;\, k) - f(r;\, k+1)G(k;\, s) = h(r;\, s), \quad r, s = 1, 2, \ldots, n, \qquad \text{.....(19)}$$

or in matric notation,

$$(f(r;\, s) - f(r;\, s+1))(G(r, s)) = (h(r;\, s)), \quad r, s = 1, \ldots, n. \qquad \text{.....(20)}$$

4. Specialisations

We consider some special cases. Let $\phi^*(n)$ denote the number of integers $b(\text{mod } n)$ such that 1 is the largest unitary divisor of n which divides b. In (**2**, (2.4)) it was proved that

$$\sum_{d\|n} \phi^*(d) = n. \qquad \text{.....(21)}$$

With $f(n) = \phi^*(n)$, $g(n) = \rho(n)$, $h(n) = n$, (16) reduces to (21), and (20) becomes, by (11),

$$(\phi^*(r;\, s) - \phi^*(r;\, s+1))([r, s]) = (r\rho(r;\, s)/s), \quad r, s = 1, 2, \ldots, n. \qquad \text{.....(22)}$$

For positive integers k, let L_k denote the set of integers n whose prime factors all have multiplicity $\geqq k$. Let $L = L_2$, so that L is the set of "square-full" integers. Also, let $l_k(n)$ denote the characteristic function of L_{k+1}, $l(n) = l_1(n)$. In (**3**, Lemma 3), it was proved that for each positive integer t,

$$\sum_{d\|n} \mu_t^*(d) = l_t(n) \overset{\text{def}}{=} \begin{cases} 1 (n \in L_{t+1}) \\ 0 (n \notin L_{t+1}), \end{cases} \qquad \text{.....(23)}$$

where $\mu_t^*(n)$ is the multiplicative function defined for primes p and non-negative integers e, by

$$\mu_t^*(p^e) = 1, \ -1, \text{ or } 0,$$

according as $e = 0$, $0 < e \leqq t$, or $e > t$.

Placing $f(n) = \mu_t^*(n)$, $g(n) = \rho(n)$, $h(n) = l_t(n)$, (20) yields, by virtue of (16) and (23), the matric relation,

$$(\mu_t^*(r; s) - \mu_t^*(r; s+1))([r, s]) = (l_t(r; s)), \quad r, s = 1, 2, \ldots, n. \qquad \text{(24)}$$

It will be observed that $\mu_1^*(n) = \mu(n)$; hence (24) becomes in the case $t = 1$,

$$(\mu(r; s) - \mu(r; s+1))([r, s]) = (l(r; s)), \quad r, s = 1, 2, \ldots, n. \qquad \text{(25)}$$

This may be viewed as a second analogue of Jacobsthal's formula. The original one also results from (24), on letting $t \to \infty$ and noting that

$$\lim_{t\to\infty} \mu_t^*(n) = \mu^*(n), \ \lim_{t\to\infty} l_t(n) = \epsilon(n).$$

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