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An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems

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Abstract. Numerical-valued cohomological index theories for G-pairs (X, A) over B, where G is a compact Lie group, have proved useful in critical point theory and in proving Borsuk-Ulam and Bourgin-Yang theorems. More information (which is lost in taking numerical values) is obtained using an ideal-valued theory, and this theory is applied to estimating the size of the zero set of a G-map from certain G-manifolds to a G-module. Parametrized versions of these theorems are also obtained by a principle which applies quite generally.

1. *Introduction*

Ljusternik-Schnirelmann category and cohomological index theory (see e.g. [**4**], [**5**] and [**6**]) are both examples of *numerical*-valued 'index theories' which are quite useful in proving 'Borsuk-Ulam' and 'Bourgin-Yang' theorems as well as in the study of critical points of invariant functionals. Our objective here is to begin the study of an ideal-valued index theory which has distinct advantages over its numerical counterparts. In the simplest situation, if X is a G-space, $\text{Index}^G X$ is an ideal in the ring $\Lambda = H^*(BG; \mathbb{K})$, where $\mathbb{K}$ is an appropriate coefficient field. The corresponding numerical index $\dim_{\mathbb{K}} (\Lambda/\text{Index}^G X)$ can be infinite (hence of limited value) even when $\text{Index}^G X$ is a proper ideal conveying useful information. In addition, two G-spaces X and Y may be distinguished by the ideals $\text{Index}^G X$ and $\text{Index}^G Y$ while the corresponding numerical values are finite and equal. As an illustration, this theory will be applied to the following problem of Lasry and Magill (see [**7**] for an alternative solution) which arose while they were working on the equilibrium of incomplete markets (mathematical economics). In its unparametrized form it is the following question of the Borsuk-Ulam type. Does an $O(k)$-equivariant map $f: V_{n,k} \to (\mathbb{R}^k)^{n-k}$ have a non-empty zero set? In this case the numerical index of $V_{n,k}$ (k-frames in $\mathbb{R}^n$) is finite while that of $(\mathbb{R}^k)^{n-k} - 0$ is infinite, which puts the problem outside the scope of numerical-valued theory. The technique employed observes first that the existence of a G-map $f: X \to Y$, where X and Y are G-spaces, forces $\text{Index}^G X \supset \text{Index}^G Y$. Then one computes the ideals in question and shows the containing relation is false in the case of $V_{n,k}$ and $(\mathbb{R}^k)^{n-k} - 0$. The parametrized version considers an $O(k)$-map $f: P \times V_{n,k} \to P \times (\mathbb{R}^k)^{n-k}$, where P is a parameter space on which $O(k)$ acts trivially.

Section 2 contains a brief description of index theory of pairs (X, A) over a base B along with basic properties, in the setting of a general multiplicative cohomology theory, and quickly specializes to the index of a single G-space employing the G-cohomology of Borel. Section 3 contains some useful computations. In § 4 we consider some general Borsuk–Ulam and Bourgin–Yang theorems together with parametrized versions. Applications, including the Lasry–Magill problem, are given in § 5. Finally § 6 gives a brief preview (without proofs) of how the ideal-valued theory fits into the Ljusternik–Schirelmann method in critical point theory. A worthwhile feature of this approach is the estimate of the ideal Index ${}^{G}K_c$ (see (3) of § 6), where K_c is a critical set. Previous estimates of the 'size of K_c' require that the numerical-valued indices be *a priori* finite (e.g. [4] and [5]) and in the contrary case fail to give any definite information.

2. *Index thoery*

The idea of an ideal-valued cohomological index theory can be formulated quite generally. We will not pursue this generality in great detail and will specialize quickly to the case of paracompact G-pairs and G-cohomology. But first consider any category $\mathcal{P}$ of topological pairs (X, A) and maps and let $h(X, A)$ denote a multiplicative cohomology theory on $\mathcal{P}$, i.e. h is a contravariant functor into graded algebras over a field $\mathbb{K}$, and h is equipped with long exact sequences, excision, the homotopy property, and a unit in $h(X)$ (where $h(X, \emptyset) = h(X)$), when $X \neq \emptyset$. Let B be a fixed space in $\mathcal{P}$ and $p:(X, A) \to (B, B)$ a map in $\mathcal{P}$, called a pair over B. We will often write $p:(X, A) \to B$ for short. Then we have $h(p): h(B) \to h(X)$ and the given multiplication $h(X, A) \otimes h(X) \to h(X, A)$, which together give $h(X, A)$, the structure of a right module over the ring $h(B)$.

Definition 2.1. $\text{Index}_B\,(X, A) = \text{Annih}\; h(X, A) \subset h(B)$, i.e. $\text{Index}_B\,(X, A)$ is the ideal in the ring $h(B)$ which annihilates $h(X, A)$.

Remark 2.2. The corresponding numerical index $|\text{Index}_B\,(X, A)| = \dim_{\mathbb{R}} [h(X, A)/\text{Index}_B\,(X, A)]$ was studied for Borel cohomology in [1] and [5].

Our main concern in this paper will be with the absolute case in definition 2.1, i.e. $A = \emptyset$. In this case we observe first that

$$\text{Index}_B X = \ker h(p), \qquad p: X \to B.$$

Next, $\text{Index}_B X$ possesses the following simple properties.

(a) (monotone) If $f: X_1 \to X_2$ is a map over B, then

$$\text{Index}_B X_1 \supset \text{Index}_B X_2.$$

(b) (additive) If $\{X_1 \cup X_2; X_1, X_2\}$ is an excisive pair in $\mathcal{P}$ and $p: X_1 \cup X_2 \to B$, then

$$\text{Index}_B X_1 \cdot \text{Index}_B X_2 \subset \text{Index}_B\,(X_1 \cup X_2)$$

or, in terms of ideal quotients [1],

$$\text{Index}_B X_2 \subset (\text{Index}_B\,(X_1 \cup X_2) : \text{Index}_B X_1).$$

The third basic property is that of continuity, which requires a continuous cohomology theory in the following sense. Suppose (X, A) in $\mathcal{P}$, where A is closed in X, X is normal, and closed subsets of X belong to $\mathcal{P}$. If $\mathcal{N}$ is the family of neighbourhoods $\mathcal{N}$ of A, $\mathcal{N} \in \mathcal{P}$ directed by inclusion, we assume $\varinjlim h(N) = h(A)$. If $\Lambda = h(\mathrm{pt})$, then $h(B)$ is a Λ-module and we further assume that X is a space over B and $h(B)$ is a Noetherian Λ-module.

