THE STONE-ČECH COMPACTIFICATION OF $\operatorname{Prim} A$

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For a C^*-algebra A, we give simple proofs of the following: $C_b(\operatorname{Prim} A)$ is isomorphic to the centre $ZM(A)$ of the multiplier algebra, $C_b(\operatorname{Prim} A)$ is isomorphic to $C(\operatorname{Prim} M(A))$ and $\operatorname{Prim} ZM(A)$ is the Stone-Čech compactification of $\operatorname{Prim} A$.

INTRODUCTION

THE DAUNS-HOFMANN THEOREM. [2, III Lemma 8.15] Let A be a C^*-algebra and let $\operatorname{Prim} A$ be the primitive ideal space of A with the hull-kernel topology. For each $a \in A$ and $f \in C_b(\operatorname{Prim} A)$, there is a unique element $f \cdot a \in A$ such that

$$(1) \qquad\qquad f \cdot a - f(P)a \in P \text{ for all } P \in \operatorname{Prim} A.$$

It is easy to see how the Dauns-Hofmann theorem gives an injection from $C_b(\operatorname{Prim} A)$ into the centre $ZM(A)$ of the multiplier algebra. It is not evident why this map should be onto. Dixmier [3, Theorem 5(iii)] proved that it was, as did Pedersen [8, Corollary 4.7]. But, we believe that the natural way to view the problem is the reverse of that commonly thought. The *easy* part is getting the injection from $ZM(A)$ into $C_b(\operatorname{Prim} A)$. Then the Dauns-Hofmann theorem is exactly what you need to show that *this* map is onto (Theorem 1.2). The simplicity of our technique means our proof is much more direct than those given before. An easy corollary of our technique provides an isomorphism from $C_b(\operatorname{Prim} A)$ onto $C_b(\operatorname{Prim} M(A))$ (Theorem 1.4).

Since $\operatorname{Prim} A$ is not necessarily Hausdorff, we carefully define the Stone-Čech compactification and the complete regularisation of such spaces. In Section 2 we deduce from Theorem 1.2 that $\operatorname{Prim} ZM(A)$ is the Stone-Čech compactification of $\operatorname{Prim} A$, specifying the formula for the canonical mapping (Theorem 2.2). This result is a generalisation of an assertion of Becker [1, p.269]. (We also discuss a problem in Becker's proof.)

The key idea in all our arguments is the continuity of the map Res_ι from ideals of B to ideals of A, dual to a homomorphism $\iota \colon A \to M(B)$. The map Res_ι, and the map

Ext$_\iota$ from ideals of A to ideals of B, were first introduced by Green [6, Proposition 9], and have been predominantly used in the study of crossed products and induced representations [5, Section 2]. However, in this paper we have used these maps in the trivial case; by letting ι be the identity map, the elementary properties of Res and Ext lead to short, elegant proofs.

In [7], we used Res$_\iota$ and Ext$_\iota$ to help give a short self-contained proof of the sectional representation theorem [7, Theorem 3.1]. This result says that a C^*-algebra can be represented as the section algebra of an upper semi-continuous C^*-bundle over the primitive ideal space of its centre. In [1], Becker asked for a simple necessary and sufficient condition on a C^*-algebra A, which ensures that the C^*-bundle produced by the sectional representation theorem will be continuous [1, p. 268]. This is answered by [7, Theorem 3.1], which says that this happens only when Res$_{\mathrm{id}}$ is open. In Section 3 we show that our condition is equivalent to the condition Becker gave (Proposition 3.1).

1. DIXMIER'S THEOREM

Let A be a C^*-algebra and let $\mathcal{I}(A)$ denote the ideal space of A with the topology which has as subbasic open sets $O_J = \{I : I \not\supseteq J\}$, where J is a closed ideal in A. The relative topology on $\mathrm{Prim}(A)$ in $\mathcal{I}(A)$ is the usual hull-kernel topology. All representations will be non-degenerate, and the extension of a representation ρ of A to the multiplier algebra $M(A)$ will be denoted by $\overline{\rho}$.

