Galton, Cayley, and the land across the river

TONY CRILLY

The practical measurement of distance on land and sea has been of enduring interest for millennia. Here we outline a brief interaction between Francis Galton (1822-1911) and Arthur Cayley (1821-95) in which they too put the ‘geo’ back into geometry.

Galton wanted to measure everything. In the adventurous life of this many-sided Victorian scientist, quantification was the hallmark of his work. In measuring geographical distances Galton used an idea suggested by Sir George Everest (1790-1866), the Surveyor-General of India in the 1830s. In the pages of the journal of the Royal Geographical Society, Everest showed how to calculate the distance from one point to an inaccessible point. One application would be to measure the distance from a point on one side of a river to a point on the other side.

In the diagram, the point $P$ is the inaccessible point, the point $A$ is on ‘our side’ of the river, and we are to calculate the distance $PA$. For the intrepid Victorian explorer such as Galton, distances are measured in ‘paces’, in which the base of the large triangle $ABP$ can be fixed at 100 paces, say. The small triangle $ADE$ is constructed by choosing $AD$ and $AE$ equal (to 10 paces, say) and pacing out the unknown side $DE$. Similarly, the other small triangle $BFG$ can be constructed with $BF = BG = 10$ paces and $FG$ is paced out.

Using the cosine rule in the small triangles, the angles at $A$ and $B$ can be determined and therefore the angle $P$ is known. Using the sine rule in the large triangle, we find

$$PA = 100 \frac{\sin B}{\sin P}.$$  

In March 1860, Galton produced a ready reckoning table whereby the distances $PA$ could be looked up. Using the trigonometrical rules as suggested, a table can easily be produced. For example, one line of it might be:
Gallon’s bushcraft advertised as ‘Rough Triangulation without the usual Instruments [such as theodolites] and without Calculation’ was included in his *Hints for Travellers* published by the Royal Geographical Society [1].

Gallon understood geometrical calculations. In his first term at Cambridge he was plunged into mathematics, as was the case with all students at that university in the nineteenth century. Working towards his impending degree examinations in the early 1840s, he went on a summer reading party to Scotland led by Cayley who was only six months older. Cayley was one of Galton’s early heroes, and he remained so; in later years to Galton he was the great Cambridge mathematician. During their lives they met, corresponded and generally kept in touch. Galton was active in African exploration in his younger days, and when he returned to England, he lived out the life of a gentleman-of-science of private means. He joined various London clubs and societies and was for many years the Secretary of the Royal Geographical Society. Until 1863 Cayley was a barrister at Lincoln’s Inn, in London [2].

In 1860, Galton was elected to the Royal Society and Cayley was one of his sponsors. Galton now had a direct line to his company, and evidently impressed by Everest’s surveying method perhaps discussed it with Cayley. Later in the same year Cayley wrote him a note about another method for calculating distance, in which he used his knowledge of the mathematical literature to arrive at a very different solution to the problem and it was one which did not require trigonometry. It was not his own, but he thought it worthwhile to make the method more widely known. To Galton he wrote:

‘The following method (given in Brianchon’s *Application de la théorie des transversals*, Paris, 1818 [3]) of finding the distance of an inaccessible point—if it is not generally known—seems really worth knowing—and I should think might often be practically useful—

The problem is to find your distance at A from an inaccessible point P. Plant stakes at A, at B anywhere in the line AP, at R, C in a line with B, and at Q in the intersection of the lines AC & RP. Then

\[ PA \cdot QC \cdot RB = PB \cdot QA \cdot RC = (PA + AB) \cdot QA \cdot RC \]

or

\[ PA = AB \cdot \frac{QA \cdot RC}{QC \cdot RB - QA \cdot RC} \]

so that PA is known in terms of the distances QA, RC, QC, RB which may be measured [4].’

<table>
<thead>
<tr>
<th>DE</th>
<th>FG</th>
<th>angle A</th>
<th>angle B</th>
<th>angle P</th>
<th>PA</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>14</td>
<td>34.92</td>
<td>88.85</td>
<td>56.23</td>
<td>120.27</td>
</tr>
</tbody>
</table>
Cayley quoted directly from Brianchon’s *Application* and uses the same notation [3, pp. 4-5]. Charles-Julien Brianchon (1783-1864) applied the theory of transversals (lines which fall across triangles) to practical problems. He was appointed as professor in the Royal Guard military school in Vincennes, near Paris, and as well as using the river example to exploit the method, he applied it to the military situation where $A$ is a cannon emplacement, $P$ is the target and it is required to find the horizontal distance between the two positions.

What is the source of Brianchon’s method? It has a Greek flavour but Brianchon does not mention this in his pamphlet, and perhaps assuming this was known, works out his own geometry of position. The method is immediately derivable from the famous theorem of Menelaus on triangles and transversals [5]. The expression quoted by Cayley

$$PA.QC.RB = PB.QA.RC$$

is the form of Menelaus’ theorem in which $PQR$ is the transversal and $ABC$ is the triangle. This is usually expressed in terms of ratios as:

$$\frac{AP}{PB} \frac{BR}{RC} \frac{CQ}{QA} = -1,$$

but in the application, the theorem is concerned with distance and the order of the letters is immaterial, as is the presence of $-1$.

If the measuring pegs are arranged so that $RB = RC$ then the formula for distance become simpler. The distance in this case becomes:

$$PA = AB \frac{QA}{(QC - QA)}$$

Actually, the method can be extended to measuring distances between points (say $T$ and $P$) in the inaccessible land beyond the river.
In the figure, let \( T \) be on the line \( PB \) on the same side of the river as \( P \), and the line from \( T \) which passes through \( Q \) intersect \( BC \) at \( S \). Then, by applying Brianchon’s method to finding the length \( TA \), and subtracting it from \( PA \) we obtain

\[
PT = AB \frac{QC \cdot QA}{(QC \cdot RB - QA \cdot RC)} \frac{(RC \cdot SB - RB \cdot SC)}{(QC \cdot SB - QA \cdot SC)}.
\]

If in equation (1) we choose \( AB \) to be the unit of measurement and designate lengths by single symbols (\( QC = x \), \( QA = y \), \( RC = a \), \( BR = b \), \( SC = c \), \( SB = d \)) we find the distance \( PT \) assumes a symmetric form when written in terms of 2 \( \times \) 2 determinants:

\[
PT = xy \begin{vmatrix} a & c \\ b & d \end{vmatrix} \begin{vmatrix} x & c \\ y & d \end{vmatrix}.
\]

This can be applied to surveying problems provided neither of the determinants in the denominator vanish (but care must be taken since the denominators can be small in value).

Apart from the intrinsic merit of Menelaus’s theorem, the method gives a practical reason for bringing it back into the geometry syllabus [6].

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References


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