(c) (continuity) There is an open set U such that $A \subset U \subset \bar{U} \subset X$ and $\mathrm{Index}_B \bar{U} = \mathrm{Index}_B A$.

Remark. For proofs of (a), (b) and (c) use the same techniques as the results in [4] for the associated numerical-valued index theory. There are also corresponding results for the relative index $\mathrm{Index}_B\,(X, A)$ as well as for another useful index theory denoted by $\delta\text{-}\mathrm{Index}_B\,(X, A)$, which is defined as the annihilator of image $\delta : h(A) \to h(X, A)$, where δ is the coboundary operator in the cohomology theory h.

The index theory primarily of interest to us here is a special case of the following. G will be a compact Lie group and $\mathcal{P}$ the category of paracompact G-pairs (X, A), i.e. X is a paracompact (Hausdorff) G-space and A is a closed G-subset. If B is a fixed paracompact G-space and $p:(X, A) \to (B, B)$ is a G-map, we have a G-pair over B, also written simply as $p:(X, A) \to B$. The cohomology theory $h(X, A) = H^*_G(X, A)$ is Borel cohomology based on Alexander–Spanier cohomology H^*. If $E = EG$, the total space of the universal G-bundle $EG \to BG$, then $H^*_G(X, A) = H^*(E \times_G X, E \times_G A)$ with coefficients in some field $\mathbb{K}$. The corresponding index theory will be denoted by $\mathrm{Index}^G_B\,(X, A)$. Thus $\mathrm{Index}^G_B\,(X, A)$ is an ideal in the ring $H^*_G(B)$. Note that $H^*_G(X, A)$ is also an ideal over $\Lambda = H^*(BG)$. When B is a point, $H^*_G(\mathrm{pt}) = H^*(E \times_G \mathrm{pt}) = H^*(BG)$. In this case $\mathrm{Index}^G_B\,(X, A)$ is denoted simply by $\mathrm{Index}^G\,(X, A)$ and is an ideal in $H^*(BG)$.

The absolute case $\mathrm{Index}^G X$ will occupy most of our attention and in this case $\mathrm{Index}^G X = \ker\,(H^*(BG) \to H^*_G(X))$, where $E \times_G X \to BG$ is induced by the projection $E \times X \to E$.

3. *Computations*

Our first computation is a product formula. We suppose $p_1 : X_1 \to B_1$, $p_2 : X_2 \to B_2$ are G_1- and G_2-spaces over B_1 and B_2, and let $p = p_1 \times p_2 : X_1 \times X_2 \to B_1 \times B_2$ denote the G-space over $B_1 \times B_2$, where $G = G_1 \times G_2$. We do not assume B_1 and B_2 are trivial G_1- and G_2-spaces. Then p induces

$$p_G : (E_1 \times E_2) \times_G (X_1 \times X_2) \to (E_1 \times E_2) \times_G (B_1 \times B_2),$$

which can be identified with

$$p_{1G_1} \times p_{2G_2} : (E_1 \times_{G_1} X_1) \times (E_2 \times_{G_2} X_2) \to (E_1 \times_{G_1} B_1) \times (E_2 \times_{G_2} B_2),$$

and hence p^*_G may be identified with (coefficients in the field $\mathbb{K}$)

$$p^*_{1G_1} \otimes p^*_{2G_2} : H^*_{G_1}(B_1) \otimes H^*_{G_2}(B_2) \to H^*_{G_1}(X_1) \otimes H^*_{G_2}(X_2)$$

in the presence of appropriate finiteness conditions, e.g. where $H^*_{G_1}(B_1)$ and $H^*_{G_1}(X_1)$ are of finite type (over $\mathbb{K}$). If we let $\not{h}_1 = \mathrm{Index}^{G_1}_{B_1} X_1$, $\not{h}_2 = \mathrm{Index}^{G_2}_{B_2} X_2$ and $\not{h} = \mathrm{Index}^{G_1 \times G_2}_{B_1 \times B_2}\,(X_1 \times X_2)$, then we have the following:

PROPOSITION 3.1. $\not{h} = \not{h}_1 \otimes H^*_{G_2}(B_2) + H^*_{G_1}(B_1) \otimes \not{h}_2$.

Proof. This is given by a simple vector space argument over $\mathbb{K}$. □

COROLLARY 3.2. *If* $H^*(BG_1)=\mathbb{K}[x_1,\ldots,x_k]$, $H^*(BG_2)=\mathbb{K}[y_1,\ldots,y_l]$, $\not{h}_1=\{f_1,\ldots,f_m\}$ *and* $\not{h}_2=\{g_1,\ldots,g_n\}$, *where* f_i *and* g_i *are polynomials in the* x *and* y *respectively, then*

$$\not{h}=\{f_1,\ldots,f_m,g_1,\ldots,g_n\},$$

the ideal generated by the polynomials f_i, g_j.

Example 3.3. Let S^p denote the p-sphere with a free $\mathbb{Z}_2$-action. Then let $\mathbb{K}=\mathbb{Z}_2$ so that in this case $H^*(BG)=\mathbb{Z}_2[t]$, $\dim t=1$. Then $\mathrm{Index}^{\mathbb{Z}_2} S^p=\{t^{p+1}\}$, the ideal generated by t^{p+1}. If $T(k)=\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$ (k times), then $T(k)$ acts freely on $S^{n_1}\times\cdots\times S^{n_k}$ and

$$\mathrm{Index}^{T(k)} S^{n_1}\times\cdots\times S^{n_k}=\{t_1^{n_1+1},t_2^{n_2+1},\ldots,t_k^{n_k+1}\}$$

in the polynomial ring $\mathbb{Z}_2[t_1,\ldots,t_k]$.