DEFINITIONS: Let A and B be C^*-algebras and $\iota : A \to M(B)$ a homomorphism. Define Res$_\iota : \mathcal{I}(B) \to \mathcal{I}(A)$ by Res$_\iota(\ker \sigma) = \ker(\overline{\sigma} \circ \iota)$, where σ is a representation of B. Define Ext$_\iota : \mathcal{I}(A) \to \mathcal{I}(B)$ by Ext$_\iota(I) = \overline{B\iota(I)B}$ [6, Proposition 9(i)].

In [7, Lemma 1.1] we showed that Res$_\iota(\ker \sigma) = \{a \in A : \iota(a)B \subseteq \ker \sigma\}$. This says that Res$_\iota$ is well-defined, and that our definition is equivalent to [6, Proposition 9(i)]. It also implies that Res$_\iota$ preserves containment and arbitrary intersections. Using this characterisation, we also showed that $I \subseteq$ Res$_\iota(K) \Leftrightarrow$ Ext$_\iota(I) \subseteq K$. It follows easily from these facts that Res$_\iota$ is continuous [7, Lemma 1.1(iii)], [6, Proposition 9(i)].

LEMMA 1.1. *Let A be a C^*-algebra and id: $ZM(A) \to ZM(A)$ the identity map. Then* Res$_{\mathrm{id}}$: Prim $A \to$ Prim $ZM(A)$ *is a continuous map with dense range, and* Res$_{\mathrm{id}}(\ker \rho) = \ker\left(\overline{\rho}|_{ZM(A)}\right)$.

PROOF: An irreducible representation ρ of A extends to an irreducible representation $\overline{\rho}$ of $M(A)$. Also $\overline{\rho}(ZM(A))$ is in the centre of Im $\overline{\rho}$, which is contained in the commutant $C1$. Thus $\overline{\rho}|_{ZM(A)}$ determines a complex homomorphism, which must be non-zero because $1 \in ZM(A)$. So $\ker \overline{\rho}|_{ZM(A)}$ is primitive and Res maps primitive ideals to primitive ideals. The range is dense because $\overline{\mathrm{Res}\,(\mathrm{Prim}\,A)} = \{Q : Q \supseteq$

$$\bigcap_P \operatorname{Res}(P)\} = \{Q : Q \supseteq \operatorname{Res}\left(\bigcap_P P\right)\} = \operatorname{Prim} ZM(A). \qquad \square$$

THEOREM 1.2 DIXMIER. *Let A be a C^*-algebra. Then there is an isomorphism $\psi \colon ZM(A) \to C_b(\operatorname{Prim} A)$ such that $\psi(z)(\ker \rho) = \widehat{z}\big(\ker\big(\overline{\rho}|_{ZM(A)}\big)\big)$, where $\wedge$ is the Gelfand transform.*

PROOF: Define $\operatorname{Res}_{\mathrm{id}}^* \colon C(\operatorname{Prim} ZM(A)) \to C_b(\operatorname{Prim} A)$ by $\operatorname{Res}_{\mathrm{id}}^*(g) = g \circ \operatorname{Res}_{\mathrm{id}}$, where $\operatorname{Res}_{\mathrm{id}}$ is as in Lemma 1. We show that $\operatorname{Res}_{\mathrm{id}}^* \circ \wedge$ is the required isomorphism ψ. Since $\operatorname{Res}_{\mathrm{id}}$ has dense range, $\operatorname{Res}_{\mathrm{id}}^*$ is an injection. The Gelfand transform is an isomorphism from $ZM(A)$ onto $C(\operatorname{Prim} ZM(A))$, so it is enough to show that $\operatorname{Res}_{\mathrm{id}}^*$ is onto. Define $L_f \colon A \to A$ by $L_f(a) = f \cdot a$, and $R_f \colon A \to A$ by $R_f(a) = f \cdot a$, where $f \cdot a$ is the unique element given by the Dauns-Hofmann theorem. One can check that (L_f, R_f) is a multiplier z_f of A. Equation (1) implies that $z_f a = a z_f$ for all $a \in A$, so $z_f \in ZM(A)$. Also from (1) we have $(z_f - f(P))a = f \cdot a - f(P)a \in P$ for all $a \in A$. Since $\operatorname{Res}_{\mathrm{id}}(P) = \{z \in ZM(A) : zA \subset P\}$, $z_f - f(P)1 \in \operatorname{Res}_{\mathrm{id}}(P)$ for all $P \in \operatorname{Prim} A$. Equivalently, $\widehat{z_f}(\operatorname{Res}_{\mathrm{id}}(P)) = f(P)$, for all P. Thus $\operatorname{Res}_{\mathrm{id}}^*(\widehat{z_f})(P) = \widehat{z_f}(\operatorname{Res}_{\mathrm{id}}(P)) = f(P)$, and $\operatorname{Res}_{\mathrm{id}}^*$ is an isomorphism. $\qquad \square$

REMARK. The following lemma is more than we need for Theorem 1.4, but it says that $\operatorname{Prim} M(A)$ is a compactification of $\operatorname{Prim} A$, in the sense of [**10**, p.1].

LEMMA 1.3. *Let A be a C^*-algebra, and $i \colon M(A) \to M(A)$ the identity map. Then $\operatorname{Res}_i \colon \operatorname{Prim} A \to \operatorname{Prim} M(A)$ is a continuous injection with dense range satisfying $\operatorname{Res}_i(\ker \rho) = \ker \overline{\rho}$, which is a homeomorphism onto its range.*

PROOF: If ρ is irreducible the extension $\overline{\rho}$ is also irreducible, so Res_i maps between primitive ideal spaces. Suppose $\ker \overline{\rho} = \ker \overline{\sigma}$. Then $\ker \overline{\rho} \circ \iota = \ker \overline{\sigma} \circ \iota$, where ι is the natural embedding of A into $M(A)$. But $\overline{\rho} \circ \iota = \rho$, and similarly for σ, so $\ker \rho = \ker \sigma$ and Res_i is one-to-one. A calculation similar to that in Lemma 1.1 shows that Res_i has dense range. It remains to check that Res_i is open onto its range. In [**7**, Lemma 1.1(iv)], we showed that Res_i is open onto its image if and only if $\operatorname{Ext}_i|_{\operatorname{Im(Res)}}$ is continuous. So it suffices to show that $\operatorname{Ext}_i \colon \mathcal{I}(M(A)) \to \mathcal{I}(A)$ is continuous. We have the embedding ι, so define $\operatorname{Res}_\iota \colon \mathcal{I}(M(A)) \to \mathcal{I}(A)$ as usual. To show Ext_i is continuous we shall show that $\operatorname{Ext}_i = \operatorname{Res}_\iota$, that is $\overline{A K A} = \{a \colon \iota(a)M(A) \subset K\}$. Suppose $\iota(a)m \in K$ for all $m \in M(A)$. Then $\iota(a) \in K$ because $1 \in M(A)$. So $a \in \overline{A K A}$ since $a \sim \mu_\lambda \iota(a)\mu_\lambda$, where μ_λ is an approximate identity in A. Conversely, suppose $a, b \in A$ and $k \in K$. Then $\iota(a)k\,\iota(b)m \in K$ for all $m \in M(A)$. Thus $A K A \subset \operatorname{Res}_\iota(K)$, and since $\operatorname{Res}_\iota(K)$ is closed, $\overline{A K A}$ is in there too. $\qquad \square$

THEOREM 1.4. *Let A be a C^*-algebra. Then there is an isomorphism*

$$\psi \colon C(\operatorname{Prim} M(A)) \to C_b(\operatorname{Prim} A)$$

satisfying $\psi(g)(\ker\rho) = g(\ker\overline{\rho})$.