COROLLARY 3.4. *if we set* $X_2=B_2=\mathrm{pt}$ *in proposition* 3.1, *we obtain*

$$\mathrm{Index}_{B_1}^{G_1\times G_2} X_1=(\mathrm{Index}_{B_1}^{G_1} X_1)\otimes H^*(BG_2),$$

where $G_1\times G_2$ *acts on* X_1 *and* B_1 *by* $(g_1,g_2)x=g_1x$.

PROPOSITION 3.5. *Let* X_1 *and* X_2 *denote* G_1- *and* G_2-*spaces* (*over* pt). *Then the join* $X_1 * X_2$ *is a* $(G_1\times G_2)$-*space via the action* $(g_1,g_2)(x_1,t,x_2)=(g_1x_1,t,g_2x_2)$ *and*

$$\mathrm{Index}^{G_1\times G_2}(X_1*X_2)\subset[(\mathrm{Index}^{G_1} X_1)\otimes H^*(BG_2)]\cap[(H^*(BG_1)\otimes\mathrm{Index}^{G_2} X_2)],$$

where $\cap$ *represents the intersection of ideals.*

Proof. X_1 and X_2 appear in X_1*X_2 as $(G_1\times G_2)$-subspaces, so that

$$\mathrm{Index}^{G_1\times G_2} X_i\supset\mathrm{Index}^{G_1\times G_2} X_1*X_2,\qquad i=1,2.$$

An application of corollary 3.4 completes the proof. □

PROPOSITION 3.6. *Let* X_1 *and* X_2 *be as in proposition* 3.5. *Then*

$$\mathrm{Index}^{G_1\times G_2}(X_1*X_2)\supset[(\mathrm{Index}^{G_1} X_1)\otimes H^*(BG_2)]\cdot[H^*(BG_1)\otimes\mathrm{Index}^{G_2} X_2],$$

where $\cdot$ *represents the product of ideals.*

Proof. Observe that $X_1*X_2=Y_1\cup Y_2$, where Y_i is a closed $(G_1\times G_2)$-subspace equivalent to X_i and the additivity property applies.

Let $\not{h}_1=\mathrm{Index}^{G_1} X_1$ and $\not{h}_2=\mathrm{Index}^{G_2} X_2$, $\Lambda_1=H^*(BG_1)$, $\Lambda_2=H^*(BG_2)$. If we identify $\not{h}_1$ with $\not{h}_1\otimes 1$ and $\not{h}_2$ with $1\otimes\not{h}_2$, then $\not{h}_1\cdot\not{h}_2=(\not{h}_1\otimes\Lambda_2)\cdot(\Lambda_1\otimes\not{h}_2)$ and the conclusion of proposition 3.6 reads

$$\mathrm{Index}^{G_1\times G_2}(X_1*X_2)\supset(\mathrm{Index}^{G_1} X_1)\cdot(\mathrm{Index}^{G_2} X_2).$$ □

PROPOSITION 3.7. *When* $(\not{h}_1\otimes\Lambda_2)\cap(\Lambda_1\otimes\not{h}_2)=\not{h}_1\cdot\not{h}_2$, *we have*

$$\mathrm{Index}^{G_1\times G_2}(X_1*X_2)=(\mathrm{Index}^{G_1} X_1)\cdot(\mathrm{Index}^{G_2} X_2).$$

As special cases, let $G=\mathbb{Z}_2$, S^1, or $SU(2)$, $\mathbb{K}_2=\mathbb{Z}_2$ or $\mathbb{Q}$ so that $H^*(BG)=\mathbb{K}[t]$, the polynomial ring on a single indeterminate t. Let $d=\dim t$ so that $d=1$, 2 or 4 and suppose G acts freely on S^p and S^q, where $p=dk-1$ and $q=dl-1$. Then

$$\mathrm{Index}^G S^p=\{t_1^k\},\qquad \mathrm{Index}^G S^q=\{t_2^l\}$$

and

$$H^*(BG)\otimes H^*(BG)=\mathbb{K}[t_1,t_2].$$

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PROPOSITION 3.8

$$\text{Index}^{G\times G}(S^p * S^q) = \{t_1^k t_2^l\} \subset \mathbb{K}[t_1, t_2].$$

Proof. Proposition 3.7 applies. □

If we represent S^N as $S^{n_1} * S^{n_2} * \cdots * S^{n_k}$ and let $\mathbb{Z}_2$ act freely on S^{n_i} so that $T(k) = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (k times) operates on S^N component-wise, then we have the following:

COROLLARY 3.9

$$\text{Index}^{T(k)}(S^{n_1} * S^{n_2} * \cdots * S^{n_k}) = \{t_1^{n_1+1} t_2^{n_2+1} \cdots t_k^{n_k+1}\},$$

the ideal generated by the single monomial $t_1^{n_1+1} t_2^{n_2+1} \cdots t_k^{n_k+1}$. *In particular*

$$\text{Index}^{T(k)}(S^{q-1} * S^{q-1} * \cdots * S^{q-1}) = (t_1 t_2 \cdots t_k)^q.$$

Proof. This is left to the reader, using proposition 3.7. □

Remark 3.10. If $G = S^1$ or $SU(2)$ and we let $T(k)$ denote $S^1 \times \cdots \times S^1$ or $SU(2) \times \cdots \times SU(2)$ (k times) and S^1 and $SU(2)$ operate freely on the S^{n_i}, the results corresponding to example 3.3 and corollary 3.9 obtain with the necessary arithmetic modifications.

Observation 3.11. The dimension of the sphere in corollary 3.9 is $n+k-1$, where $n = \sum n_i$. In the spectral sequence of the fibration $S^{n+k-1} \to E \times_{T(k)} S^{n+k-1} \to BT(k)$ the differential operator d_{n+k} must send the generator of S^{n+k-1} to the monomial $t_1^{n_1+1} t_2^{n_2+1} \cdots t_k^{n_k+1}$ and this forces $H^*(BT(k)) \to H^*_{T(k)}(S^{n+k-1})$ to be surjective. This fact will be used later.

Now we consider G-spaces X_1 and X_2 (over points) and let G act on $X \times Y$ and $X * Y$ diagonally, i.e. $g(x, y) = (gx, gy)$, $g(x, t, y) = (gx, t, gy)$.

PROPOSITION 3.12

$$\text{Index}^G(X_1 \times X_2) \supset (\text{Index}^G X_1) \cap (\text{Index}^G X_2).$$

Proof. Consider the projection $X_1 \times X_2 \to X_i$, $i = 1, 2$. The monotone property implies $\text{Index}^G(X_1 \times X_2) \supset \text{Index}^G X_i$. □

PROPOSITION 3.13

$$\text{Index}^G X_1 * X_2 \supset (\text{Index}^G X_1) \cdot (\text{Index}^G X_2).$$

Proof. This is left to the reader. □

Remark 3.14. In general,

$$\text{Index}^G(X_1 * X_2) \neq (\text{Index}^G X_1) \cdot (\text{Index}^G X_2).$$

As an example let G act trivially on X_1. Then

$$\text{Index}^G X_1 = H^*(BG) = \text{Index}^G X_1 * X_2 \supset \text{Index } X_1 \cdot \text{Index}^G X_2 = \text{Index}^G X_2.$$

Now choose any X_2 such that $\text{Index}^G X_2$ is a proper ideal.

Remark 3.15. We note here that if X is a free G-space, then $\text{Index}^G X$ coincides with ker $H^*(BG) \to H^*(X/G)$, where $X/G \to BG$ is the classifying map [2].

We now consider a basic computation which is important to an application which we give later on. $V_{n,k}$ will denote the space of orthonormal k-frames in $\mathbb{R}^n$ and $O(k)$

will denote the orthogonal group on $\mathbb{R}^k$. Then $O(k)$ operates freely on $V_{n,k}$ by the usual action gv, $g \in O(k)$ and v a column vector representing the k-frame. We restrict this action to the subgroup $T(k)$ of diagonal matrices with entries ± 1 and compute $\operatorname{Index} T^{(k)}\, V_{n,k}$. The computation will be based on the fibration

$$V_{n-k+1,1} \xrightarrow{i} V_{n,k} \xrightarrow{\pi} V_{n,k-1}, \tag{1}$$

where π is the projection on the first $k-1$ coordinates. Consider the sequence

$$\mathbb{Z}_2 \to T(k) \to T(k-1), \tag{2}$$

where $\mathbb{Z}_2$ injects on the last coordinate and $T(k)$ projects on the first $k-1$ coordinates. Dividing out the action of (2) on (1), we obtain a fibration

$$\mathbb{R}P^{n-k} \xrightarrow{\tilde{i}} \tilde{V}_{n,k} \xrightarrow{\tilde{\pi}} \tilde{V}_{n,k-1}. \tag{3}$$

We then have an induced diagram of fibrations

$$\begin{array}{ccccc}
S^{n-k} & \longrightarrow & \mathbb{R}P^{n-k} & \xrightarrow{p_{n-k+1,1}} & BT(1) = B\mathbb{Z}_2 \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{i}} & & \downarrow{\scriptstyle \tilde{i}_B} \\
V_{n,k} & \longrightarrow & \tilde{V}_{n,k} & \xrightarrow{p_{n,k}} & BT(k) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi_B} \\
V_{n,k-1} & \longrightarrow & \tilde{V}_{n,k-1} & \xrightarrow{p_{n,k-1}} & BT(k-1)
\end{array}$$

where the $p_{i,j}$ are classifying maps. Recall that our coefficients are $\mathbb{Z}_2$, and since $\tilde{i}_B^*$ and $p_{n-k+1,1}^*$ are surjective, $\tilde{i}^* : H^*(\tilde{V}_n, k) \to H^*(\mathbb{R}P^{n-k})$ is surjective and the Leray-Hirsch theorem [8] applies. If $\theta : H^*(\mathbb{R}P^{n-k}) \to H^*(\tilde{V}_{n,k})$ is a right inverse for $\tilde{i}^*$ and $u \in H^1(\mathbb{R}P^{n-k})$ is the generator, we may assume that if $H^*(BT(k)) = \mathbb{Z}_2[t_1, \ldots, t_k]$, then $\theta(u) = p_{n,k}^*(t_k)$ and, in fact, $\theta(u^i) = p_{n,k}^*(t_k^i)$, $1 \le i \le n-k$. We then have an isomorphism of $H^*(\tilde{V}_{n,k-1})$-modules

$$\varphi_k : H^*(\tilde{V}_{n,k-1}) \otimes H^*(\mathbb{R}P^{n-k}) \to H^*(\tilde{V}_{n,k})$$

given by $\varphi_k(x \otimes y) = \tilde{\pi}^*(x) \cdot \theta(y)$.

Note that $\tilde{i}^*(\tilde{\pi}^*(x) \cdot \theta(y)) = 0$ if $\dim x > 0$ and $\tilde{i}^*(\tilde{\pi}^*(1) \cdot \theta(y)) = y$, so that the kernel of $\tilde{i}^*$ corresponds precisely $\tilde{H}^*(\tilde{V}_{n,k}) \otimes H^*(\mathbb{R}P^{n-1})$. Set $\bar{t}_i = p_{n,k}(t_i)$, $1 \le i \le k$, and note that if we assume inductively that $p_{n,k-1}^*$ is surjective, then $p_{n,k}^*$ is surjective and $\tilde{H}^*(\tilde{V}_{n,k})$ is generated by $\bar{t}_1, \ldots, \bar{t}_k$. We are now in a position to state our computational result on the index of $V_{n,k}$.