PROOF: Lemma 1.3 gives us a continuous map $\mathrm{Res}_i \colon \mathrm{Prim}\,A \to \mathrm{Prim}\,M(A)$, so define $\mathrm{Res}_i^* \colon C(\mathrm{Prim}\,M(A)) \to C_b(\mathrm{Prim}\,A)$, by $\mathrm{Res}_i^*(g) = g \circ \mathrm{Res}_i$, which is injective because Res_i has dense range. Let $M \colon ZM(A) \to ZM(A)$ be the identity map, and define $\mathrm{Res}_M \colon \mathrm{Prim}\,M(A) \to \mathrm{Prim}\,ZM(A)$ as usual. From Lemma 1.1 we have the map $\mathrm{Res}_{\mathrm{id}} \colon \mathrm{Prim}\,A \to \mathrm{Prim}\,ZM(A)$. From the definitions, $\mathrm{Res}_{\mathrm{id}} = \mathrm{Res}_M \circ \mathrm{Res}_i$, so that $\mathrm{Res}_{\mathrm{id}}^* = \mathrm{Res}_i^* \circ \mathrm{Res}_M^*$. By the proof of Theorem 1.2, $\mathrm{Res}_{\mathrm{id}}^*$ is onto, so Res_i^* must be onto. Choose $\psi = \mathrm{Res}_i^*$.   ☐

2. STONE-ČECH COMPACTIFICATION AND COMPLETE REGULARISATION

DEFINITION: The *Stone-Čech compactification* of a topological space X is a compact Hausdorff space βX together with a continuous map $\beta \colon X \to \beta X$ such that, for every compact Hausdorff space Y and every continuous map $k \colon X \to Y$, there is a unique continuous map $k' \colon \beta X \to Y$ with $k = k' \circ \beta$ [**9**, 14.1.1].

REMARK. It is well known that $\mathrm{Prim}\,C_b(X)$ is the Stone-Čech compactification of X in the case where X is Hausdorff. The point of the next lemma, is that this is so even when X is *not* Hausdorff. Although this proof is similar the standard existence proof [**9**, 14.1.2], this shows how our techniques give a neat exposition.

LEMMA 2.1. *Let X be a topological space and define $\varepsilon \colon X \to \mathrm{Prim}\,C_b(X)$ by $\varepsilon(x) = \ker\varepsilon_x$, where ε_x is evaluation at x. Then $(\mathrm{Prim}\,C_b(X), \varepsilon)$ is the Stone-Čech compactification of X.*

PROOF: Let $k \colon X \to Y$ be a continuous map, with Y compact Hausdorff. Define $k^* \colon C(Y) \to C_b(X)$ such that $k^*(f) = f \circ k$, and define $\mathrm{Res}_{k^*} \colon \mathcal{I}(C_b(X)) \to \mathcal{I}(C(Y))$ as usual. Lemma 1.1 tells us that it restricts to a map between primitive ideal spaces, so we have $\mathrm{Res}_{k^*} \colon \mathrm{Prim}\,C_b(X) \to \mathrm{Prim}\,C(Y)$. Since Y is compact Hausdorff, there exists a homeomorphism $h \colon \mathrm{Prim}\,C(Y) \to Y$. Define $k' \colon \mathrm{Prim}\,C_b(X) \to Y$ by $k' = h \circ \mathrm{Res}_{k^*}$. We need to show that $k(x) = k'(\varepsilon(x))$ for all $x \in X$. Well, $k'(\varepsilon(x)) = h \circ \mathrm{Res}_{k^*}(\ker\varepsilon_x) = h(\ker\varepsilon_x \circ k^*)$, so it suffices to see that $\ker\varepsilon_{k(x)} = \ker(\varepsilon_x \circ k^*)$. But $\varepsilon_x \circ k^*(f) = \varepsilon_x(f \circ k) = f \circ k(x) = f(k(x))$, so $f \in \ker(\varepsilon_x \circ k^*)$ means $f(k(x)) = 0$, which is equivalent to $f \in \ker(\varepsilon_{k(x)})$. The map k' is unique, because the range of ε is dense in $\mathrm{Prim}\,C_b(X)$.   ☐