THEOREM 3.16. $\operatorname{Index}^{T(k)} = \{f_1, \ldots, f_k\} \subset \mathbb{Z}_2[t_1, \ldots, t_k]$ *where* $\{f_1, \ldots, f_k\}$ *is the ideal generated by polynomials* $f_1, \ldots, f_k$. *Here* f_i *has the form*

$$f_i = t_i^{n-i+1} + w_{i,n-1} t_i^{n-i} + \cdots + w_{i,0}, \tag{4}$$

where $w_{i,j}$ *are polynomials in* $t_1, \ldots, t_{i-1}$ *and* $w_{i,j} t^j$ *is of degree* $n-i+1$. *In particular* $(t_1 t_2 \cdots t_k)^{n-k}$ *is not contained in* $\operatorname{Index}^{T(k)}(V_{n,k})$.

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Proof. We proceed by induction on k. Consider the element $\bar{t}_k^{n-k+1}$. Since $\tilde{i}^*(\bar{t}_k^{n-k+1}) = 0$, $\bar{t}_k^{n-k+1}$ has the form

$$\bar{t}_k^{n-k+1} = \bar{w}_{k,n-k}\bar{t}_k^{n-k} + \cdots + \bar{w}_{k,0}.$$

Hence

$$\bar{t}_k^{n-k+1} - \bar{w}_{k,n-k}\bar{t}_k^{n-k} - \cdots - \bar{w}_{k,0}$$

or

$$f_k = t_k^{n-k+1} + w_{k,n-k}t_k^{n-k} + \cdots + w_{k,0}$$

is in the kernel of $p_{n,k}^*$, where $p_{n,k}^*(w_{k,j}) = -\bar{w}_{k,j}$. By induction, $\{f_1, \ldots, f_k\} \subset \ker p_{n,k}^* = \mathrm{Index}^{T(k)} V_{n,k}$. Now a simple algebraic argument shows the reverse inclusion. It is also easy to see that $(t_1 t_2 \cdots t_k)^{n-k}$ cannot be expressed in terms of $f_1, \ldots, f_k$. The latter can also be seen inductively as follows. Write

$$(t_1 t_2 \cdots t_k)^{n-k} = (t_1 t_2 \cdots t_{k-1})^{n-k} t_k^{n-k},$$

and since $(\bar{t}_1 \bar{t}_2 \cdots \bar{t}_{k-1})^{n-k+1} \neq 0$, $(\bar{t}_1 \bar{t}_2 \cdots \bar{t}_{k-1})^{n-k} \cdot \bar{t}_k^{n-k} \neq 0$ using the isomorphism φ_k. Thus $(t_1 t_2 \cdots t_k)^{n-k}$ is not in the kernel of $p_{n,k}^* = \mathrm{Index}^{T(k)} V_{n,k}$. □

4. *Borsuk–Ulam–Bourgin–Yang (BUBY) theorems and the parameter principle*

We first formulate and prove a general Bourgin–Yang theorem as follows. Let V and V' denote G-spaces and Z' a closed G-set in V'. We assume through this section that $\Lambda = H^*(BG)$ is Noetherian.

THEOREM 4.1. *Let $f: V \to V'$ be a G-map and $Z = f^{-1}(Z')$. Then Z is a G-set and*

$$\mathrm{Index}^G Z \subset (\mathrm{Index}^G V : \mathrm{Index}^G (V' - Z')).$$

Proof. The proof uses the three basic properties as follows. Using the continuity property, choose a G-neighbourhood U of Z such that $Z \subset U \subset \bar{U}$ and $\mathrm{Index}^G \bar{U} = \mathrm{Index}^G Z$. Then additivity implies that

$$\mathrm{Index}^G \bar{U} \subset (\mathrm{Index}^G V : \mathrm{Index}^G (V - U)).$$

Finally, the monotone property implies that

$$\mathrm{Index}^G (V - U) \supset \mathrm{Index}^G (V' - Z')$$

and hence

$$\mathrm{Index}^G Z \subset (\mathrm{Index}^G V : \mathrm{Index}^G (V' - Z')).$$ □

A special case of the above theorem which is also an immediate consequence of the monotone property is the following:

PROPOSITION 4.2. *Let $f: V \to V'$ be a G-map and $Z = f^{-1}(Z')$. Then, if* $\mathrm{Index}^G V \not\supset \mathrm{Index}^G (V' - Z')$, $Z \neq \varnothing$.

We prove next a parametrized version of the above theorem. We are indebted to A. Dold for a suggestion which led to this general formulation. Our parameter space P will be compact, connected, smooth manifold (possibly with boundary), orientable over our coefficient field $\mathbb{K}$.

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PROPOSITION 4.3 (parameter principle). *Let P denote a parameter space and $p_0 \in P - \partial P$. Let V denote a compact, connected, smooth manifold, orientable over $\mathbb{K}$. $f:(P, \partial P) \times V \to (P, \partial P)$ is a given G-map such that $f_y:(P, \partial P) \to (P, \partial P)$ has non-zero degree, where $f_y(x) = f(x, y)$, $x \in P$, $y \in V$. Then $W = f^{-1}(p_0)$ is a G-set and* $\text{Index}^G\, W = \text{Index}^G\, V$.

Proof. (a) We first prove the proposition in the case when V is a free G-space. If $\tilde{V}$ denotes the orbit space V/G, then f induces

$$\begin{array}{ccc} P \times V & \xrightarrow{f} & P \\ & {\scriptstyle \eta} \searrow \quad \nearrow {\scriptstyle \tilde{f}} & \\ & P \times \tilde{V} & \end{array}$$

and $\eta(W) = \tilde{W} = \tilde{f}^{-1}(p_0)$. Choose a G-neighbourhood U of W such that $W \subset U \subset \bar{U}$ and $\text{Index}^G\, \bar{U} = \text{Index}^G\, W$. Let $\tilde{g}$ denote an approximation of $\tilde{f}$ so that $\tilde{g} \pitchfork p_0$, i.e. p_0 is a regular value for $\tilde{g}$ and $\tilde{N} = \tilde{g}^{-1}(p_0) \subset \tilde{U}$. Then $\tilde{N}$ is a submanifold with $\dim \tilde{N} = \dim \tilde{V}$. Let $N = \eta^{-1}(\tilde{N})$ and $g = \tilde{g}\eta$. Then we have a diagram

$$\begin{array}{ccc} N & \xrightarrow{\eta} & \tilde{N} \\ {\scriptstyle k_1} \downarrow & & \downarrow {\scriptstyle k_2} \\ V & \xrightarrow{\eta} & \tilde{V} \end{array}$$

where k_1 and k_2 are the usual projections. Note that $\deg f_y = \deg g_y$, $y \in V$. Also, it is easy to check, because $k_1(u, v) = y$ if and only if $v = y$ and $g(u, y) = p_0$, that $\deg k_1 = \deg k_2 = \deg f_y \neq 0$. We may apply the 'umkehr homomorphism' (Hopf transfer [3]) to the map k_2 to conclude that $k_2^*: H^*(\tilde{V}) \to H^*(\tilde{N})$ is injective. Since the action is free, we have that

$$k_{1G}^*: H_G^*(V) \to H_G^*(N)$$

is also injective and hence $\text{Index}^G\, N = \text{Index}^G\, V$. But $N \subset \bar{U}$ implies $\text{Index}^G\, N \supset \text{Index}^G\, \bar{U} = \text{Index}^G\, W$. Hence $\text{Index}^G\, V \supset \text{Index}^G\, W$. On the other hand the projection map $W \to V$ is a G-map and hence $\text{Index}^G\, W \supset \text{index}^G\, V$. Thus in the free case $\text{Index}^G\, W = \text{Index}^G\, V$. □

(b) To prove the general case, we reduce to the free case as follows. Recall that $E = EG$ is the limit of compact, smooth, free G-manifolds $E^1 \subset E^2 \subset \cdots \subset E^m \subset \cdots$, with the property that the inclusion map i_m induces *isomorphisms*

$$H^q(E \times_G X) \to H^q(E^m \times_G X)$$

for $q < m$, X a paracompact G-space [**6**]. Thus

$$\bigcap_m \text{Index}^G\,(E^m \times X) = \text{Index}^G\, X.$$

Now, in the general case, for each value of m we may apply (a) to the map

$$\bar{f}: P \times (E^m \times V) \to P,$$

where $\bar{f}(x, e, y) = f(x, y)$, to obtain

$$\text{Index}^G (E^m \times V) = \text{Index}^G (E^m \times W)$$

for each m, and hence

$$\text{Index}^G V = \text{Index}^G W. \qquad \square$$

We now have the parametrized version of theorem 4.1.

THEOREM 4.4. *Let P denote a parameter space (as described above) and V a compact, connected manifold which is orientable over $\mathbb{K}$. Let V' denote a G-space and*

$$\varphi = (f, g) : (P, \partial P) \times V \to (P, \partial P) \times V'$$

a G-map such that $f_y : (P, \partial P) \to (P, \partial P)$, $y \in V$, has non-zero degree over $\mathbb{K}$. If $Z' \subset V'$ is a closed subset of V', $p_0 \in P - \partial P$, and $Z(p_0) = \varphi^{-1}(p_0 \times Z')$, then

$$\text{Index}^G Z(p_0) \subset (\text{Index}^G V : \text{Index}^G (V' - Z')).$$

Proof. Let $W = f^{-1}(p_0)$. Then by the parameter principle $\text{Index}^G W = \text{Index}^G V$. Consider

$$g_0 = g \,|\, W : W \to V'$$

and note that $g_0^{-1}(Z') = Z(p_0)$. Applying theorem 4.1 gives

$$\text{Index}^G Z(p_0) \subset (\text{Index}^G V : \text{Index}^G (V' - Z')). \qquad \square$$

Remark. As pointed out by the referee, proposition 4.3, and consequently theorem 4.4, can be extended to the case where V is a compact G-subset of some $\mathbb{R}^N$, but this extension is not required here.

Remark 4.5. We note that the right-hand side of the inclusion

$$\text{Index}^G Z(p_0) \subset (\text{Index}^G V : \text{Index}^G (V' - Z')) \qquad \square$$

is independent of P and p_0. Thus, for appropriate V, the parametrized version follows immediately after one has established the result for the single space V.

5. *Applications*

We will give two applications, both involving the Euclidean space $(\mathbb{R}^k)^{n-k}$. $O(k)$ acts in the standard way on $\mathbb{R}^k$ and hence $O(k)$ acts on $\mathbb{R}^k \times \cdots \times \mathbb{R}^k$ ($n-k$ times) in a coordinate-wise manner. Let $T(k) \subset O(k)$ denote the subgroup of diagonal matrices $\text{diag}[\varepsilon_1, \ldots, \varepsilon_k]$, where $\varepsilon_i = \pm 1$. Thus $T(k) = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (k times) and $(\mathbb{R}^k)^{n-k}$ is a $T(k)$-space. We may also represent $(\mathbb{R}^k)^{n-k}$ as $(\mathbb{R}^{n-k})^k$, where $\mathbb{Z}_2$ acts freely on each $\mathbb{R}^{n-k} - 0$. Thus, as a $T(k)$-space $(\mathbb{R}^k)^{n-k} - 0$ is the same $T(k)$-homotopy type as the $T(k)$-space,

$$S^{k(n-k)-1} = S^{n-k-1} * \cdots * S^{n-k-1} \qquad (k \text{ times}).$$

Applying corollary 3.9, we have

$$\text{Index}^{T(k)} ((\mathbb{R}^k)^{n-k} - 0) = \{(t_1 t_2 \cdots t_k)^{n-k}\} \subset \mathbb{Z}_2[t_1, \ldots, t_k].$$