THEOREM 2.2. *Let A be a C^*-algebra. Then $(\mathrm{Prim}\,ZM(A), \mathrm{Res}_{\mathrm{id}})$ is the Stone-Čech compactification of $\mathrm{Prim}\,A$, where $\mathrm{Res}_{\mathrm{id}} \colon \mathrm{Prim}\,A \to \mathrm{Prim}\,ZM(A)$ satisfies $\mathrm{Res}_{\mathrm{id}}(\ker\rho) = \ker(\overline{\rho}|_{ZM(A)})$, for all $\ker\rho \in \mathrm{Prim}\,A$.*

PROOF: Theorem 1.2 gives an isomorphism $\psi \colon ZM(A) \to C_b(\mathrm{Prim}\,A)$, so one can show that $\mathrm{Res}_\psi \colon \mathrm{Prim}(C_b(\mathrm{Prim}\,A)) \to \mathrm{Prim}\,ZM(A)$ is a homeomorphism. Lemma 1.2

says that for every continuous map $k: \operatorname{Prim} A \to Y$ (Y compact Hausdorff), there is a unique continuous map $k'': \operatorname{Prim}(C(\operatorname{Prim} A)) \to Y$ with $k = k'' \circ \varepsilon$. So define $k': \operatorname{Prim} ZM(A) \to Y$ by $k' = k'' \circ (\operatorname{Res}_\psi)^{-1}$. It remains to show that $k = k' \circ \operatorname{Res}_{\mathrm{id}}$, and for this it is enough to verify $\operatorname{Res}_\psi \circ \varepsilon = \operatorname{Res}_{\mathrm{id}}$:

$$
\begin{aligned}
\operatorname{Res}_\psi \circ \varepsilon (\ker \rho) &= \{ z \in ZM(A) : \psi(z) f \in \varepsilon(\ker \rho) \ \text{ for all } \ f \in C_b(\operatorname{Prim} A)\} \\
&= \{ z : \psi(z) f(\ker \rho) = 0 \ \text{ for all } \ f\} \\
&= \{ z : \widehat{z}(\operatorname{Res}_{\mathrm{id}}(\ker \rho)) = 0\} = \operatorname{Res}_{\mathrm{id}}(\ker \rho).
\end{aligned}
$$

Thus $(\operatorname{Prim} ZM(A), \operatorname{Res}_{\mathrm{id}})$ is the Stone-Čech compactification.   ☐

DEFINITIONS: A topological space X is *completely regular* if singleton sets are closed and, given a closed set C and a point $x \notin C$, there exists an $f \in C_b(X)$ such that $f(x) = 0$ and $f(C) = \{1\}$ [**10**, p.4].

The *complete regularisation* of a topological space X is a completely regular space ρX together with a continuous surjection $\rho: X \to \rho X$ such that, for every completely regular space Y and every continuous surjection $k: X \to Y$, there is a unique continuous surjection $k': \rho X \to Y$ with $k = k' \circ \rho$.

LEMMA 2.3. *Let X be a topological space. There is an equivalence relation on X defined by $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$ for all $f \in C_b(X)$, and $(X/\!\sim, q)$ is the complete regularisation of X, where $q: X \to X/\!\sim$ is the quotient map.*

PROOF: Suppose $k: X \to Y$ is a continuous surjection into a completely regular space. We claim that the map $k': X/\!\sim \to Y$ satisfying $k'(q(x)) = k(x)$ is well defined. Suppose $q(x_1) = q(x_2)$, that is, $f(x_1) = f(x_2)$ for all $f \in C_b(X)$. Since $g \circ k \in C_b(X)$ for all $g \in C_b(Y)$, $g(k(x_1)) = g(k(x_2))$ for all g. But Y is completely regular, so $k(x_1)$ must equal $k(x_2)$, and k' is well-defined. The map k' is continuous by the definition of the quotient topology.   ☐

REMARKS. *1.* When A is unital, $\operatorname{Prim} A$ is compact, so the Stone-Čech compactification of $\operatorname{Prim} A$ *is* the complete regularisation by definition. Thus, Theorem 2.2 says that $(\operatorname{Prim} ZA, \operatorname{Res}_{\mathrm{id}})$ is the complete regularisation of $\operatorname{Prim} A$. Since the complete regularisation map is a surjection, when A is unital $\operatorname{Res}_{\mathrm{id}}$ maps onto $\operatorname{Prim} ZA$.