We recall next the $T(k)$-space $S^{n-1} \times S^{n-2} \times \cdots \times S^{n-k}$ of example 3.3, i.e. each $\mathbb{Z}_2$ factor of $T(k)$ acts freely on the corresponding sphere. Then

$$\text{Index}^{T(k)} S^{n-1} \times \cdots \times S^{n-k} = \{t_1^n, t_2^{n-1}, \ldots, t_k^{n-k+1}\}.$$

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THEOREM 5.1. *Let* $f: S^{n-1}\times\cdots\times S^{n-k}\to(\mathbb{R}^k)^{n-k}$ *denote a* $T(k)$*-map. Then* f *has an orbit of zeros.*

Proof. Since $(t_1t_2\cdots t_k)^{n-k}\notin\{t_1^n, t_2^{n-1},\ldots, t_k^{n-k+1}\}$, corollary 4.2 applies and the proof is complete. □

The Bourgin–Yang theorem 4.1 provides more information. Let $Z=f^{-1}(0)$, where f is as in theorem 5.1. Then

$$\text{Index}^{T(k)}Z\subset(\{t_1^n,\ldots,t_k^{n-k+1}\}:\{(t_1t_2\cdots t_k)^{n-k}\}).$$

It is easy to see that the monomial

$$\mu=t_1^{k-1}t_2^{k-2}\cdots t_k^0\notin\text{Index}^{T(k)}Z$$

or alternatively μ is not in the kernel ($\mathbb{Z}_2$-coefficients) of

$$H^*_{T(k)}(Z)\leftarrow H^*(BT(k)),$$

the homomorphism which defines $\text{Index}^{T(k)}Z$. Thus the cup length of $H^*_{T(k)}(Z)$ is at least $k(k-1)/2$. Since Z is a free $\mathbb{Z}_2$-space, $H^*_{T(k)}(Z)=H^*(\tilde{Z})$, where $\tilde{Z}=Z/T(k)$.

THEOREM 5.2. *Let* $f: S^{n-1}\times\cdots\times S^{n-k}\to(\mathbb{R}^k)^{n-k}$ *denote a* $T(k)$*-map and* $Z=f^{-1}(0)$*. Then, if* $\tilde{Z}=Z/T(k)$*, the cup length of* $\tilde{Z}$ *is at least* $k(k-1)/2$ *and the category of* $\tilde{Z}$ *is at least* $k(k-1)/2+1$*.*

Now, as noted in § 4, we may by the parameter principle extend theorem 5.2 to the corresponding parametrized version as follows.

THEOREM 5.3. *Let* P *denote a parameter space (see* § 4*) and*

$$\varphi:(P,\partial P)\times S^{n-1}\times\cdots\times S^{n-k}\to(P,\partial P)\times(\mathbb{R}^k)^{n-k}$$

a $T(k)$*-map. Take* $p_0\in P-\partial P$ *and let* $W=\varphi^{-1}(p_0,0)$*. If* $\varphi=(f,g)$ *and* $f_y:(P,\partial P)\to(P,\partial P)$ *has non-zero degree (in* $\mathbb{Z}_2$*), where* $f_y(x)=f(x,y)$, $x\in P$, $y\in S^{n-1}\times\cdots\times S^{n-k}$*, then*

$$\text{category }\tilde{W}\geq\frac{k(k-1)}{2}+1,$$

where $\tilde{W}=W/T(k)$*.*

Our next example involves a problem of Lasry and Magill (see [7] for an alternative solution). Let $V_{n,k}$ denote orthonormal k-frames in $\mathbb{R}^n$, which is an $O(k)$-space as in § 3. The Lasry–Magill problem may be formulated as follows. Let D^l denote an l-cell on which $O(k)$ acts trivially. Let

$$\varphi: D^l\times V_{n,k}\to D^l\times(\mathbb{R}^k)^{n-k}$$

denote an $O(k)$-map with the following property. If $\varphi=(f,g)$, where $f: D^l\times V_{n,k}\to D^l$, then $f|\partial d^l\times V_{n,k}$ is just a projection on the first factor.

Lasry–Magill problem. Show that there exists an $(x_0,y_0)\in D^l\times V_{n,k}$ such that $\varphi(x_0,y_0)=(0,0)$.

Note that $f_y:(D^l,\partial D^l)\to(D^l,\partial D^l)$, where $f_y(x)=f(x,y)$, is the identity on ∂D^l, so that $\deg f_y=1$. The reader will recognize then that the Lasry–Magill problem is just the parametrized version of the following.

THEOREM 5.4. *Let $f: V_{n,k} \to (\mathbb{R}^k)^{n-k}$ denote an $O(k)$-map and $Z = f^{-1}(0)$. Then f is also a $T(k)$-map and*

$$\operatorname{Index}^{T(k)} Z \subset (\operatorname{Index}^{T(k)} V_{n,k} : \operatorname{Index}^{T(k)} ((\mathbb{R}^k)^{n-k} - 0)).$$

Since $\operatorname{Index}^{T(k)} V_{n,k} \not\supset \operatorname{Index}^{T(k)} ((\mathbb{R}^k)^{n-k} - 0)$, $Z \neq \emptyset$.

Proof. Applying theorem 3.16, $(t_1 \cdots t_k)^{n-k}$ is not in $\operatorname{Index}^{T(k)} V_{n,k}$.

Theorem 5.4 may be strengthened as follows. Recall that in theorem 3.16 $\operatorname{Index}^{T(k)} V_{n,k}$ is generated by polynomials

$$\begin{aligned} f_1 &= t_1^n, \\ f_2 &= t_2^{n-1} + w_{2,n-2} t_2^{n-2} + \cdots + w_{2,0}, \\ &\vdots \\ f_k &= t_k^{n-k+1} + w_{k,n-k} t_n^{n-k} + \cdots + w_{k,0}, \end{aligned}$$

and the monomial $\mu = t_1^{k-1} t_2^{k-2} \cdots t_k^0$ is not in the ideal quotient

$$(\{f_1, \ldots, f_k\} : \{(t_1 \cdots t_k)^{n-k}\}).$$

Thus $\mu \notin \operatorname{Index}^{T(k)} Z$. □

THEOREM 5.5. *Let $f: V_{n,k} \to (\mathbb{R}^k)^{n-k}$ denote a $T(k)$-map and $Z = f^{-1}(0)$. Then*

$$\text{category } \tilde{Z} \geq \frac{k(k-1)}{2} + 1,$$

where $\tilde{Z} = Z / T(k)$, the orbit space of Z.

Using the parameter principle, we have the following parametrized version.

THEOREM 5.6. *Let P denote a parameter space and*

$$\varphi : (P, \partial P) \times V_{n,k} \to (P, \partial P) \times (\mathbb{R}^k)^{n-k}$$

a $T(k)$-map. Take $p_0 \in P - \partial P$ and let $W = \varphi^{-1}(p_0, 0)$. If $\varphi = (f, g)$ and $f_y : (P, \partial P) \to (P, \partial P)$ has non-zero degree (in $\mathbb{Z}_2$), where $f_y(x) = f(x, y)$, $x \in P$, $y \in V_{n,k}$, then

$$\text{category } \tilde{W} \geq \frac{k(k-1)}{2} + 1,$$

where $\tilde{W}$ is the orbit space of W with respect to $T(k)$.

Returning to theorem 5.4, we note that when $k = 1$ we have the classical result. It is also interesting to observe that for each value of k

$$(t_1 t_2 \cdots t_k)^{n-k+1} \subset \operatorname{Index}^{T(k)} V_{n,k}.$$

One way to see this is to show that $(t_1 t_2 \cdots t_k)^{n-k}$ actually maps onto the top-dimensional cohomology class of the manifold $G_{n,k}$ under $H^*(G_{n,k}) \leftarrow H^*(BO(k))$ (induced by the classifying map). This shows that our proof fails for $n-k$ increased by one, for any k.

Indeed the following example, suggested by a referee, shows that $p = n - k$ is the best possible for a $T(k)$-map $f: V_{n,k} \to (\mathbb{R}^k)^p$ to have zeros. Represent an element of $V_{n,k}$ as a $k \times n$ matrix A and for $1 \leq p \leq n$ let $f: V_{n,k} \to (\mathbb{R}^k)^p$ denote the map which sends A into the last p columns of A. Then f is a G-map for any subgroup of $O(k)$ and for $p = n - k + 1$, f has no zeros.

Remark 5.7. If in the preceding results $\mathbb{R}$ is replaced by the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$, then corresponding results obtain. The proofs are essentially identical.

As our final application of the use of the ideal-valued index in obtaining theorems of the Bourgin-Yang type, we mention the following general result. The proof is analogous to the proof of theorem 4.1 with δ-Index_B^G replacing Index^G. It is particularly useful in proving Bourgin-Yang theorems for maps of G-sphere bundles to G-vector bundles. In particular, the results in [**5**] which employed the numerical analogues of index theory follow readily using it.

THEOREM 5.8. *Let $f:(X, A)\to(X', A')$ denote a G-map over B, where $\Lambda = H^*(BG)$ is Noetherian and $H^*(B)$ is finitely generated over $\mathbb{K}$. Assume Z' is a closed G-set in X' and $f(A)\subset A'-Z'$. If*

$$(f|A)_G^*: H_G^*(A'-Z')\to H_G^*(A)$$

is surjective and $Z=f^{-1}(Z')$, then

$$\mathrm{Index}_B^G\, Z\subset(\delta\text{-}\mathrm{Index}_B^G\,(X, A)\colon \delta\text{-}\mathrm{Index}_B^G\,(X'-Z', A'-Z')).$$

6. $\mathrm{Index}^G(\cdot)$ *and the Ljusternik-Schnirelmann method*

We close with a brief preview on the use of the ideal-valued index theory in the Ljusternik-Schnirelmann method in critical point theory.

For simplicity, our setting will be a compact G-manifold M. Let Σ denote the family of closed G-sets in M and $\Lambda = H^*(BG) = H^*(BG, \mathbb{K})$, where $\mathbb{K}$ is the field of coefficients, usually suppressed in the notation. $I(M)$ will denote the ideals $\not{p}$ in Λ such that $\mathrm{Index}^G(M)\subset \not{p}$. Then let

$$\Sigma_{\not{p}} = \{X\in\Sigma\colon \mathrm{Index}^G X\subset \not{p}\},\qquad \not{p}\in I(M).$$

Then, if $f: M\to\mathbb{R}$ is a C^1-functional, we set

$$c_{\not{p}} = \inf_{X\in\Sigma_{\not{p}}}\sup_{x\in X} f(x).$$

It is not difficult to show that each $c_{\not{p}}$ is a critical value, called an ideal critical value for f. Now let c denote an ideal critical value and $I(c)=\{\not{p}\in I(M)\colon c_{\not{p}}=c\}$. Also let $M_c=\{x: f(x)\le c\}$ and K_c be the critical set with critical value c. Then $\not{p}_* = \mathrm{Index}^G M_c$ is the minimal ideal in $I(c)$. Furthermore, the multiplicity result on the size of K_c takes the following form: there is an ideal $\mathscr{a}_c\subset\Lambda$ such that:

(1) $\mathscr{a}_c \supsetneq \not{p}_*$;

(2) $\mathscr{a}_c \not\subset \not{p}$, $\not{p}\in I(c)$;

(3) $\mathrm{Index}^G K_c\subset(\not{p}_*:\mathscr{a}_c)$.

Under certain conditions this forces K_c to have positive G-cohomology. Finally, if M admits a proper filtration of ideals

$$\Lambda\supset\hat{\Lambda}=\not{p}_1\supset\not{p}_2\supset\cdots\supset\not{p}_N=\mathrm{Index}^G M$$

with an appropriate property, then any C^1 G-map $f: M\to\mathbb{R}$ has at least N critical orbits. This suggests a 'Morse theory' based upon ideal-valued index theory and this will be the subject of a forthcoming article.

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