2. Becker proves that $\operatorname{Res}_{\mathrm{id}}$ is the complete regularisation for unital A [**1**, p.268]. But he seems to assert that if $P_1, P_2 \in \operatorname{Prim} A$ can be separated by open neighbourhoods in $\operatorname{Prim} A$, then there exists $f \in C_b(\operatorname{Prim} A)$ such that $f(P_1) = 0$ and $f(P_2) = 1$. We show that this can fail in general by providing an example of a C^*-algebra A, with P_1 and P_2 in $\operatorname{Prim} A$ which can be separated, but there exists P_3 which can't be separated from either P_1 or P_2. Thus a function in $C_b(\operatorname{Prim} A)$ must agree at P_1 and P_3, and at P_2 and P_3, and hence agrees at P_1 and P_2. This example is due to Dana Williams.

Let

$$A = \{f \in C([0,1], M_2 \oplus M_2) : f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \oplus \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \text{ for some } \alpha, \beta, \gamma \in \mathbf{C}\}.$$

Let p_1 and p_2 be the projections $\text{id} \oplus 0$ and $0 \oplus \text{id}$ respectively. Then for each $t \in [0,1)$, let $\pi_i^t(f) = p_i f(t)$. We also have representations defined by $\pi_x(f) = \alpha$, $\pi_y(f) = \beta$ and $\pi_z(f) = \gamma$, where $f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \oplus \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}$. These are all the irreducible representations. Note that as $t \to 1$, π_1^t converges to both π_x and π_y, while π_2^t converges to both π_y and π_z. It follows that neither π_x and π_y, nor π_y and π_z, can be separated by disjoint open sets. Since A is a Type I C^*-algebra, neither $\ker \pi_x$ and $\ker \pi_y$, nor $\ker \pi_y$ and $\ker \pi_z$, can be separated by disjoint open sets. Now let $I_1 = \{f \in A : p_2 f(t) = 0 \text{ for all } t \in [0,1]\}$, and similarly for I_2. Then $O_{I_i} = \{Q \in \text{Prim}\, A : Q \not\supseteq I_i\}$, for $i = 1, 2$, are disjoint open sets with $\ker \pi_x \in O_{I_1}$ and $\ker \pi_z \in O_{I_2}$. Thus $\ker \pi_x$ and $\ker \pi_z$ can be separated.

3. BECKER ON SECTIONAL REPRESENTATION

Becker asks for a simple condition on A which ensures that the the C^*-bundle produced by the sectional representation theorem is continuous. From [7, Theorem 3.1] we know that this happens only when $\text{Res}_{\text{id}} : \text{Prim}\, A \to \text{Prim}\, ZM(A)$ is open. We shall now show that our condition is equivalent to the condition Becker gives [1, p.268].

PROPOSITION 3.1. *Let A be a C^*-algebra and $\text{id} : ZM(A) \to ZM(A)$ be the identity map. The map $\text{Res}_{\text{id}} : \text{Prim}\, A \to \text{Prim}\, ZM(A)$ is open, if and only if*

$$I \supseteq \bigcap\{J \in \text{Prim}\, ZM(A) : a \in \text{Ext}_{\text{id}}(J)\} \Rightarrow a \in \text{Ext}_{\text{id}}(I),$$

where $a \in A$ and $I \in \text{Prim}\, ZM(A)$.

PROOF: Suppose Res_{id} is open and $I \supseteq \bigcap\{J : a \in \text{Ext}(J)\}$. Dixmier says $\{P \in \text{Prim}\, A : a \notin P\}$ is open in $\text{Prim}\, A$ [4, 3.1.2]. Since Res is open, $\{\text{Res}(P) : a \notin P\}$ is open in $\text{Prim}\, ZM(A)$. We claim that this open set is equal to $\{J : a \notin \text{Ext}(J)\}$. Suppose J is in the first set. That is, there exists a P such that $J = \text{Res}(P)$ and $a \notin P$. This is the same as $P \supseteq \text{Ext}(J)$ and $a \notin P$, and this implies $a \notin \text{Ext}(J)$. Conversely, suppose J is in the second set. That means $a \notin \text{Ext}(J)$. So, there must exist at least one primitive ideal $P \supseteq \text{Ext}(J)$ such that $a \notin P$, because every ideal is the intersection of all primitive ideals containing it. But $P \supseteq \text{Ext}(J)$ is equivalent to $\text{Res}(P) = J$ (since J is maximal), so J is in the first set. Thus we have that $\{J : a \in \text{Ext}(J)\}$ is closed. Since $I \supseteq \bigcap\{J : a \in \text{Ext}_{\text{id}}(J)\}$, I is in the closure of the set, so I is actually an element of it, which means that $a \in \text{Ext}(I)$.

Conversely, suppose $I \supseteq \bigcap \{J \in \operatorname{Prim} ZM(A) : a \in \operatorname{Ext_{id}}(J)\} \Rightarrow a \in \operatorname{Ext_{id}}(I)$. We begin by proving $I_\lambda \to I$ in $\operatorname{Im}(\operatorname{Res_{id}})$ implies $\operatorname{Ext}(I) \supseteq \bigcap_\lambda \operatorname{Ext}(I_\lambda)$. Suppose $I_\lambda \to I$ and $a \in \operatorname{Ext}(I_\lambda)$ for all λ. Then $I_\lambda \in \{J : a \in \operatorname{Ext}(J)\}$ for all λ, so $I \supseteq \bigcap_\lambda I_\lambda \supseteq \bigcap \{J : a \in \operatorname{Ext}(J)\}$. By the assumption then, $a \in \operatorname{Ext}(I)$ and $\operatorname{Ext}(I) \supseteq \bigcap_\lambda \operatorname{Ext}(I_\lambda)$.

To show Res_ι is open, its enough to show that $\operatorname{Comp} \operatorname{Res}(O_K)$ is closed. Suppose $I_\lambda \to I$ in $\operatorname{Im}(\operatorname{Res})$ and $I_\lambda \in \operatorname{Comp} \operatorname{Res}(O_K)$ for all λ. That means $\operatorname{Ext}(I_\lambda) \in \operatorname{Comp}(O_K)$ for all λ (since $J \in \operatorname{Res}(O_K) \Leftrightarrow \operatorname{Ext}(J) \in O_K$ [7, Equation 2]). That is, $\operatorname{Ext}(I_\lambda) \supseteq K$ for all λ, so $\bigcap_\lambda \operatorname{Ext}(I_\lambda) \supset K$, and then we have $\operatorname{Ext}(I) \supseteq K$. Hence $I \in \operatorname{Comp} \operatorname{Res}(O_K)$ and $\operatorname{Comp} \operatorname{Res}(O_K)$ is closed. ▯

References

[1] T.Becker, 'A few remarks on the Dauns-Hofmann theorems for C^*-algebras', *Arch. Math.* **43** (1984), 265–269.

[2] J. Dauns and K.H. Hofmann, 'Representations of rings by sections', *Mem. Amer. Math. Soc.* **83** (1968).

[3] J. Dixmier, 'Ideal center of a C^*-algebra', *Duke Math. J.* **35** (1968), 375–382.

[4] J. Dixmier, C^*-*algebras* (North-Holland, New York, 1977).

[5] E.C. Gootman and A.J. Lazar, 'Crossed products of type I AF algebras by abelian groups', *Israel J. Math.* **56** (1986), 267–279.

[6] P. Green, 'The local structure of twisted covariance algebras', *Acta Math.* **140** (1978), 191–250.

[7] M. Nilsen, 'C^*-bundles', (preprint, 1995).

[8] G. Pedersen, 'Applications of weak* semicontinuity in C^*-algebra theory', *Duke Math. J.* **39** (1972), 431–450.

[9] Z. Semadeni, *Banach spaces of continuous functions*, Volume 1 (PWN, Polish Scientific, Warsaw, 1971).

[10] R.C. Walker, *The Stone-Čech compactification* (Springer Verlag, Berlin, Heidelberg, New York, 1974).

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